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Cell-dynamics modeling of oscillator systems

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We propose a computationally efficient discrete model with short-range coupling for a system of limit cycle oscillators. Depending on the intensity of the coupling and of the attraction to the limit cycle, the system exhibits synchronization, amplitude death, or incoherence. We consider the behavior of a pair, a chain, and a two-dimensional lattice with randomly distributed frequencies. The onset of synchronization in the two-dimensional lattice is analyzed as a nonequilibrium phase transition.

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Systems of coupled limit cycle oscillators have been intensively studied by scientists due to its wide application in physical [1], chemical [2,3] and biological systems [4,5]. One of the main points is the possibility of a collective behavior by means of phase locking or synchronization among oscillatory elements. The phase model proposed by Kuramoto [2] has been one of the most frequently used tools to investigate the possibility of a macroscopic synchronization [6–11] due to its simple and generic form. Despite its simplicity, we find that this model is inadequate for large scale simulations and our main purpose in this paper is to propose a computer efficient model for systems of coupled limit cycle oscillators.

There are several nonlinear equations presenting Höpf bifurcations to a stable limit cycle, but we would like to consider a generic system in which details have been washed away. In this spirit, we mention two basic models: (i) The phase model, which results from the averaging of weakly coupled equations of motion with a limit-cycle attractor and (ii) the Ginzburg-Landau equation, which is a partial differential equation for an oscillatory field near the Höpf bifurcation [2,12]. Here we propose a cell-dynamical system model which is a model discrete in space and time, for the amplitude of oscillation.

The dynamics of coupled limit-cycle oscillators is very rich. Here we would like to focus on two aspects of the problem: amplitude death and synchronization. Amplitude death, or the stabilization of the rest state, is a phenomenon observed in systems of coupled differential equations with mean field coupling, [3,13,14], and appears whenever coupling strength and attraction to the limit cycle are of the same order for a sufficiently wide frequency distribution. Because the component oscillators in nature would never possess identical natural frequencies, mutual synchronization appears to be the unique possible mechanism for producing and maintaining macroscopic rhythmicity. In this case, we have macroscopic clusters of oscillators with a common asymptotic frequency, characterizing an oscillatory behavior in a larger scale. Many authors have suggested that the onset of synchronization at some critical finite coupling strength could be considered a nonequilibrium phase transition if the phenomenon is possible in the thermodynamical limit [4,6,8,9,11].

A cell-dynamical system is a map from a set of discrete spatial patterns to itself. The name cell-dynamical system actually extends to all discrete time-space models, like cellular automata and coupled map lattices, so we should point out what is different in our model. Regarding cellular automata, our model considers continuous pattern variables with a time evolution given by a function chosen to have the correct flow. The exact form of this function is not important, so, in this sense, our model differs from coupled map lattice models in which the precise form of the map is stressed. Cell-dynamical system models have been extensively used to study spatial structures in far from equilibrium systems [15–19]. The main feature of this kind of approach is to provide computationally efficient models that give the correct pattern formation dynamics. The construction of cell-dynamical system models consists of three steps: (i) construction of a local map that describes the local fluctuation of the system without any constraints, (ii) imposing a penalty to spatial discontinuous changes, and (iii) application of auxiliary conditions (say, symmetry, conservation, etc.).

Here we consider that each cell in the lattice represents a mesoscale volume, which has oscillatory behavior. It is clear that each cell may represent an average over a finer scale. We describe the dynamics of each limit cycle oscillator by a complex continuous field $z(n,t)$ defined in cell n in the (discrete) time t. The dynamics for each oscillator is then given by

$$z(n,t+1) = F(z(n,t)), \qquad (1)$$

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where the function F is defined as

$$F(A,\omega,z)=e^{i\omega}f(A,z),\quad 0<\omega<2\pi. \tag{2}$$

The exact form of f is not important, as has been pointed out before [15,16]. For example, consider two possible forms

