Delayed random walks

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The fluctuations about the stable point in a delayed dynamical system are modeled as a delayed random walk: i.e., a random walk in which the transition probability depends on the position of the walker at a time τ in the past and transitions in the direction of the stable point are more probable. It is shown that, depending on the magnitude of the delay, the root mean square displacement $\sqrt{\langle X^2(t)\rangle}$ versus time interval approaches a limiting value in either an oscillatory or nonoscillatory fashion. This limiting value of $\sqrt{\langle X^2(t)\rangle}$ is a linear function of τ.

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Complex fluctuations are ubiquitous in nature. Disentangling the relative role played by stochastic and deterministic processes in shaping the observed fluctuations has been problematic particularly when time delays exist. Recently emphasized examples include blood cell production [1], optical bistability [2], electrical circuits [3], neural reflexes [4], and the control of chaos with delayed feedback [5]. A fundamental problem is that the concepts and tools of statistical physics, e.g., random walk, Langevin's, and Fokker-Planck analysis, have not yet been adapted for noisy dynamical systems possessing time delays. Consequently, current investigations have been confined to numerical simulations [4]. Here we introduce the concept of a delayed random walk as a possible direction for theoretically studying dynamical systems with both fluctuations and delay. A delayed random walk is defined as a random walk in which the transition probability depends on the position of the walker at some time τ in the past.

First, we formulate a random walk approximation for a stable dynamical system subjected to noisy perturbations. We assume that the attractiveness of a stable point in a noisy dynamical system can be approximated by a random walk in which movements in the direction of the stable point are more probable. The position of the walker at time t is $X(t)$. Identify the origin of a one-dimensional random walk with the stable point $[X(0) = 0]$ and let the random walker take a step of unit length in unit time. The probability $P(t)$ for the walker to take a step at time t to the right (positive direction) is given by

$$P(t) = \begin{cases} p & [X(t) > 0] \\ 0.5 & [X(t) = 0] \\ 1-p & [X(t) < 0], \end{cases} \tag{1}$$

where $0 < p < 1$. The origin is attractive when $p < 0.5$. By symmetry with respect to the origin the average position is $\langle X(t)\rangle = 0$.

As a consequence of $p < 0.5$, the root mean square displacement $\sqrt{\langle X^2(t)\rangle}$ as a function of time interval approaches a limiting value (Fig. 1). This limiting behavior can be understood analytically. By using symmetry arguments it is possible to show that the probability distribution of (1) when $\tau = 0$ can be obtained by solving the following set of equations:

$$\begin{aligned} P_0(t+1) &= 2(1-p)P_1(t), \\ P_1(t+1) &= \frac{1}{2}P_0(t) + (1-p)P_2(t), \\ P_X(t+1) &= pP_{X-1}(t) + (1-p)P_{X+1}(t) \quad (2 \le X), \end{aligned} \tag{2}$$

where $P_X(t)$ is the probability to be at position X at time t. We can solve this set of equations for the stationary probability distributions P_X^s by using the trial function $P_X^s = Z^X$ [6]. The solutions are given as [7]

$$P_0^s = 2C_0 p, \tag{3}$$

$$P_X^s = C_0\left(\frac{p}{1-p}\right)^X \quad (1 \le X), \tag{4}$$

where

$$C_0 = \frac{1-2p}{4p(1-p)}. \tag{5}$$

With this solution, we can calculate $\sigma^2(0) \equiv \sigma_X^2(\tau = 0) \equiv \lim_{t\to\infty}\langle X^2(t)\rangle$ as [7]

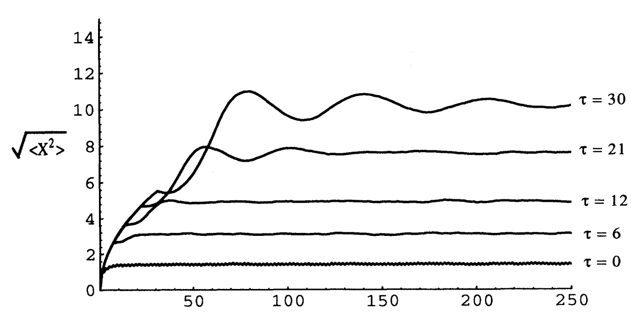

FIG. 1. Dynamics of $\sqrt{\langle X^2(t)\rangle}$ for various τ with fixed p $(= 0.25)$.

