Solution moment stability in stochastic differential delay equations

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(Received 11 May 1995)

We study the behavior of the first and second solution moments for linear stochastic differential delay equations in the presence of additive or multiplicative white and colored noise. In the presence of additive noise (white or colored), the stability domain of both moments is identical to that of the unperturbed system. When these moments lose stability, there is a Hopf bifurcation and the first moment oscillates with a period identical to the solution of the unperturbed equation, while the oscillation period of the second moment is exactly one half the period of the unperturbed solution and the first moment. When perturbations are of the parametric (or multiplicative) type and white noise is assumed, under the Itô interpretation the first moment of the solution preserves properties of the solution of the deterministic equation, while the behavior of the second moment depends on the amplitude of the stochastic perturbation. The critical delay value at which the second moment loses stability and becomes oscillating is derived, and it is less than the critical delay for the first moment. Under the Stratonovich interpretation, quite different properties were observed for the moment equations, namely, various critical values of the delay and period of oscillations. For the case of parametric colored noise perturbations, sufficient (p-stability) conditions are derived which are independent of the value of delay, and it is shown that colored noise has a stabilizing effect with respect to white noise.

PACS number(s): 02.50.Fz, 02.30.Ks, 02.50.Ey, 02.50.Wp

I. INTRODUCTION

The foundations of the mathematical theory of differential equations with retarded arguments have been extensively developed in [1–12], and functional differential equations have been studied intensively in the past two decades (see surveys in [11,13]), but in spite of the efforts of many contributors this field is still in its infancy. The areas of application of differential delay equations include the dynamics of laser systems [14,15], physiological control systems [16–18], liquid crystals [19], dynamical diseases [20–24], neural network models [25–30], and agricultural economics [31,32].

Often in applied areas where delays are important, deterministic differential delay equations are inadequate to capture the essence of the real situation, and one must instead frame models in terms of stochastic differential equations, which take into account the perturbations often present in the real world. These random perturbations have the property that in numerical simulations they can imply not only quantitative changes in the dynamics but also qualitative ones. However, in trying to verify these results analytically, one often encounters serious difficulties due to the complexity of probabilistic models. At the present time the theory of stochastic *ordinary* differential equations has been fundamentally shaped by the work of [33–46]. Unfortunately, in a comparative sense the solution behavior of stochastic differential delay equations is in a relatively undeveloped situation. Investigations in this direction can be found in [12,13,47–56]. Results establishing the existence and uniqueness of solutions to stochastic functional differential equations appear in [13,48,52].

Further, these investigations have usually dealt with *linear* stochastic differential delay equations, whereas in the real world the equations modeling actual processes are not only stochastic but also highly nonlinear. However, because of the paucity of available techniques for dealing with nonlinear differential delay equations, not to mention nonlinear stochastic differential delay equations, a common and useful (often the *only*) strategy is to linearize the system describing the process of interest about the fixed points and then to examine the local stability of these fixed points in response to small perturbations. When external perturbations are important, this leads in a natural way to the examination of the stability of linear stochastic differential delay equations.

Thus, linear stochastic differential delay equations mirror the local properties of mathematical models describing the behavior of real systems with delays in the presence of random perturbations. Further, this class of equations represents a bridge between linear differential delay equations and linear stochastic ordinary equations.

An interesting problem arising in applications is the investigation of the stability of the moments of the solutions of these linear stochastic differential decay equations, which can be reduced to a study of the deterministic linear differential delay equations. The purpose of this paper is to derive moment equations for the solutions of linear stochastic differential delay equations and to investigate the onset of oscillations in their first and second moments. The results presented here generalize those in [57].

1063-651X/95/52(4)/3366(11)/$06.00

The paper is organized as follows. In Sec. II we briefly present the mathematical preliminaries needed for the rest of the paper. Section III examines the effect of additive and multiplicative (or parametric) white noise on the stability behavior of the first and second moments of linear differential delay equations. We extend these results to the case of colored noise in Sec. IV. The paper concludes with a brief discussion in Sec. V.

II. MATHEMATICAL PRELIMINARIES

Let the probability space (Ω,Σ,P) be given, and $w(t)\in\mathbb{R}^1$ be a scalar Wiener process defined on Ω having independent stationary Gaussian increments with $w(0)=0$, $E\{w(t)-w(s)\}=0$, and $E\{w(t)w(s)\}=\min(t,s)$. The symbol E denotes the mathematical expectation. The sample trajectories of $w(t)$ are continuous, nowhere differentiable, and have infinite variation on any finite time interval. The upper limit of samples of a Wiener process approaches $+\infty$ with probability 1 for $t\rightarrow\infty$, while the lower limit is $-\infty$.

We denote by $\xi(t)$ a stationary Gaussian white noise process with $E\{\xi(t)\}=0$ and covariance function $E\{\xi(t)\xi(s)\}=\delta(t-s)$, where δ is the Dirac delta function. From the theory of stochastic differential equations we understand that, formally, a white noise process $\xi(t)$ is the derivative of the Wiener process $w(t)$ [41]. A colored noise process will be denoted by $\eta(t)$ and described in Sec. IV.

Our central interest is the oscillating properties of the solution moments of the stochastic differential delay equation driven by white noise $\xi(t)$

$$dx(t)=f(t,x_t)dt+g(t,x_t)\xi(t)dt\ ,\quad t\geq 0 \tag{2.1a}$$

or by colored noise η (defined in Sec. IV)

$$dx(t)=f(t,x_t)dt+g(t,x_t)\eta(t)dt\ ,\quad t\geq 0\ , \tag{2.1b}$$

where $x_t=x(t+\theta)$, $-\tau\leq\theta\leq 0$, $x(t)\in\mathbb{R}^1$. The initial condition for (2.1a) and (2.1b) is

$$x(\theta)=\phi(\theta)\ ,\quad -\tau\leq\theta\leq 0\ , \tag{2.2}$$

where ϕ is an arbitrary continuous deterministic function. A stochastic process $x(t)$ is called a *solution* of the stochastic differential equation (2.1) when it satisfies, with probability 1, the integral equation

$$x(t)=x(0)+\int_0^t f(s,x_s)ds+\int_0^t g(s,x_s)dw(s)\ ,$$

where the second integral is a stochastic integral (understood in either the Itô or Stratonovich sense; see the discussion in Sec. III [39,40].