$$f_a(A,z)=A\frac{z}{\sqrt{1+|z|^2(A^2-1)}}, \tag{3a}$$

$$f_b(A,z)=A\frac{z}{|z|}\tanh(|z|). \tag{3b}$$

Seeing the behavior of these maps is easy if we write $z=Re^{i\theta}$. We want that, asymptotically, $R(t)=R(t+1)=R_l$. For f_a we have that, for $A>1$, there is a stable limit cycle of radius $R_l=1$. For $A<1$, the asymptotic situation corresponds to $R_l=0$, that is, the rest state is stable. The same analysis can be made for f_b, in which case for $A>1$ the limit cycle is given by the condition $A\tanh(R_l)=R_l$, and for $A<1$ the rest state is stable. In both cases we have a Höpf bifurcation at $A=1$. We choose to work with f_a because it gives R_l independent of A, which is convenient but not necessary, and because it is not singular at the origin. As will be seen below, coupling between oscillators can stabilize the rest state, and then f_b will diverge.

To accomplish step (ii) we add a stabilizing term that penalizes gradients and couples cells diffusively. Considering that relaxation is faster than diffusion we get

$$z(n,t+1)=F(z(n,t))+D[\langle\langle F(z(n,t))\rangle\rangle-F(z(n,t))], \tag{4}$$

where $\langle\langle F\rangle\rangle$ is an isotropic spatial average. Step (iii) is not necessary, and Eqs. (3a) and (4) define our model. We should point out that the Euler scheme used in the phase model simulations [6,10,11], leads to a cell-dynamical system model for the phases only. It is important to stress that a cell-dynamical system is not a discretization of partial differential equation, although the opposite is true. The discretization of a partial differential equation gives a cell-dynamical system that is often inefficient and may generate a dynamical behavior different from the one expected from the continuous formulation. A cell-dynamical system is an *ab initio* modeling with the aid of intuitive ideas about the system.

In our model D is a measure of the coupling intensity and A is a measure of the attraction to the limit cycle. If a group of oscillators is asymptotically phase locked, they will oscillate with a common frequency. To identify this situation we define the asymptotic frequency $\Omega(n)$ of the oscillator in site n as usual [2]

$$\Omega(n)=\lim_{T\to\infty}\frac{1}{T}\sum_t\left[\theta(n,t+1)-\theta(n,t)\right], \tag{5}$$

T is the number of time steps after discarding the transient.

Before analyzing a chain of oscillators we consider a pair of coupled oscillators with different frequencies:

$$z_j(t+1)=F(A,w_j,z_j)+D[F(A,w_k,z_k)-F(A,w_j,z_j)],\quad j,k=1,2. \tag{6}$$

We can analyze the asymptotic state of the pair as we vary A and D by measuring R_i and Ω_i. We find three possibilities: (a)$R_i\neq 0$ and $\Omega_1\neq\Omega_2$, which we call the nonsynchronized state, (b)$R_i=0$, corresponding to amplitude death, and (c) $R_i\neq 0$ and $\Omega_1=\Omega_2$, which we call the synchronized state.

This result is shown in Fig. 1 where we plot the $A\times D$ phase diagram for two oscillators with frequencies $\omega_1=\pi/6$ and $\omega_2=\pi/2$. In the synchronized state z_1 and z_2 have the same asymptotic time-independent frequency defined by (5), but different phases.

The region corresponding to amplitude death can be calculated by a linear stability analysis of the state $(z_1,z_2)=(0,0)$. The linearized equations are

$$z_j(t+1)=A(1-D)e^{i\omega_j}z_j(t)+ADe^{i\omega_k}z_k(t),\quad j,k=1,2. \tag{7}$$

The Jacobian matrix can then be easily diagonalized.