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$$\sigma^2(0) = \frac{1}{2(1-2p)^2}. \quad (6)$$

Second, we introduce the effect of a time delay into the random walk by assuming that $P(t)$ depends on the position of the walker at a time τ in the past; hence $X(t)$ in (1) is replaced by $X(t-\tau)$. The probability distribution when $\tau > 0$ can be obtained by solving the set of equations

$$P_X(t+1) = pP_{X-1|X>0}(t|t-\tau) + \frac{1}{2}P_{X-1|X=0}(t|t-\tau) + (1-p)P_{X-1|X<0}(t|t-\tau) + pP_{X+1|X<0}(t|t-\tau) + \frac{1}{2}P_{X+1|X=0}(t|t-\tau) + (1-p)P_{X+1|X>0}(t|t-\tau) \quad (0 \le X < \tau + 2), \quad (7)$$

$$P_X(t+1) = pP_{X-1}(t) + (1-p)P_{X+1}(t) \quad (\tau + 2 \le X), \quad (8)$$

where, for example, the notation $P_{X+1|X<0}(t|t-\tau)$ signifies the probability that the walker is positioned at $X+1$ and $X < 0$ at times t and $t-\tau$.

For arbitrary τ this equation can be solved by a tedious consideration of the different cases. For example, when $\tau = 1$ and $\tau = 2$ we obtain, respectively,

$$\sigma^2(1) = \frac{1}{2(1-2p)^2}\frac{(7-24p+32p^2-16p^3)}{(3-4p)}, \quad (9)$$

$$\sigma^2(2) = \frac{1}{2(1-2p)^2}\frac{(25-94p+96p^2+64p^3-160p^4+64p^5)}{(5+2p-24p^2+16p^3)}. \quad (10)$$

The values of σ^2 for $\tau > 2$ were obtained from numerical simulations. Figure 2 shows for a given value of p, σ is approximately a linear function of τ. Moreover, the constant slope m of the plot of σ vs τ is also approximately a linear function of p (data not shown). These results together with (6) suggest that

$$\sigma(\tau) \sim (0.59 - 1.18p)\tau + \frac{1}{\sqrt{2}(1-2p)}. \quad (11)$$

Thus it is possible to estimate either τ or p by measuring $\sigma^2(\tau)$.

The approach of $\sqrt{\langle X^2(t)\rangle}$ to its limiting value σ as a function of τ is shown in Fig. 1. For short τ there is a nonoscillatory approach to σ, whereas for longer τ damped oscillations occur. The period T of these oscillations is approximately twice the delay. Oscillations with $2\tau < T < 4\tau$ can be observed in first-order differential

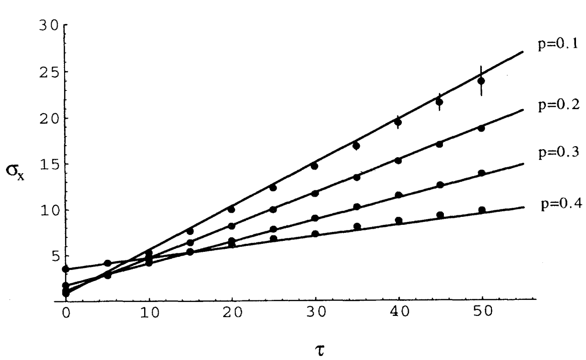

FIG. 2. σ_X as a function of τ for various p. (The error bars are to be the root mean square error of data points.)

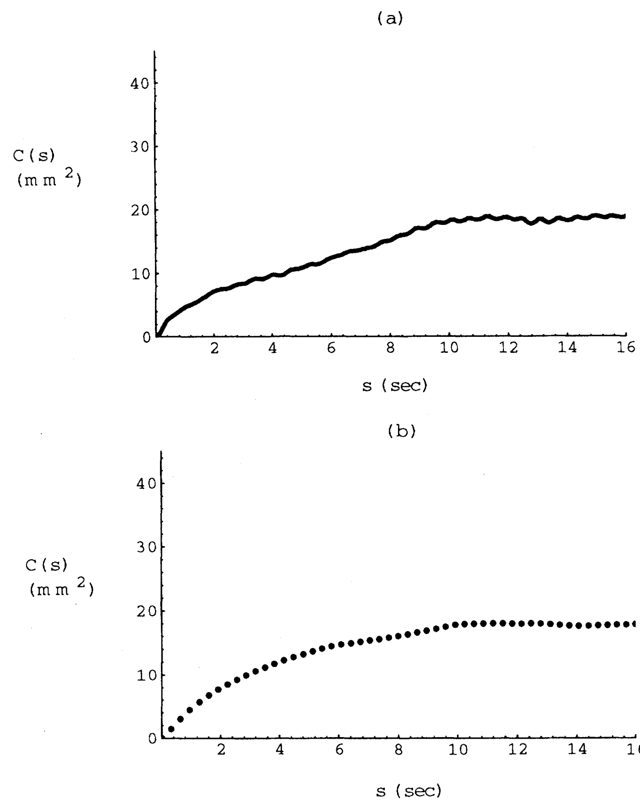

FIG. 3. Comparison of two-point correlation function $C(s)$ for (a) human postural sway during quiet standing and (b) a delayed random walk with $p = 0.35$ and $\tau = 1$. Experimental data were supplied by and are published with permission of C. Chow and J. Collins. In order to compare (b) with (a) we estimated a unit step length as 1.2 mm and a unit time as 320 msec.

equations with delayed negative feedback [8].