Using both the Itô and Stratonovich calculus for stochastic differentials and properties of stochastic integrals, we derive the moment equations for the solutions of the stochastic equations (2.1a) and (2.1b) and give necessary and sufficient conditions for the stability of solutions. For equations with stochastic perturbations given by multiplicative colored noise, we have only been able to prove sufficient conditions for exponential mean-square stability (see Sec. IV B).

Our technique will usually involve asking when the solution of the linear differential delay equation

$$\dot{y}(t)=\alpha y(t)+\beta y(t-\tau)\ , \tag{2.3}$$

with the initial condition $y(\theta)=\phi(\theta)$, where $-\tau\leq\theta\leq 0$, approaches 0 as $t\rightarrow\infty$. The assumption that there exists a solution of (2.3) of the form $y(t)\sim e^{\lambda t}$ gives the characteristic quasipolynomial for the eigenvalues of (2.3):

$$\lambda-\alpha-\beta e^{-\lambda\tau}=0\ . \tag{2.4}$$

It is well known [9,11,58] that the necessary and sufficient condition for $\mathrm{Re}\lambda<0$, and thus for $\lim_{t\rightarrow\infty}y(t)=0$, is given by

$$\tau<\tau_c\equiv\frac{\cos^{-1}\left[-\frac{\alpha}{\beta}\right]}{\sqrt{\beta^2-\alpha^2}}\ , \tag{2.5}$$

which is shown graphically in the (α,β) parameter space in Fig. 1. Furthermore, when λ is pure imaginary, i.e., $\lambda=i\omega$, it is known that there may be a Hopf bifurcation in (2.3) corresponding to a pair of complex conjugate eigenvalues crossing the imaginary axis, separating the left- and right-hand complex plane. This Hopf bifurcation may be either subcritical or supercritical. Substituting $\lambda=i\omega$ into Eq. (2.4), we obtain the equation of the Hopf bifurcation boundary in parametric form:

$$\omega=-\beta\sin\omega\tau\ , \tag{2.6a}$$

$$\alpha=-\beta\cos\omega\tau\ . \tag{2.6b}$$

From Eqs. (2.6a) and (2.6b) it follows that the value of delay τ_c given by the expression

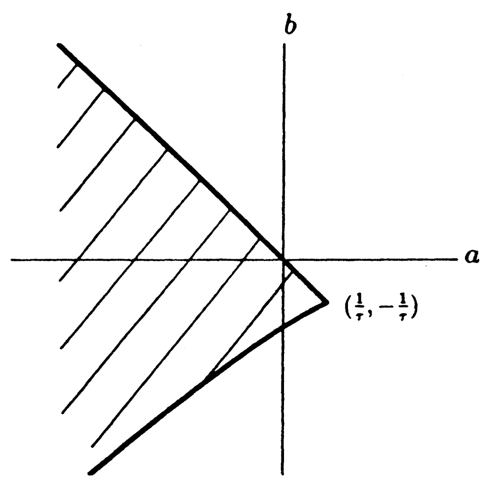

FIG. 1. Necessary and sufficient conditions for asymptotic stability of the trivial solution of (2.3). The hatched region denotes the region of parametric space in which the trivial solution of (2.3) is asymptotically stable. On the boundary defined by (2.7) there is a periodic solution of (2.3) with the period given by (2.9).

$$\tau=\tau_c\equiv\frac{\cos^{-1}\left[-\dfrac{\alpha}{\beta}\right]}{\sqrt{\beta^2-\alpha^2}} \qquad (2.7)$$

is the critical value at which the stability guaranteed by (2.5) is lost and the solution will demonstrate oscillatory behavior. From (2.6a) and (2.6b), it further follows that

$$\omega=\sqrt{\beta^2-\alpha^2}\ . \qquad (2.8)$$

Thus, Eq. (2.3) has a solution of the form $y(t)=e^{i\omega t}$, where the angular frequency ω is given by (2.8) if and only if the parameters α, β, τ satisfy (2.7). Because of the connection $\omega=2\pi/T$ between ω and the period T of oscillations, when these oscillations occur they have period

$$T=\frac{2\pi\tau_c}{\cos^{-1}\left[-\dfrac{\alpha}{\beta}\right]}=\frac{2\pi}{\sqrt{\beta^2-\alpha^2}}\ . \qquad (2.9)$$

III. MOMENT EQUATIONS WITH WHITE NOISE

A. Additive white noise

Consider the scalar linear stochastic differential delay equation with additive white noise

$$dx(t)=[ax(t)+bx(t-\tau)]dt+\sigma\, dw(t)\ ,\quad t\geq 0\ , \qquad (3.1)$$

where $w(t)$ is the standard scalar Wiener process, $\tau>0$ is a constant delay, and the initial function satisfies (2.2). The solution $x(t)$ of (3.1) is given by

$$x(t)=x(0)+\int_0^t[ax(s)+bx(s-\tau)]ds+\int_0^t\sigma\, dw(s)\ ,$$

where the second integral is a stochastic integral. For noise entering additively there is no difference between the Itô and Stratonovich interpretation of stochastic integrals.

Let $m(t)=Ex(t)$ denote the mathematical expectation of the solution of (3.1). Then

$$Ex(t)=Ex(0)+E\int_0^t[ax(s)+bx(s-\tau)]ds\ , \qquad (3.2)$$

since $E\int_0^t\sigma\, dw(s)=0$. We therefore obtain a differential delay equation for $m(t)$:

$$\dot m(t)=am(t)+bm(t-\tau)\ , \qquad (3.3)$$

with the initial condition $m(\theta)=Ex(\theta)=\phi(\theta)$ for $-\tau\leq\theta\leq 0$. Thus, the expectation value of the solution for the linear stochastic equation (3.1) satisfies the deterministic equation without noise. From the preceding remarks in Sec. II, we therefore know that a necessary and sufficient condition for the stability of the first moment $m(t)$ of (3.3) is given by (2.5) if we identify $\alpha=a$ and $\beta=b$. Further, when the parameters a,b,τ satisfy (2.7), then we know that a Hopf bifurcation in $m(t)$ will take place and the first moment will oscillate with a period given by (2.9). Thus, in the mean, the solution of the linear stochastic differential delay equation with additive white noise behaves precisely like the solution of the unperturbed deterministic equation (see Fig. 2).