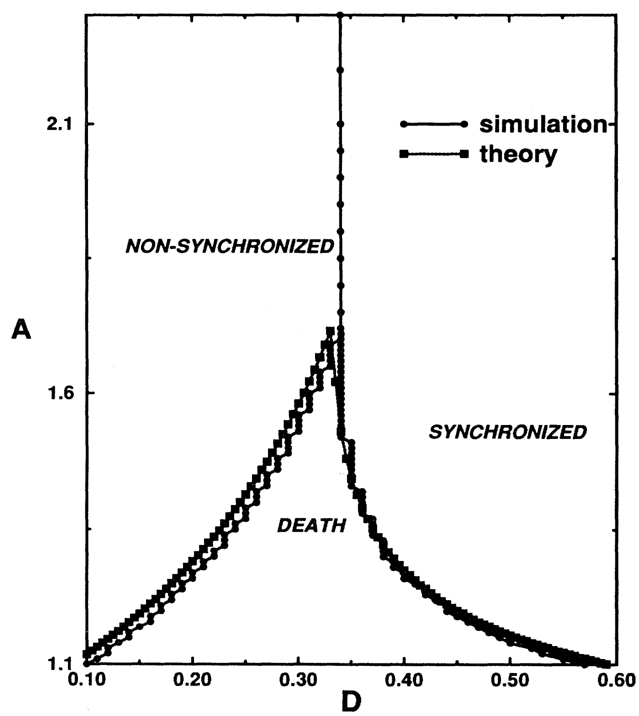

FIG. 1. Phase diagram for a pair of coupled oscillators with frequencies $\pi/6$ and $\pi/2$. A is a parameter that measures the attraction to limit cycle and D is the coupling strength. The nonsynchronized and synchronized regions correspond to parameter values such that the oscillators have different and equal asymptotic frequencies, respectively. In the region labeled as death, the rest state is stable. The boundary of this region can be calculated by a linear stability analysis.

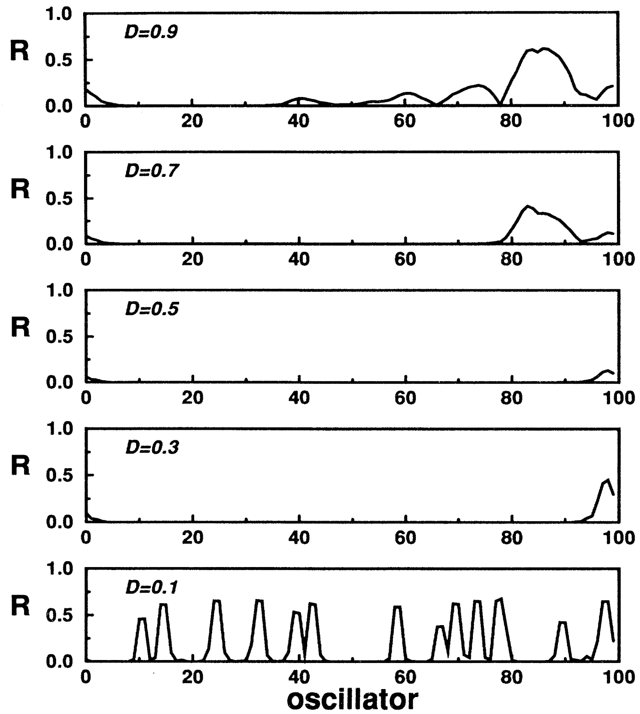

FIG. 2. The amplitude of oscillation in a chain with periodic boundary conditions, $A = 1.1$, and different values of the coupling strength. R is the amplitude of oscillation along the chain. Amplitude death occurs for D near 0.5.

