Bistable kinetic model driven by correlated noises: Unified colored-noise approximation

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A Fokker-Planck equation for a general one-dimensional non-Markovian system driven by correlated Gaussian noises is derived by means of an extended unified colored-noise approximation. The general stationary probability distribution (SPD) is obtained. The SPD contains three important limits: the uncorrelated noise limit, the white noise limit, and the usual uncorrelated white noise limit. The following important physical aspects have been revealed by virtue of the above-mentioned SPD. (1) In contrast to the well known case of uncorrelated white noises where the parameter of additive noise cannot enter the extremal equation of SPD, now the additive noise parameter does enter the extremal equation as a non-Markovian effect even if the system is driven by uncorrelated noises. (2) When the correlation between the noises does exist, the SPD contains information caused by both correlation and color of the noises. The general results obtained in this Brief Report are applied to a bistable kinetic model. We find for the steady state of the model that in the case of correlated noises, the symmetry of SPD under the reflection of the state variable x with respect to the origin is destroyed. However in the case of non-Markovian processes driven by uncorrelated noises, the above symmetry is preserved.

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I. INTRODUCTION

Recently, Fulinski and Telejko have investigated the bistable system driven by correlated additive and multiplicative white noises [1]. They have shown that the presence of correlation between the noises changes the dynamics of the system. The authors of Ref. [1] pointed out correctly that the transition between unimodal and bimodal stationary distribution is strongly influenced by the correlation between the noises. However, the statistical properties for the systems driven by correlated additive and multiplicative noises have still not been investigated because the method given in Ref. [1] cannot provide a correct foundation to study the effects of correlation of the noises quantitatively [2]. Singh showed that the correlation between the quantum noises for a homogeneously broadened two-mode ring laser at line center gives a nonzero contribution of the order n_0^{-1} [3]; here n_0 denotes the mean number of photons in the laser cavity at threshold. In our opinion the effects of the correlation of quantum noises between the laser modes may be of interest for the problem of laser physics. More recently, Zhu investigated theoretically the statistical fluctuations of a single-mode laser with correlations between additive and multiplicative with noise terms. The mean, variance, and skewness of the steady-state laser intensity are calculated by Zhu [4] through a one-dimensional laser equation [5].

In our previous works [2,6], the bistable systems driven by correlated white noises have been studied. In this Brief Report, we extend the unified colored-noise approximation [7,8] (UCNA) [7] to such non-Markovian systems in which the noises are correlative. The general Langevin equation (LE) and Fokker-Planck equation (FPE) are derived under the extended UCNA in Sec. II. In Sec. III, the general theory is applied to the bistable kinetic model with correlated additive and multiplicative noises. In Sec. IV, the effects of τ (the noise correlation time) and λ (the strength of correlation between the noises) on the statistical properties of the bistable kinetic model are discussed.

II. GENERAL THEORY

We consider the following set of stochastic differential equations:

$$\dot{x}=h(x)+g_1(x)\epsilon(t)+g_2(x)\Gamma(t) , \tag{1}$$

$$\dot{\epsilon}=-\frac{1}{\tau}\epsilon+\frac{1}{\tau}\Gamma_1(t) \tag{2}$$

with δ-correlated Gaussian white noises $\Gamma(t)$ and $\Gamma_1(t)$,

$$\langle\Gamma_1'(t)\rangle=\langle\Gamma(t)\rangle=0 ,$$
$$\langle\Gamma_1(t)\Gamma_1(t')\rangle=2D\delta(t-t') , \tag{3}$$
$$\langle\Gamma(t)\Gamma(t')\rangle=2\alpha\delta(t-t')$$

$$\langle\Gamma_1(t)\Gamma(t')\rangle=\langle\Gamma(t)\Gamma_1(t')\rangle=2\lambda\sqrt{D\alpha}\delta(t-t') . \tag{4}$$

It has been shown that the above two-dimensional Markovian processes (1)–(4) are stochastically equivalent to one-dimensional non-Markovian process described by (1) with Gaussian colored noise (the *O-U* noise) $\epsilon(t)$ [8]

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$$\langle \epsilon(t)\rangle=0\ , \quad \langle \epsilon(t)\epsilon(t')\rangle=\frac{D}{\tau}\exp\left[-\frac{1}{\tau}|t-t'|\right] , \tag{5}$$

and (3) and (4); here $\Gamma_1(t)$ is defined by (2)–(4). Below we will treat the problem by extending the UCNA to obtain a one-dimensional Markovian approximation.