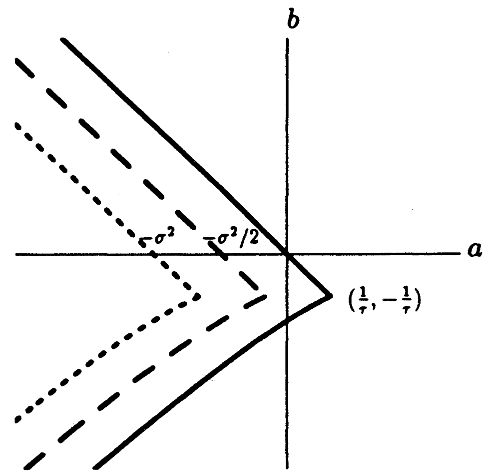

FIG. 2. Necessary and sufficient conditions for the stability of (2.3) in the presence of white noise. The rightmost solid line denotes the stability boundaries of the first and second moments for additive white noise and the stability boundary for the first moment in the presence of multiplicative white noise under the Itô interpretation. The dashed line corresponds to the stability boundary of the second moment with multiplicative white noise in the Itô interpretation and the stability boundary for the first moment in the Stratonovich interpretation. The dotted line marks the stability region for the second moment, multiplicative white noise (the Stratonovich interpretation).

The stability situation for the second moment of (3.1) is identical to that for the first moment. To show this, we derive the differential equation for $Ex^2(t)$, using the Itô differential rule:

$$dx^2(t)=2x(t)[ax(t)+bx(t-\tau)]dt$$
$$+\sigma^2dt+2\sigma x(t)dw(t)\ .$$

Integrating from 0 to t, taking the mathematical expectation of both parts, using the properties of the stochastic integral, and then finally differentiating with respect to t, we obtain

$$\frac{d}{dt}Ex^2(t)=2aEx^2(t)+2bE[x(t)x(t-\tau)]+\sigma^2\ . \qquad (3.4)$$

Introducing the notation $K(t,s)=E[x(t)x(s)]$, so that $K(t,t)=E[x(t)x(t)]=Ex^2(t)$, Eq. (3.4) becomes

$$\dot K(t,t)=2aK(t,t)+2bK(t,t-\tau)+\sigma^2\ , \qquad (3.5)$$

whose steady-state solution K^* satisfies the equation $2aK^*+2bK^*+\sigma^2=0$, or

$$K^*=-\frac{\sigma^2}{2(a+b)}\ . \qquad (3.6)$$

Defining a new variable $Z(t,s)=K(t,s)-K^*$, so that Z measures the deviation of K from K^*, Eq. (3.5) takes the form

$$\dot Z(t,t)=2aZ(t,t)+2bZ(t,t-\tau)\ . \qquad (3.7)$$

To examine the stability of (3.7), we take the same approach as for the first moment and examine the consequences of the assumption that (3.7) has a solution of the form

$$Z(t,s)\sim e^{\lambda t}e^{\lambda s} , \tag{3.8}$$

so that $Z(t,t)\sim e^{2\lambda t}$. We wish to obtain conditions on λ that will guarantee that (3.8) is indeed a solution. Again, a necessary and sufficient condition is that λ be a solution of the characteristic quasipolynomial

$$\lambda-a-be^{-\lambda\tau}=0 . \tag{3.9}$$

Using the material of Sec. II, we see that the second moment of (3.1) will be stable $[\lim_{t\to\infty}Ex^2(t)=K^*]$ if and only if (2.5) is satisfied. Further, for $\tau>\tau_c$, the second moment oscillates about K^* with an exponentially increasing amplitude. Finally, if (2.7) holds, there is again a Hopf bifurcation and the second moment is oscillatory (again about K^*) with a period *that is different from the period of the first moment.* More precisely, at the critical delay τ given by (2.7), the solution of (3.7) oscillates with a period

$$T'=\frac{2\pi}{\omega'}=\frac{\pi\tau_c}{\cos^{-1}(-a/b)}=\frac{\pi}{\sqrt{b^2-a^2}} \tag{3.10}$$

that is precisely one-half of the period of the oscillation of the first moment when stability is lost. This result is, of course, intuitively what one would expect.

Thus, we observe the same qualitative behavior for the moments of orders 1 and 2 for the solutions of (3.1) with additive white noise as for the deterministic undisturbed equation. This means that perturbing with additive white noise in a differential delay equation does not change the stability behavior of the mean, while for the second moment it is only the period of oscillations and the center that are changing. Therefore additive white noise does not have any influence on the stability and oscillating behavior of the solution of a linear differential delay equation, as shown in Fig. 2.

B. Multiplicative (parametric) white noise

In this section we consider the stability properties of the first and second solution moments of the stochastic differential delay equation

$$dx(t)=[ax(t)+bx(t-\tau)]dt+\sigma x(t)dw(t) , \quad t\geq 0 \tag{3.11}$$

with parametric white noise, where $\tau\geq 0$, and an initial function is given by (2.2). The solution $x(t)$ of (3.11) satisfies, with probability 1, the integral equation

$$x(t)=x(0)+\int_0^t[ax(s)+bx(s-\tau)]ds+\int_0^t\sigma x(s)dw(s) . \tag{3.12}$$

The last integral in (3.12) is a stochastic integral, which can be interpreted in either the Itô or Stratonovich sense.

Remark 1. In the Stratonovich calculus, as in the normal calculus, the stochastic integral

$$\int_0^t g(s,x_s(w))dw_s(w)$$

is defined as the limit of the midpoint approximation

$$\sum_{i=1}^{n} g\left[t_{i-1},\frac{(w_i+w_{i-1})}{2}\right][w_i-w_{i-1}]$$

for all partitions $p=t_0\leq t_1\leq\cdots\leq t_n=t$, where $w_i=w_{t_i}(w)$ as the maximum step size $\delta=\max_i(t_i-t_{i-1})\to 0$. In the Itô calculus, the integrand of the stochastic integral is approximated by a left-hand limit, so the integral is defined as the limit of

$$\sum_{i=1}^{n} g(t_{i-1},w_{i-1})[w_i-w_{i-1}] .$$

The main difference between the Itô and Stratonovich calculus occurs in their corresponding chain rules and in the calculation of stochastic integrals. For a stochastic process $x(t)$ interpreted by the Itô calculus, a special Itô formula must be used to calculate a stochastic differential of the complex function $f(t,x(t))$, while in the Stratonovich calculus the normal rules of classical calculus hold. From a purely mathematical viewpoint both the Itô and Stratonovich interpretations are correct, but in different modeling contexts one may be more appropriate than the other. Thus, for the modeler the central question is what stochastic differential equation must be chosen in order to describe accurately a given physically realizable process of interest. It has been pointed out [41,59–64] that the Stratonovich interpretation of a stochastic differential equation is the appropriate one when white noise can be considered as the limiting case of the colored noise actually existing in the system. This observation has been confirmed experimentally [65]. On the one hand, this situation can arise in many of the biological, engineering, and physical sciences. On the other hand, many systems are discrete in either time or state or both. In these cases the stochastic equation, obtained as a continuous time limit of a discrete time problem, would be appropriately interpreted according to Itô. Fortunately, as will become clear later, there is a strong connection between the two, and we are able to shift from one type of integral to the other, thus exploiting the advantages of each.