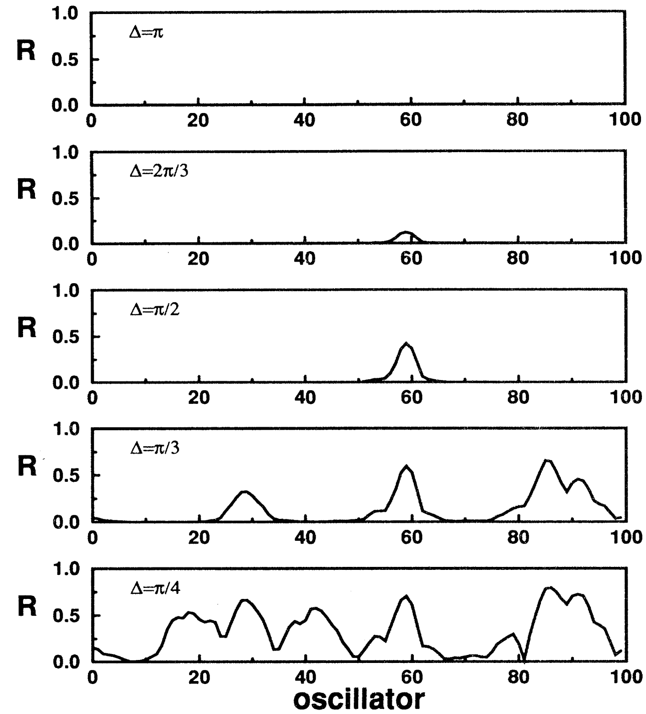

FIG. 3. Amplitude death as a function of the frequency spread. Here we plot the amplitude of oscillation, R, for a chain of 100 oscillators with periodic boundary conditions and a uniform distribution of frequencies with widths π, $\frac{2\pi}{3}$, $\frac{\pi}{2}$, $\frac{\pi}{3}$, and $\frac{\pi}{4}$. The other parameters are $D = 0.5$ and $A = 1.1$. As the frequency spread is increased, the rest state is stabilized.

The stability of the rest state requires that both eigenvalues have absolute value smaller than one. The calculated line in Fig. 1 corresponds to the first eigenvalue that becomes unstable.

We consider now a one-dimensional array defined by (4) with periodic boundary conditions. Each oscillator is coupled to its nearest neighbors. First we choose a uniform distribution of frequencies between 0 and $\frac{\pi}{2}$, and $z(t = 0) = 1$. The length of each run is the number of iterations after discarding the first 3000 iterations. After 3000 iterations, we read the values of $R(n,t)$ and $\theta(n,t)$. Again we find the possibility of amplitude death. In Fig. 2 we plot the value of $R(n)$ along the chain, for $A = 1.1$ and different values of D. To see the role of the frequency spread in the amplitude death, we choose different widths Δ for the frequency distribution, and read the values of R after 3000 iterations for $D = 0.5$ and $A = 1.1$. The results can be seen in Fig. 3. We see that by increasing the frequency difference among oscillators, the stabilization of the rest state is favored. For $\Delta = \pi$, $R = 0$ everywhere along the chain.

To investigate the possibility of synchronization, we choose a uniform distribution of frequencies and calculate $\Omega(n)$ for each oscillator after 3000 iterations. For example, consider a distribution centered in $\omega = 1.0$ with width $\Delta = 0.1$ and $A = 2.0$. Depending on the coupling intensity, the whole chain is phase locked. Figure 4, shows the values of the natural frequencies $\omega(n)$ and asymptotic frequencies $\Omega(n)$ for $D = 0.1, 0.3, 0.5$, and 0.7. We observe the formation of frequency plateaus that grow with increasing D. This kind of structure has already been reported in rotor chains [6] with Gaussian distribution of frequencies. There, the tails of the distribution prevent the formation of a single cluster in infinite chains for finite coupling strength. This effect is observed here when we plot the asymptotic frequency of each oscillator in the chain for different values of the natural frequency spread Δ as seen in Fig. 5. For Δ=0.05 and 0.1, the chain is practically synchronized while for Δ=0.5 and 1.0 clusters are oscillating with different frequencies. Oscillator death appears for Δ=1.5. Oscillators with R=0 are represented as having Ω=0.