Applying the extended UCNA to the case of correlated noises, we obtain from (1)–(5) the following one-dimensional Markovian process (7):

$$\dot{x}=\frac{h(x)}{C(\tau,x)}+\frac{1}{C(\tau,x)}[g_1(x)\Gamma_1(t)+g_2(x)\Gamma(t)] , \tag{6}$$

where

$$C(\tau,x)=1-\tau\left[h'(x)-\frac{g_1'(x)}{g_1(x)}h(x)\right] . \tag{7}$$

In (7) and in the following equations the prime denotes the differentiation with respect to x.

To get the FPE from (6) and (7), we start from the stochastic equivalent Stratonovich stochastic differential equation

$$\dot{x}=\frac{h(x)}{C(\tau,x)}+\frac{1}{C(\tau,x)}g(x)\tilde{\Gamma}(t) , \tag{8a}$$

$$g(x)=[Dg_1^2(x)+2\lambda\sqrt{D\alpha}g_1(x)g_2(x)+\alpha g_2^2(x)]^{1/2} , \tag{8b}$$

in which $\tilde{\Gamma}(t)$ is Gaussian white noise with zero mean and

$$\langle \tilde{\Gamma}(t)\tilde{\Gamma}(t')\rangle=2\delta(t-t') . \tag{9}$$

The proof of the stochastic equivalence of (6) and (8) is analogous to the Appendix of Ref. [6].

It must be pointed out that the results of (8) is the LE in which the correlation between the noises has been considered. This LE is the extension of the case of uncorrelated noises. When $\lambda=0$, the case of uncorrelated noises, Eqs. (8) is simplified to Eq. (20) of Ref. 7(b).

Now we can obtain the FPE corresponding to Eqs. (8). For convenience, we rewrite (8) as

$$\dot{x}=\tilde{h}(x)+\tilde{g}(x)\tilde{\Gamma}(t) , \tag{10}$$

with

$$\tilde{h}(x)=\frac{h(x)}{C(\tau,x)} , \tag{11}$$

$$\tilde{g}(x)=\frac{g(x)}{C(\tau,x)}=\frac{1}{C(\tau,x)}[Dg_1^2(x)+2\lambda\sqrt{D\alpha}g_1(x)g_2(x)+\alpha g_2^2(x)]^{1/2} . \tag{12}$$

The FPE reads

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}A(x)P(x,t)+\frac{\partial^2}{\partial x^2}B(x)P(x,t) , \tag{13}$$

with

$$A(x)=\tilde{h}(x)+\tilde{g}(x)\tilde{g}'(x) , \tag{14}$$

$$B(x)=\tilde{g}^2(x) . \tag{15}$$

Combining Eqs. (11) and (12), and (14) and (15), we get the expressions of drift function,

$$A(x)=\frac{h(x)}{C(\tau,x)}+\frac{1}{[C(\tau,x)]^2}[Dg_1(x)g_1'(x)+\lambda\sqrt{D\alpha}(g_1(x)g_2'(x)+g_2(x)g_1'(x))+\alpha g_2(x)g_2'(x)] -\frac{C'(\tau,x)}{[C(\tau,x)]^3}[Dg_1^2(x)+2\lambda\sqrt{D\alpha}g_1(x)g_2(x)+\alpha g_2^2(x)] , \tag{16}$$

and diffusion function,

$$B(x)=\frac{1}{[C(\tau,x)]^2}[Dg_1^2(x)+2\lambda\sqrt{D\alpha}g_1(x)g_2(x)+\alpha g_2^2(x)] . \tag{17}$$

Equations (8) and (13), with (16) and (17), are the main results of this paper.