1. Itô interpretation of parametric white noise

First we examine the behavior of the first and second moments of the solution $x(t)$, assuming that the stochastic integral in (3.12) is interpreted as an Itô stochastic integral. For the mathematical expectation $m(t)=Ex(t)$ of the solution, we have the differential delay equation

$$\dot m(t)=am(t)+bm(t-\tau)$$

with initial function

$$m(\theta)=\phi(\theta) , \quad -\tau\leq\theta\leq 0 ;$$

i.e., $m(t)$ behaves as a solution of the deterministic delay equation. When a, b, and τ satisfy (2.5), the trivial solution $m=0$ is stable; and when they satisfy (2.7), the first moment of the solution of (3.11) becomes unstable and starts oscillating about 0 with period T given by (2.9), as we have shown graphically in Fig. 2.

Considering the second moment $Ex^2(t)$ of the solution $x(t)$ of (3.11), we obtain from Itô's rule

$$dx^2(t)=2x(t)[ax(t)+bx(t-\tau)]dt + \sigma^2 x^2(t)dt + 2\sigma x^2(t)dw(t) .$$

Integrating from 0 to t, taking the mathematical expectation of both parts, using the properties of stochastic integral, and then differentiating with respect to t, we get

$$\frac{d}{dt}Ex^2(t)=(2a+\sigma^2)Ex^2(t)+2bEx(t)x(t-\tau) . \quad (3.13)$$

Using the same procedure as in Sec. II A when we examined the stability of the second moment in the presence of additive white noise, we see that (3.13) becomes

$$\dot{K}(t,t)=(2a+\sigma^2)K(t,t)+2bK(t,t-\tau) , \quad (3.14)$$

with a corresponding characteristic quasipolynomial

$$\nu-(2a+\sigma^2)-2be^{-\frac{1}{2}\nu\tau}=0 , \quad (3.15)$$

where $\nu=2\lambda$. Thus we conclude (see Fig. 2) that the stochastic differential delay equation (3.11), with parametric white noise, will have a stable second moment if and only if

$$\tau<\tau_c^{\mathrm{sq}}=\frac{\cos^{-1}\left[-\frac{a+\sigma^2/2}{b}\right]}{\sqrt{b^2-(a+\sigma^2/2)^2}} . \quad (3.16)$$

At $\tau=\tau_c^{\mathrm{sq}}$ there is a Hopf bifurcation, and the second moment $Ex^2(t)$ of the solution of (3.11) loses its stability and oscillates with a period

$$T^{\mathrm{sq}}=\frac{\pi}{\sqrt{b^2-(a+\sigma^2/2)^2}} . \quad (3.17)$$

The oscillation period of the second moment when it becomes unstable, T^{sq}, no longer bears a simple relationship to the period of the first moment of its instability, as in the case of additive white noise. However, it is easy to show that $T^{\mathrm{sq}}<T$ whenever

$$0<\frac{\sigma^2}{2}<\sqrt{(b/a)^2-1} ;$$

and when the right-hand side of this inequality is violated, then $T^{\mathrm{sq}}>T$.

Comparing the behavior of the first and second moments of the solution $x(t)$ of (3.11), we can easily see that under the Itô interpretation the first moment $Ex(t)$ behaves like the solution of the deterministic equation, which means that, in the mean, the solution of the differential delay equation driven by parametric white noise does not differ from the properties of the solution of the undisturbed equation, while the behavior of the second moment does depend on the amplitude σ of the stochastic fluctuations. The critical value of the delay at which oscillations of the second moment $Ex^2(t)$ occur is explicitly dependent on the noise intensity σ^2 and obviously is less than the white noise critical delay for the first moment (which coincides with the critical delay value for the deterministic equation). Thus, there exists some interval between τ_c^{sq} and $\tau_c^m=\tau_c$ when the first moment of the solution is still stable while the second is oscillating with unbounded increasing amplitude. This is clearly illustrated in Fig. 2.

2. *Stratonovich interpretation of parametric white noise*

We now consider the Stratonovich interpretation of the stochastic differential equation (3.11). Assuming that the stochastic integral in (3.12) is a Stratonovich integral, and using the connection between Itô and Stratonovich representations [63], we can transform (3.11) into the corresponding Itô stochastic differential equation:

$$dx(t)=\left[\left[a+\frac{\sigma^2}{2}\right]x(t)+bx(t-\tau)\right]dt+\sigma x(t)dw(t) , \quad t\geq 0 . \quad (3.18)$$

As before, the differential equation for the first moment $m(t)=Ex(t)$ is given by

$$\dot{m}(t)=\left[a+\frac{\sigma^2}{2}\right]m(t)+bm(t-\tau) ,$$
$$m(\theta)=\phi(\theta) , \quad -\tau\leq\theta\leq 0 , \quad (3.19)$$

so the characteristic quasipolynomial is

$$\lambda-\left[a+\frac{\sigma^2}{2}\right]-be^{-\lambda\tau}=0 . \quad (3.20)$$

Consequently, the Itô stability condition is replaced by the first moment Stratonovich stability condition:

$$\tau<\tau_c\equiv\frac{\cos^{-1}\left[\frac{a+\sigma^2/2}{b}\right]}{\sqrt{b^2-(a+\sigma^2/2)^2}} . \quad (3.21)$$

When $\tau=\tau_c$ the mean value of the solution $x(t)$ of (3.18) begins to oscillate with a period

$$T=\frac{2\pi}{\sqrt{b^2-(a+\sigma^2/2)^2}} . \quad (3.22)$$

In a similar fashion we find that constant amplitude oscillations of the second moment $Ex^2(t)$ of the solution of (3.18) occur at a critical value of τ given by

$$\tau_c^{\mathrm{sq}}=\frac{\cos^{-1}\left[-\frac{a+\sigma^2}{b}\right]}{\sqrt{b^2-(a+\sigma^2)^2}} , \quad (3.23)$$

with a period

$$T^{\mathrm{sq}}=\frac{\pi}{\sqrt{b^2-(a+\sigma^2)^2}} . \quad (3.24)$$

Thus it is clear that the conditions for oscillation of the moments under the Itô interpretation are entirely equivalent to those of Stratonovich if we replace the coefficient a by $a+\sigma^2/2$, or more explicitly $a_I=a_S+\sigma^2/2$, as illustrated in Fig. 2.