For two-dimensional lattices we couple each oscillator to its first and second neighbors by defining the average $\langle\langle F\rangle\rangle$ as:

$$\langle\langle F\rangle\rangle = \frac{1}{6}\sum_{\mathcal{N}} F_{\mathcal{N}} + \frac{1}{12}\sum_{\mathcal{NN}} F_{\mathcal{NN}}, \tag{8}$$

where $\mathcal{N}$ represents nearest-neighbor cells and $\mathcal{NN}$, next nearest-neighbor cells. We study the macroscopic synchronization in this system by means of the frequency order parameter r defined by Kuramoto [2]:

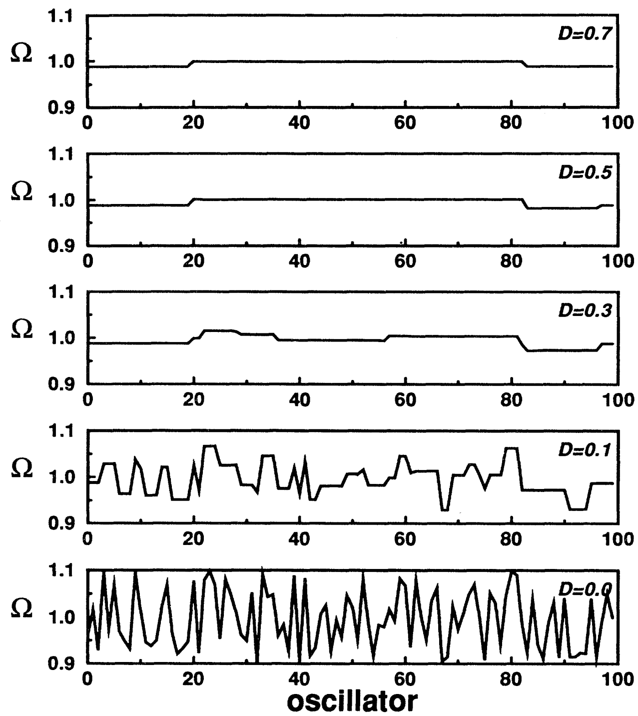

FIG. 4. Effect of the coupling strength in synchronization. Here we plot the asymptotic frequency Ω along a chain of 100 oscillators with periodic boundary conditions and uniform distribution of frequencies centered at 1.0 with width 0.1, A = 2.0, and D = 0.1, 0.3, 0.5, and 0.7. The initial condition z is $z(t = 0) = 1$. The length of the run is 3000 iterations, after having discarded 3000 iterations. The solid line corresponds to the asymptotic frequency.

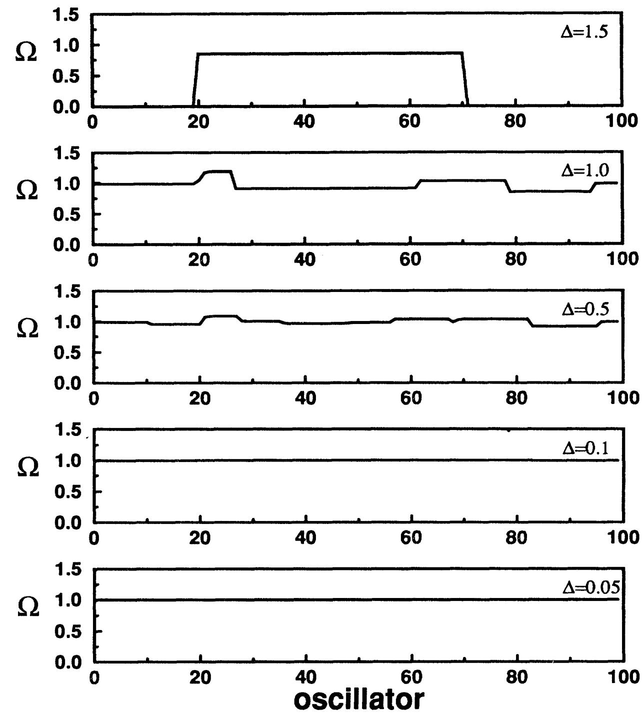

FIG. 5. The asymptotic frequency Ω of a chain after 3000 time steps for frequency distributions of different widths, D=0.8 and A=1.1. Dead oscillators are represented by Ω=0.

$$r = \frac{N_s}{N}, \tag{9}$$

where N is the oscillator population and N_s is the number of oscillators in the largest synchronized cluster. If, for finite coupling strength, r is $O(1)$ in the limit $N \to \infty$ we say that there is macroscopic synchronization, which can be interpreted as a non-equilibrium phase transition from an incoherent to a coherent or ordered time-dependent state [2,6].