The stationary probability distribution (SPD) $P_s(x)$ can be obtained from [9]

$$P_s(x)=N\frac{1}{B(x)}\exp\left\{\int\frac{A(x)}{B(x)}dx\right\} \tag{18}$$

and (16) and (17) by integration,

$$P_s(x)=N\frac{C(\tau,x)}{[Dg_1^2(x)+2\lambda\sqrt{D\alpha}g_1(x)g_2(x)+\alpha g_2^2(x)]^{1/2}}\exp\left\{\int\frac{h(x)C(\tau,x)}{Dg_1^2(x)+2\lambda\sqrt{D\alpha}g_1(x)g_2(x)+\alpha g_2^2(x)}dx\right\} ; \tag{19}$$

and the extremal equation of $P_s(x)$ may be written directly from

$$A(x)=B'(x) \quad \text{or} \quad \tilde{h}(x)-\tilde{g}(x)\tilde{g}'(x)=0 . \tag{20}$$

That is,

$$h(x)[C(\tau,x)]^2-C(\tau,x)[Dg_1(x)g_1'(x)+\lambda\sqrt{D\alpha}(g_1(x)g_2'(x)+g_2(x)g_1'(x))+\alpha g_2(x)g_2'(x)] +C'(\tau,x)[Dg_1^2(x)+2\lambda\sqrt{D\alpha}g_1(x)g_2(x)+\alpha g_2^2(x)]=0 . \tag{21}$$

It is of interest to point out that the general results (19) and (21) contain the following important special cases: (i) the non-Markovian process driven by uncorrelated noises ($\tau \neq 0$ and $\lambda = 0$); (2) the Markovian process driven by correlated noises ($\tau = 0$ and $\lambda \neq 0$) (this case had been investigated in our previous work [2,6]).

It would be of interest to mention that when we take $\tau = 0$ and $\lambda = 0$ in Eqs. (19) and (21), these equations reduce to the results of the usual dynamics system driven by uncorrelated Gaussian white noises,

$$P_s(x) = N \frac{1}{[D g_1^2(x) + \alpha g_2^2(x)]^{1/2}} \times \exp\left\{ \int \frac{h(x)}{D g_1^2(x) + \alpha g_2^2(x)} dx \right\} \quad (22)$$

and

$$h(x) - [D g_1(x) g_1'(x) + \alpha g_2(x) g_2'(x)] = 0 \ . \quad (23)$$

III. APPLICATION TO THE BISTABLE KINETIC MODEL

We apply the colored-noise theory of correlated noises to a typical one-dimensional system, the bistable kinetic system driven by additive and multiplicative noises simultaneously, and assume the dimensionless form

$$\dot{x} = x - x^3 + x\epsilon(t) + \Gamma(t) \ , \quad (24)$$

where the noises $\epsilon(t)$ and $\Gamma(t)$ are the same as in Eq. (1). Equation (24) is a special case of Eq. (1), with $h(x) = x - x^3$, $g_1(x) = x$, $g_2(x) = 1$. Hence expression (7) reduces to $C(\tau, x) = 1 + 2\tau x^2$, and the result (19) for SPD reduces to [10]

$$P_s(x) = N(1 + 2\tau x^2)[D x^2 + 2\lambda\sqrt{D\alpha}\, x + \alpha]^{\tilde{C} - 1/2} \times \exp\left\{ f(x) + \frac{\tilde{E}}{\sqrt{D\alpha}(1-\lambda^2)^{1/2}} \times \tan^{-1} \frac{D x + \lambda\sqrt{D\alpha}}{\sqrt{D\alpha}(1-\lambda^2)^{1/2}} \right\} , \quad (25)$$

where

$$0 \le \lambda < 1 \ , \quad (26a)$$

$$f(x) = \tilde{A}\frac{x}{D} + \tilde{B}\frac{x^2}{2D} + 4\tau\lambda \left[\frac{\alpha}{D}\right]^{1/2} \frac{x^3}{3D} - 2\tau\frac{x^4}{4D} \ , \quad (26b)$$

$$\tilde{A} = 2(1-2\tau)\lambda \left[\frac{\alpha}{D}\right]^{1/2} + 8\tau\lambda \left[\frac{\alpha}{D}\right]^{1/2} \frac{\alpha}{D}(2\lambda^2 - 1) \ , \quad (26c)$$