Remark 2. Clearly, the Itô and Stratonovich interpretations lead to different predictions of the critical value of the delay and period of oscillations, which is also observed in stochastic differential equations [77]. The connection between them is easily seen by setting a_I and a_S as the values of the parameter a under the Itô and Stratonovich interpretations, respectively, and noting that one can pass between the results by using the connection $a_I = a_S + \sigma^2/2$; this shift is evident in the stability regions of Fig. 2. Different global properties of the stochastic processes defined by the Itô and Stratonovich stochastic equations have been discussed [41,61,64,65].

IV. MOMENT EQUATIONS FOR COLORED NOISE

In Sec. III we studied the effect of Gaussian white noise perturbations on the solution behavior of a linear differential delay equation. However, Gaussian white noise is an unattainable idealization of the real random perturbations and is an inappropriate representation of external noise when the effect of a nonzero correlation time (colored noise) in the noise needs to be taken into account.

Colored noise is modeled by the Ornstein-Uhlenbeck process $\eta(t)$ [41], which satisfies the Langevin equation

$$\frac{d\eta(t)}{dt} = -\alpha\eta(t) + \alpha\xi(t) \ , \quad t > 0 \ , \quad \eta(0) = \eta_0 \ , \tag{4.1}$$

where $\alpha > 0$ and $\xi(t)$ is a scalar white noise process.

Recalling some properties of the Ornstein-Uhlenbeck process $\eta(t)$, we note that the stochastic differential equation

$$d\eta(t) = -\alpha\eta(t)dt + \alpha\, dw(t) \ , \quad t > 0 \ , \quad \eta(0) = \eta_0 \ , \tag{4.2}$$

is linear in the narrow sense, is autonomous, and has a unique solution [41]

$$\eta(t) = \eta_0 e^{-\alpha t} + \alpha \int_0^t e^{-\alpha(t-s)} dw(s) \ , \tag{4.3}$$

where $w(t)$ is a Wiener process. Suppose that $E\eta_0(t) < \infty$; then

$$E\eta(t) = e^{-\alpha t} E\eta_0 \ . \tag{4.4}$$

For the correlation function we have

$$E\{\eta(t)\eta(s)\} = e^{-\alpha(t+s)} \left[E\eta_0^2 - \frac{\alpha}{2} \right] + \frac{\alpha}{2} e^{-\alpha|t-s|} \ . \tag{4.5}$$

In particular,

$$E\{\eta^2(t)\} = e^{-2\alpha t} \left[E\eta_0^2 - \frac{\alpha}{2} \right] + \frac{\alpha}{2} \ . \tag{4.6}$$

Thus for an arbitrary η_0, we have $\lim_{t\to\infty} e^{-\alpha t}\eta_0 = 0$, which means that at long times the distribution of $\eta(t)$ approaches a normal distribution, with a zero mean and a variance $\alpha/2$ for arbitrary constant η_0. When η_0 is normally distributed, $N(0,\alpha/2)$, then $\eta(t)$ is a stationary Gaussian process with $E\eta(t) = 0$ and exponentially decreasing correlation function

$$E\{\eta(t)\eta(s)\} = \frac{\alpha}{2} e^{-\alpha|t-s|} \ . \tag{4.7}$$

From (4.7) we have, in particular, $E\eta^2(t) = \alpha/2$.

Next, recall the properties of the integrated Ornstein-Uhlenbeck process

$$\mu(t) = \int_0^t \eta(s) ds \ , \tag{4.8}$$

started at time $t = 0$ at the origin $\mu(0) = 0$. For η_0 normally distributed $N(0,\alpha/2)$, the expectation of μ is given by

$$E\mu(t) = 0 \ , \tag{4.9}$$

and the variance is given by

$$E\mu^2(t) = t + \frac{1}{\alpha}(e^{-\alpha t} - 1) \ . \tag{4.10}$$

A. Additive colored noise

With these preliminary remarks, we now turn to a study of the effect of additive colored noise on the stability behavior of a linear differential delay equation. Consider a stochastic process $x(t)$ that satisfies the equation

$$dx(t) - [ax(t) + bx(t-\tau)]dt + \sigma\eta(t)dt \ , \quad t \ge 0 \ , \tag{4.11}$$

where $x(t) \in \mathbb{R}^1$, with the deterministic initial function $x(\theta) = \phi(\theta)$, $-\tau \le \theta \le 0$. Here $\eta(t)$ is a colored noise modeled by the Ornstein-Uhlenbeck process (4.2), and $\tau > 0$ is a constant delay.

To study the behavior of the moments of the solution $x(t)$ of the differential delay equation (4.11), we consider a two-component stochastic process $y(t) \equiv (x(t), \eta(t))$. We introduce the notation

$$A = \begin{bmatrix} a & \sigma \\ 0 & -\alpha \end{bmatrix} , \quad B = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} , \quad c = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} ,$$

so that we can rewrite the original equations (4.2) and (4.11) as

$$dy(t) = [Ay(t)dt + By(t-\tau)]dt + c\, dw(t) \ , \quad t > 0 \ . \tag{4.12}$$

By a solution of (4.12) we mean the stochastic process $y(t)$ defined by

$$y(t) = y(0) + \int_0^t [Ay(s) + By(s-\tau)]dt + \int_0^t c\, dw(s) \ , \tag{4.13}$$

where the last integral is a stochastic internal. To define an initial function $y(\theta) = \phi(\theta)$, $-\tau \le \theta \le 0$ for (4.12), we consider formally that $\eta(\theta) = \eta_0$, where η_0 is assumed to be a normally distributed $N(0,\alpha/2)$ random variable.

Denoting the mathematical expectation of the solution $y(t)$ of (4.12) by $m(t) = Ey(t)$, we obtain

$$\dot{m}(t) = Am(t) + Bm(t-\tau) \ . \tag{4.14}$$

Thus in the presence of colored noise the stability properties of the first moment are identical to those in the presence of white noise, which in turn are identical to the unperturbed system. This is shown in Fig. 3 by the solid line.