To decide whether a transition exists, we have calculated $\langle r \rangle$ in ensembles of 20 samples of $L \times L$ oscillators as follows. The native frequencies were chosen from a Gaussian distribution centered in 1.0 with width 0.1, the initial condition is $z = 1.0$, $A = 2.0$ and we consider periodic boundary conditions. Equation (4) was iterated 1000 times, 500 of which were discarded as a transient. The asymptotic frequencies were then calculated and the frequency clusters were analyzed with an algorithm used in percolation problems [20] as in [11]. Two oscillators were considered synchronized whenever the differences in their asymptotic frequencies were less than 0.0001. This procedure was then repeated for different values of D. The resulting curves of $\langle r \rangle$ versus D for different lattices sizes are plotted in Fig. 6. The phase transitionlike be-

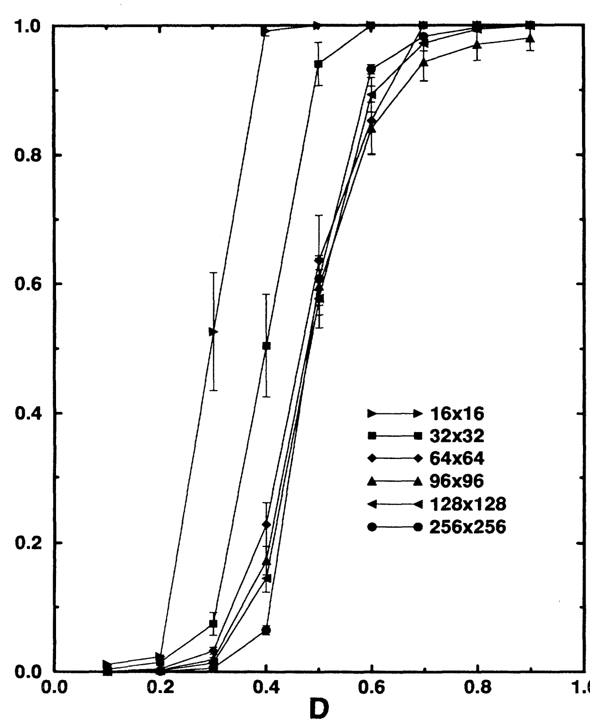

FIG. 6. Frequency order parameter r averaged over 20 samples as a function of the coupling strength D. The native frequencies are chosen from a Gaussian distribution centered in 1.0 with variance 0.1. The asymptotic frequencies were calculated after 1000 iterations, 500 of which were discarded as a transient.

havior observed with the phase model in two-dimensional lattices [10,11] is also present here: as the system size is increased, the curves converge to a single curve, but the convergence is better observed with this model. Therefore, we confirm that frequency order is possible in the thermodynamical limit for two-dimensional systems.

As to the efficiency of the model, we compare the CPU time used for obtaining a curve in Fig. 6 to the time required to draw an equivalent curve using the phase model, as in [11]. The simulation of a 256 × 256 lattice with the CDS model required the same amount of time as the simulation of a 96 × 96 lattice of rotors, using the same Cray Y-MP computer. Considering that with the cell-dynamical system model we have information about the phase and the amplitude, we conclude that it is a more powerful model for simulations.