$$\tilde{B} = -(1-2\tau) + 2\tau(1-4\lambda^2)\frac{\alpha}{D} \ , \quad (26d)$$

$$\tilde{C} = -2(1-2\tau)\lambda^2 \frac{\alpha}{D^2} + 8\tau\lambda^2 \frac{\alpha^2}{D^3} - 16\tau\lambda^4 \frac{\alpha^2}{D^3} + \frac{1}{2D} + (1-2\tau)\frac{\alpha}{2D^2} - \tau(1-4\lambda^2)\frac{\alpha^2}{D^3} \ , \quad (26e)$$

$$\tilde{E} = -\lambda \left[\frac{\alpha}{D}\right]^{1/2} \left[1 + (1-2\tau)\frac{\alpha}{D} - 2\tau(1-4\lambda^2)\frac{\alpha^2}{D^2} \right] + 2(1-2\tau)\lambda(2\lambda^2-1)\left[\frac{\alpha}{D}\right]^{1/2}\frac{\alpha}{D} - 4\tau\lambda(1-4\lambda^2)(2\lambda^2-1)\left[\frac{\alpha}{D}\right]^{1/2}\frac{\alpha^2}{D^2} - 4\tau\lambda(2\lambda^2-1)\left[\frac{\alpha}{D}\right]^{1/2}\frac{\alpha^2}{D^2} \ . \quad (26f)$$

At the same time, the extremal equation (21) of SPD becomes

$$-4\tau^2 x^7 + 4\tau(\tau-1)x^5 + (4\tau + 2\tau D - 1)x^3 + 6\tau\lambda\sqrt{D\alpha}\, x^2 + (1 + 4\tau\alpha - D)x - \lambda\sqrt{D\alpha} = 0 \ . \quad (27)$$

IV. CONCLUSION

A. General results

Equations (13) with (16) and (17) and its steady-state results (19) and (21) are main results of this Brief Report. The above-mentioned results are valid for a large class of one-dimensional non-Markovian systems driven by correlated noises, and they contain a great deal of information for the statistical properties of the system.

The following physical aspects have been revealed by the above-mentioned equations. In contrast to the well known case of uncorrelated white noises, where the parameter of additive noise cannot enter the extremal equation of SPD, as can be seen from Eq. (23); now due to $C'(\tau, x) \neq 0$ for a nonlinear system, Eq. (21) indicates that the additive noise parameter does enter the extremal equation as a non-Markovian effect even if the system is driven by uncorrelated noises.

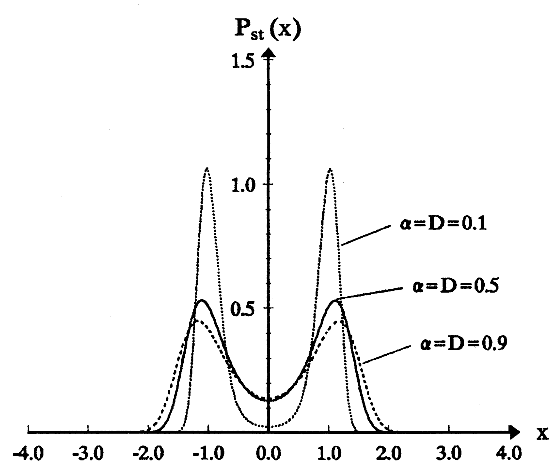

FIG. 1. The SPD of the bistable kinetic model for $\lambda = 0$ and $\tau = 1$ are fixed. $\alpha = D = 0.1$, 0.5, and 0.9, respectively.

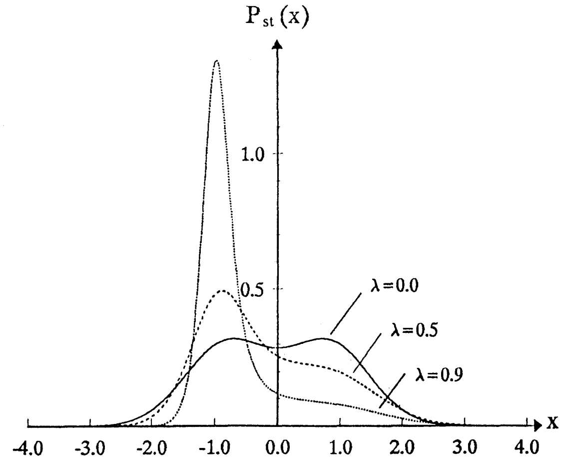

FIG. 2. The SPD of the bistable kinetic model for $\tau=0$ and $\alpha=D=0.5$ are fixed. $\lambda=0.0$, 0.5, and 0.9, respectively.