To examine the stability of the second moment of $y(t)$ in the presence of additive colored noise, we use Itô's rule to give the stochastic differential of $y(t)y^T(t)$:

$$\frac{d}{dt}E\{y(t)y^T(t)\}=E\{dy(t)y^T(t)+y(t)dy^T(t)+cc^T\}$$
$$=E\{[Ay(t)+By(t-\tau)]y^T(t)+y(t)[y^T(t)A^T+y^T(t-\tau)B^T]+cc^T\}$$
$$=E\{[Ay(t)y^T(t)+y(t)y^T(t)A^T+By(t-\tau)y^T(t)+y(t)y^T(t-\tau)B^T]+cc^T\}\ .$$

Let $R(t,s)=E\{y(t)y^T(s)\}$ be the covariance matrix of the process $y(t)$, so that $R(t,t)$ satisfies

$$\dot{R}(t,t)=AR(t,t)+R(t,t)A^T+BR(t-\tau,t)+R(t,t-\tau)B^T+cc^T\ . \tag{4.15}$$

This can be rewritten as the system

$$\frac{d}{dt}Ex^2(t)=2aEx^2(t)+2\sigma Ex(t)\eta(t)+2bEx(t-\tau)x(t)\ , \tag{4.16a}$$

$$\frac{d}{dt}Ex(t)\eta(t)=(a-\alpha)Ex(t)\eta(t)+\sigma E\eta^2(t)+bEx(t-\tau)\eta(t)\ , \tag{4.16b}$$

$$\frac{d}{dt}E\eta^2(t)=-2\alpha E\eta^2(t)+\alpha^2\ . \tag{4.16c}$$

Let us denote $Ex(q)\eta(t)=P(q,t)$, which gives

$$Ex(t)\eta(t)=P(t,t)\ ,\quad Ex(t-\tau)\eta(t)=P(t-\tau,t)\ ,$$

and let $K(t,s)=Ex(t)x(s)$, which gives

$$K(t,t)=Ex^2(t)\ ,\quad K(t-\tau,t)=Ex(t-\tau)x(t)\ .$$

Solving (4.16c) with an initial condition $E\eta_0$ [remember η_0 is a normally distributed $N(0,\alpha/2)$ random variable], we obtain, after substitution of $E\eta^2(t)$ in (4.16b), the pair of equations

$$\frac{dP(t,t)}{dt}=(a-\alpha)P(t,t)+\frac{\sigma\alpha}{2}+bP(t-\tau,t)\ , \tag{4.17a}$$

$$\frac{dK(t,t)}{dt}=2aK(t,t)+2\sigma P(t,t)+2bK(t-\tau,t)\ . \tag{4.17b}$$

Solving (4.17b) for $P(t,t)$, we obtain

$$P(t,t)=\frac{1}{2\sigma}\left[\frac{dK(t,t)}{dt}-2aK(t,t)-2bK(t-\tau,t)\right]; \tag{4.18}$$

so from (4.18) we have

$$P(t-\tau,t)=\frac{1}{2\sigma}\left[\frac{dK(t-\tau,t)}{dt}-2aK(t-\tau,t)-2bK(t-2\tau,t)\right] \tag{4.19}$$

and

$$\frac{dP(t,t)}{dt}=\frac{1}{2\sigma}\left[\frac{d^2K(t,t)}{dt^2}-2a\frac{dK(t,t)}{dt}-2bK(t-\tau,t)dt\right]. \tag{4.20}$$

Substituting (4.18) and (4.19) in (4.17a), we obtain

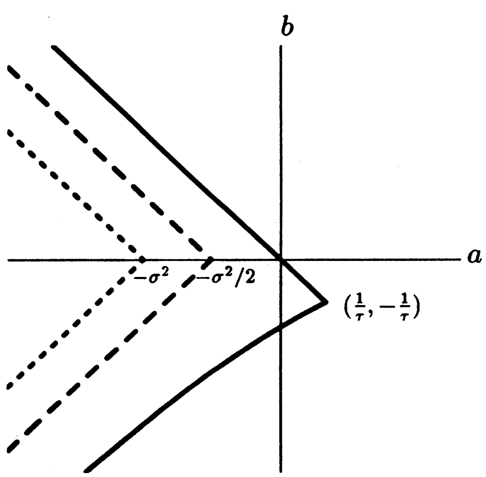

FIG. 3. Stability conditions for (2.3) in the presence of colored noise. The solid line denotes the stability boundaries of the first and second moments for additive colored noise. The dashed line shows sufficient conditions for exponential stability of the first moment for multiplicative colored noise. The dotted line denotes the sufficient mean-square stability region for multiplicative colored noise.

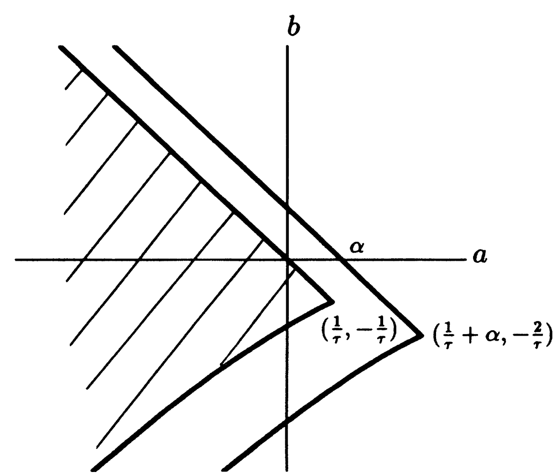

FIG. 4. Necessary and sufficient conditions for the solutions of (4.25) and (4.26) to satisfy $\mathrm{Re}\lambda\leq 0$. The hatched region corresponds to (4.25). The right solid line corresponds to the case of (4.26).

$$\frac{dP(t,t)}{dt}=\frac{a-\alpha}{s\sigma}\left[\frac{dK(t,t)}{dt}-2aK(t,t)\right.$$
$$\left.-2bK(t-\tau,t)\right], \tag{4.21}$$

whose steady state satisfies

$$0=K^*\frac{a-\alpha}{2\sigma}\{-2a-2b\}+\frac{\sigma\alpha}{2}+\frac{b}{2\sigma}\{-2a-2b\} .$$

Explicitly K^* is given by

$$K^*=\frac{\sigma^2\alpha}{2}\frac{1}{(a+b)(a+b-\alpha)} . \tag{4.22}$$

Introducing the new variable

$$F(s,t)=K(s,t)-K^*$$

in (4.21) and using (4.22), we get the equivalent relation

$$\frac{d^2F(t,t)}{dt^2}-2a\frac{dF(t,t)}{dt}-2bF(t-\tau,t)dt$$
$$=(a-\alpha)\left[\frac{dF(t,t)}{dt}-2aF(t,t)-2bF(t-\tau,t)\right]$$
$$+b\left[\frac{dF(t-\tau,t)}{dt}-2aF(t-\tau,t)\right.$$
$$\left.-2bF(t-2\tau,t)\right] . \tag{4.23}$$