The only published experimental paradigm in which the noisy fluctuations about a stable point in a delayed dynamical system were measured is postural sway in quietly standing adults [9]. In these experiments the investigators measured the center of pressure (COP) using a force platform. Stabiliogram-diffusion plots were obtained by plotting the two-point correlation of COP, $C(s) \equiv \langle [X(t) - X(t-s)]^2 \rangle$, as a function of time interval s. A variety of different trends in $C(s)$ were observed [Figs. 3(a) and 4(a)]. For subject A there is a smooth approach [Fig. 3(a)] of $C(s)$ to its limiting value, for subject B an oscillatory approach [Fig. 4(a)]. In both cases there is excellent agreement with the random walk model [Figs. 3(b) and 4(b)]. The delay can be estimated from the data as $\tau \sim 300-500$ msec for subject A and ~ 400–700 msec for subject B. These delays are consistent with measured values of the time for a corrective movement in response to a sway [10].

The identification of three scaling regions in stabiliogram-diffusion plots for postural sway in many, but not all, human subjects [Fig. 3(a)] lies at the basis of previous suggestions that sway can be modeled as two distinct bounded correlated random walks [9]. Our results suggest that the same phenomena can be qualitatively accounted for by a single delayed random walk. It is possible that better quantitative agreement between prediction and observation could be obtained by incorporating second-order effects [11] into the random walk, e.g., allowing $P(t)$ to depend on both position and direction of movement at some time in the past (i.e., a delayed random walk with persistence).

The applications of statistical mechanical techniques have traditionally been limited to the description of phenomena in which time delays are thought to be of little significance. Here we have shown that it is possible to understand fluctuations in a stable delayed dynamical system by modeling it as a delayed random walk. Although this model is simple, it nonetheless provides an explanation for a wide range of qualitatively different stabiliogram-diffusion plots, all of which are observed experimentally. Moreover, a method for measuring either p or τ from experimental observations is obtained. These observations provide a strong motivation for the development of appropriate delayed random walk models for the characterization of noisy delayed dynamical systems.

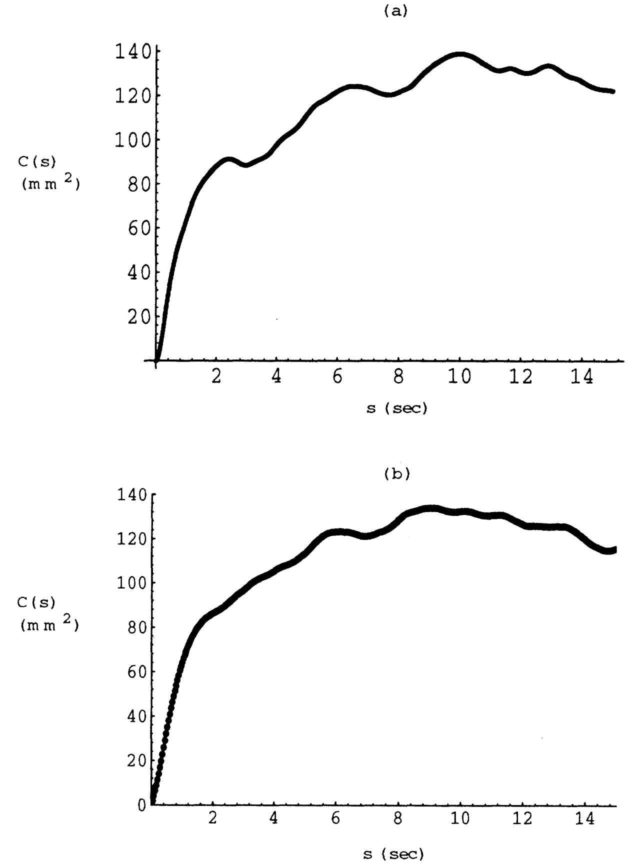

FIG. 4. Comparison of two-point correlation function $C(s)$ for (a) human postural sway during quiet standing and (b) a delayed random walk with $p = 0.40$ and $\tau = 10$. Experimental data were supplied by and are published with permission of C. Chow and J. Collins. Estimations of unit step length and unit step time are 1.4 mm and 40 msec.

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[7] The general solution of (1) is $P_X^s = C_1 + C_2(\frac{p}{1-p})^X$. Since $1 = \sum_{-\infty}^{\infty} P_X^s$, we must have $C_1 = 0$. By making use of the symmetry and (2), $C_2 = \frac{1}{2}\left[p - 1 + \sum_{X=0}^{\infty}(\frac{p}{1-p})^X\right]^{-1}$ and we obtain (5). With this solution, we can calculate $\sigma(\tau)$ by taking the partial derivative of the generating function to obtain (6) [6].

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