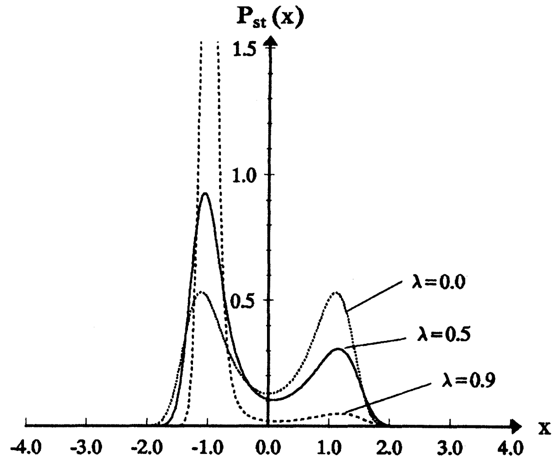

FIG. 3. The SPD of the bitable kinetic model for $\tau=1$ and $\alpha=D=0.5$ are fixed. $\lambda=0.0$, 0.5, and 0.9, respectively.

B. Conclusion for the bistable kinetic model

Equations (25) and (27) are also main results of this paper. In the bistable kinetic model, we see from (25)–(27) that, in the presence of correlation between the noises, the symmetry of SPD under the reflection of state variable x with respect to the origin is destroyed. However, the color of the noise itself does not affect the above symmetry. This may be seen clearly from the figures. In the case of an uncorrelated non-Markovian system, i.e., $\lambda=0$ and $\tau\neq 0$, the above symmetry is preserved, as shown by Fig. 1. When the noises are correlative, the above symmetry is destroyed, as shown in Fig. 2 for the white noise case and in Fig. 3 for the colored-noise case.

Comparing the curves in Figs. 2 and 3 for $\lambda=0$ (i.e., the case of uncorrelated noises), we see that the noisy color causes the peak of SPD to become narrow and grow in height. At the same time, the minimum of SPD drops in height. When we compare the curves in Figs. 2 and 3 for $\lambda=0.5$ (or $\lambda=0.9$), i.e., the case of correlated noises, we find an evident difference between the following cases: case (i), $\lambda=0.5$ and $\tau=0$; case (ii), $\lambda=0.5$ and $\tau=1$. The combination of correlation and color of the noises [case (ii)] causes the SPD to become narrow and the peak to grow in height; at the same time, it causes the transition from the unimodal to the bimodal structure of the SPD.

It must be pointed out that Eq. (25) is valid only in the range $0\leq\lambda<1$. For $\lambda=1$, instead of (25) we obtain from Eq. (19) the following corresponding result by integration [10]:

$$P_s(x)=N(1+2\tau x^2)[\sqrt{D}x+\sqrt{\alpha}]^{2(\tilde{C}-1/2)}\times\exp\left\{f(x)-\frac{\tilde{E}}{\sqrt{D}}\frac{1}{\sqrt{D}x+\sqrt{\alpha}}\right\},$$

$$f(x)=\tilde{A}\frac{x}{D}+\tilde{B}\frac{x^2}{2D}+4\tau\left[\frac{\alpha}{D}\right]^{1/2}\frac{x^3}{3D}-2\tau\frac{x^4}{4D},$$

$$\tilde{A}=2+4\tau\left[\frac{\alpha}{D}\right]^{1/2}\left[2\frac{\alpha}{D}-1\right],$$

$$\tilde{B}=-1-2\tau\left[-1+3\frac{\alpha}{D}\right],$$

$$\tilde{C}=\frac{1}{2D}\left[1-3\frac{\alpha}{D}\right]+\tau\frac{\alpha}{D^2}\left[3-5\frac{\alpha}{D}\right],$$

$$\tilde{E}=\left[\frac{\alpha}{D}\right]^{1/2}\left[1+2\tau\frac{\alpha}{D}\right]\left[\frac{\alpha}{D}-1\right].$$

For reasons of length, we do not describe this case in this paper.

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