Again we examine the consequences of the assumption that the solution $F(s,t)$ to (4.23) has the form $F(s,t)\equiv e^{\lambda s}e^{\lambda t}$, so that $F(t,t)\equiv e^{2\lambda t}$. If this is the case, then the quasipolynomial for (4.23) has the form

$$(\lambda-a-be^{-\lambda\tau})[2\lambda-(a-\alpha)-be^{-\lambda\tau}]=0 . \tag{4.24}$$

It follows from (4.24) that either

$$\lambda-a-be^{-\lambda\tau}=0 \tag{4.25}$$

or

$$2\lambda-(a-\alpha)-be^{-\lambda\tau}=0 \tag{4.26}$$

is satisfied. For stability considerations we are obviously interested in knowing which of the relations (4.25) or (4.26) must be taken in order to ensure that all the eigenvalues of (4.24) have negative real parts. The criteria for the solutions λ of (4.25) and (4.26) to satisfy $\text{Re}\lambda\leq 0$ are well known [11] and presented graphically in Fig. 4. As is obvious from Fig. 4, and which may easily be checked analytically, the necessary and sufficient conditions for local stability of $F(t,t)=Ex^2(t)-K^*$ are equivalent to the condition that eigenvalues of (4.24) have $\text{Re}\lambda\leq 0$. This in turn is analytically expressed in Eq. (2.5). Note, however, from our assumption that $F(t,t)=e^{2\lambda t}$, that when (2.5) is an equality, so that $\lambda=i\omega$, then the predicted period of oscillations in $F(t,t)$ is given by $T=\pi/\sqrt{b^2-a^2}$. Going back to the original variables, we conclude that $K(t,t)=Ex^2(t)$ will oscillate about some positive value that depends on K^* given by (4.22), with a period that is half of the period of the first moment oscillations when stability is lost. Hence, the results of this section in conjunction with those of Sec. III show that with respect to additive noise, the stability and oscillation conditions for the first and second moments of the solution are completely independent of the noise color, as illustrated in Fig. 3.

B. Multiplicative colored noise

We now turn to a study of the stability properties of the solution of a differential delay equation with parametric colored noise. Consider a stochastic process $x(t)$ that satisfies the differential equation

$$dx(t)=[a+\sigma\eta(t)]x(t)dt+bx(t-\tau)dt , \quad t\geq 0 \tag{4.27}$$

where $x(t)\in\mathbb{R}^1$, the initial function $x(\theta)=\phi(\theta)$, $-\tau\leq\theta\leq 0$ is deterministic, $\tau>0$ is a constant delay, and $\eta(t)$ is the Ornstein-Uhlenbeck process defined by (4.2).

For equations with stochastic perturbations given by multiplicative colored noise, we have only been able to derive sufficient conditions for exponential mean-square stability and stability in the mean. Thus, we require the notion of p stability for stochastic differential delay equations, which we introduce following [13].

Definition 4.1. The trivial solution $x=0$ of (4.29) is called p stable if, for any $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that for any initial function $\phi(\theta)$ the inequality

$$\sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|^p<\delta(\varepsilon)$$

implies $E\{|x(t,\phi)|^p\}<\varepsilon$ for $t\geq 0$, and is exponentially p stable if there exist positive constants c_1 and c_2 such that

$$E\{|x(t,\phi)|^p\}\leq c_1\sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|^p\exp(-c_2t) , \quad t\geq 0 .$$

If $p=1$, we speak of stability in the mean; in the case $p=2$ we talk about mean-square stability. Since $|EX|\leq E|X|$, stability in the mean implies stability of the expectation value of the solution $m(t)=Ex(t)$. Mean-square stability is equivalent to stability of the second moment [41].

The method used in this section for the investigation of stability and asymptotic properties is Liapunov's second method [66]. This well-known method, proposed by Liapunov for ordinary differential systems, is based on the following idea. A positive definite function $v(x)$ or $v(t,x)$ is selected, which plays the role of a generalized distance from the origin ($x=0$) to a point x. If along trajectories of the equation this function is nonincreasing ($dv/dt\leq 0$), then the trivial solution $x\equiv 0$ is stable.

For differential delay equations, Liapunov's direct method was extended in two ways. The first uses the method of Liapunov-Krasovskii functionals [3,12], which requires a functional defined on the trajectory segments instead of a Liapunov function.

Another approach, initially proposed by Razumikhin [67,68] to extend the Liapunov function method to deterministic differential delay equations, and clarified in [10], is based on the following idea. If a solution of a differential delay equation begins in a ball and is to leave this ball at some time, T, then $|x(T+\theta)|<|x(T)|$ for all

$\theta \in [-\tau, 0]$. This method was also applied in [69–73] to derive mean-square stability conditions for matrix stochastic differential delay equations.

Using this idea we will obtain mean-square stability conditions for stochastic delay differential equations with multiplicative colored noise. We will consider the solution of the appropriate equation with a deterministic initial function (2.2) and *assume that the solution is not stable.* By this we mean that there must exist some moment of time $T > \tau$ that is a first exit time of the solution from the stability domain (the neighborhood of size ε about zero); i.e.,

$$T = \inf\{t > \tau : |x(T)| = \varepsilon\} .$$

From this it follows that, except for a subset of probability zero, trajectories satisfy

$$|x(T-\tau)| < |x(T)| = \varepsilon , \tag{4.28a}$$

so that

$$E\{|x(T-\tau)|^2\} < E\{|x(T)|^2\} = \varepsilon^2 . \tag{4.28b}$$

Calculating the differential of a Liapunov function $v(x(T))$, where the Liapunov function is chosen to be $v = |x|^p$, we then derive conditions under which the assumption that at time T the solution leaves the stability domain leads to a contradiction. In this way we derive sufficient conditions for stability in the mean and mean-square stability for the solution of the stochastic differential delay equation driven by colored noise.

We pick a Liapunov function $v(x) = |x|^p$ and use the direct Liapunov method. Then for all time t

$$\begin{aligned} d|x(t)|^p &= \{p|x(t)|^p[a + \sigma\eta(t)] + pb|x(t)|^{p-1}x(t-\tau)\}dt \\ &\le \{p|x(t)|^p[a + \sigma\eta(t)] + p|b|\,|x(t)|^{p-1}|x(t-\tau)|\}dt . \end{aligned} \tag{4.29}$$

Now assume that $x(t)$ is not stable, which implies that there is a time T such that (4.28a) and (4.28b) hold. From (4.29) and (4.28a) we obtain at time T, which is assumed to be the first exit time, that

$$d|x(T)|^p \le p[a + \sigma\eta(T) + |b|]x(T)|^p dT .$$

Therefore

$$|x(T)|^p \le |x_0|^p \exp\left[p(a+|b|)T + p\int_0^T \sigma\eta(s)ds\right] .$$

For $E|x(T)|^p$ we have the estimate

$$E|x(T)|^p \le |x_0|^p e^{p(a+|b|)T} E \exp\left[p\sigma\int_0^T \eta(s)ds\right] .$$

Using (4.8), the last expression takes the form

$$E|x(T)|^p \le |x_0|^p e^{p(a+|b|T)} E\, e^{p\sigma\mu(T)} .$$

To continue the estimation, we use the following representation for the Gaussian stochastic process $z(t)$:

$$Ee^{z(t)} = e^{Ez(t) + \frac{1}{2}Ez^2(t)} . \tag{4.30}$$

Applying (4.30) and using the properties of the integrated Ornstein-Uhlenbeck process $\mu(t)$, yields

$$E|x(T)|^p \le |x_0|^p e^{p(a+|b|)T} e^{p\sigma E\{\mu(T)\} + \frac{1}{2}p^2\sigma^2 E\{\mu^2(T)\}} .$$

Suppose that η_0 is normally distributed. Then taking into account the expressions (4.9) and (4.10) for the mean and variance of $\mu(t)$, we obtain the estimation for $E|x(T)|^p$:

$$E|x(T)|^p \le |x_0|^p e^{p(a+|b|+\frac{1}{2}p\sigma^2)T} e^{(p^2\sigma^2/2\alpha)(e^{-\alpha T}-1)} . \tag{4.31}$$

Let us now consider two possibilities. In the first, the first exit time $T \gg 0$, so

$$e^{p(a+|b|+\frac{1}{2}p\sigma^2)T} e^{(p^2\sigma^2/\alpha)(e^{-\alpha T}-1)} \to e^{p(a+|b|+\frac{1}{2}p\sigma^2)T - (p^2\sigma^2/2\alpha)} . \tag{4.32}$$

Consequently if the condition

$$p(a+|b|+\tfrac{1}{2}p\sigma^2) < 0 \tag{4.33}$$

is satisfied, then the pth moment of the trivial solution is exponentially stable in the sense of Definition 4.1, with $c_1 = e^{-(p^2\sigma^2/2\alpha)}$, $c_2 = p(a+|b|+\frac{1}{2}p\sigma^2)$, which contradicts the assumption that there is a first exit time, i.e., (4.28a). Thus condition (4.33) is sufficient for the exponential p stability of the trivial solution of (4.27).

Alternately, if the first exit time is sufficiently small so that the approximation $e^{-\alpha T} - 1 \simeq -\alpha T$ holds, then from (4.31),

$$E|x(T)|^p \le |x_0|^p e^{p(a+|b|)T}$$

results, which implies that the solution $x(t)$ of (4.27) is exponentially p stable when $a < -|b|$.

Thus from (4.33) it follows (see Fig. 3) that sufficient stability conditions for the first and second moments of the trivial solution of (4.27) are, respectively, $a < -|b| - \frac{1}{2}\sigma^2$ and $a < -|b| - \sigma^2$. If we compare the estimation (4.31) to the white noise case by taking the limit $\alpha \to \infty$, we have

$$E|x(T)|^p \le |x_0|^p e^{p(a+|b|+\frac{1}{2}p\sigma^2)T} ,$$

which implies exponential p stability for $a < -|b| - p\sigma^2$, if the equation

$$dx(t) = [ax(t) + bx(t-\tau)] + \sigma x(t)dw(t)$$

is interpreted in the Stratonovich sense.

Finally we note that when the color of the noise is increasing ($\alpha\downarrow$) the stability properties are changing from the white noise stability region ($a < -|b| - \frac{1}{2}p\sigma^2$) to the deterministic region ($a < -|b|$). Colored noise has a stabilizing effect with respect to white noise, which can be understood from (4.31). Namely,

$$E|x(T)|^p \le |x_0|^p e^{p(a+|b|+\frac{1}{2}p\sigma^2)T} e^{(p^2\sigma^2/2\alpha)(e^{-\alpha T}-1)}$$

$$< |x_0|^p e^{p(a+|b|+\frac{1}{2}p\sigma^2)T} ,$$

which means that the relaxation of $E|x(T)|^p$ to zero is accelerated in the presence of colored noise as compared to the white noise case.

V. CONCLUSIONS

We have investigated the effects of additive and multiplicative white and colored noise on the stability of the trivial solution of a linear differential delay equation by deriving equations for the first and second order moments and examining the onset of oscillations. For stochastic ordinary differential equations, the Fokker-Planck analysis for the evolution of densities can be applied to study solution stability and bifurcations [62,74–77]. However, there is no analog to the Fokker-Planck equation for stochastic equations with a retarded argument. Concerning the influence of colored noise on the density behavior of stochastic differential delay equations, some numerical results are known [27,28].

We have shown that in the presence of additive noise (white or colored), the stability domain of both moments is identical to that of the unperturbed system. When these moments lose stability, there is a Hopf bifurcation and the first moment oscillates with a period identical to the solution of the unperturbed equation, while the oscillation period of the second moment is exactly one-half the period of the unperturbed solution and the first moment. When perturbations are of the parametric (or multiplicative) type and white noise is assumed, under the Itô interpretation the first moment of the solution preserves properties of the solution of the deterministic equation, while the behavior of the second moment depends on the amplitude of the stochastic perturbation. The critical delay value at which the second moment loses stability and becomes oscillating has been derived, and it is less than the critical delay for the first moment. Under the Stratonovich interpretation, quite different properties have been observed for the moment equations.

For the case of parametric colored noise perturbations, sufficient p stability conditions have been derived that are independent of the value of delay, and the stabilizing effect of colored noise with respect to white noise has been observed for both short-time and asymptotic behavior of the solutions.

ACKNOWLEDGMENTS

We thank Professor L. Arnold (Bremen) for helpful comments. This work was supported by grants from the North Atlantic Treaty Organization (M.C.M), the Natural Science and Engineering Research Council of Canada (M.C.M and I.G.N.), and the Alexander von Humboldt Stiftung (M.C.M.).

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