We now push the analogy with equilibrium phase transitions even further and try to determine the order of this transition. Daido [8] have suggested that the onset of macroscopic synchronization could be compared to a second-order phase transition in equilibrium cooperative systems, but he has not explicitly determined the order of the transition in systems with short-range coupling. In temperature-driven equilibrium first-order transitions there is a discontinuity of the order parameter at some finite value of the temperature, in contrast to a smooth variation in second-order transitions. This is the expected behavior in infinite systems. The effect of a finite lattice is to round and shift the transition region, making it difficult to locate and to classify the transition [21]. A more precise tool to identify the order of the transition is the fourth-order reduced cumulant of the order parameter Ψ [22,21] defined as

$$U_L = 1 - \frac{\langle \Psi^4 \rangle_L}{3\langle \Psi^2 \rangle_L^2}. \tag{10}$$

The label L is a reminder of the finiteness of the system. In the thermodynamic limit $U_\infty = \frac{2}{3}$ in a second-order transition, and passes through a minimum at the transition temperature in a first-order transition. To understand this behavior, suppose a Gaussian distribution for Ψ. A continuous transition corresponds to a dislocation of the center of the Gaussian and to alterations of its width, so we expect Gaussian averages all the time. In a discontinuous transition there is phase coexistence, so at some point we have the superposition of two Gaussians centered at the values corresponding to each phase which causes the deviation from the single Gaussian value $\frac{2}{3}$. This is an oversimplified explanation, but it gives the idea behind the quantity U_L. The fourth-order cumulant has been successfully used to determine the order in many equilibrium phase transitions (see, for example [22,23]). We have calculated U_L for the samples plotted in Fig. 6. As defined in (10), the quantity U_L is not well defined numerically when $\psi \to 0$, which is the case here. To overcome this difficulty we follow [23] and add

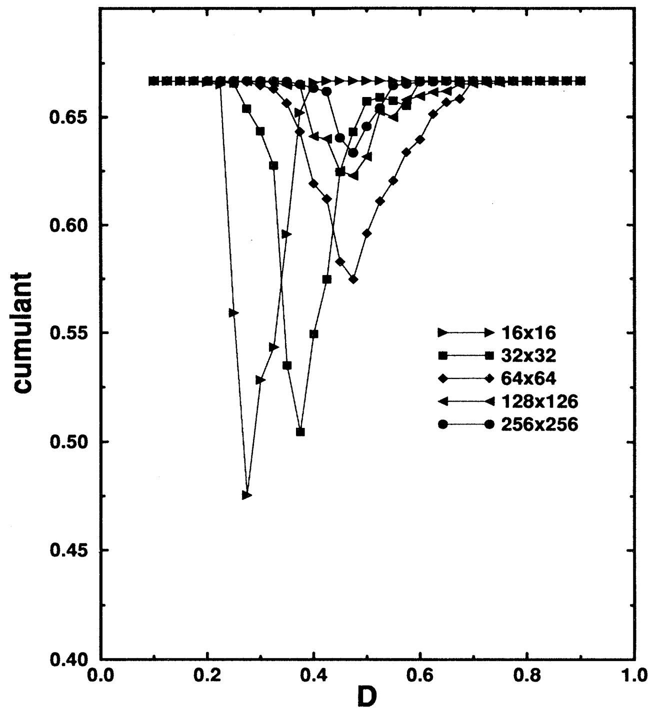

FIG. 7. Fourth-order reduced cumulant of the order parameter as a function of the coupling strength for lattices of different sizes. The presence of a minimum suggests that the transition is of first order.

an arbitrary constant to all values of r, that is, we rigidly shift the ψ distribution. The results can be seen in Fig. 7. The presence of a minimum shows that we have a first-order transition.

In summary , we have proposed a discrete model for the dynamics of the amplitude and phase of coupled limit-cycle oscillators. The model is computationally efficient and specially adequate for vector compilers, which are important qualities when one needs to investigate the asymptotic behavior of large systems. With this model we observed the phenomena of amplitude death and synchronization. Analysis of two-dimensional lattices indicated that there is a nonequilibrium phase transition between a nonsynchronized state and a synchronized state, possibly of first order.

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[1] D. S. Fisher, Phys. Rev. B **31**, 1396 (1985).
[2] Y. Kuramoto, Chemical Oscillations Waves and Turbulence (Springer-Verlag, Berlin, 1984).
[3] K. Bar-Eli, Physica D **14**, 242 (1985).
[4] A. T. Winfree, The Geometry of Biological Time (Springer-Verlag, New York, 1967).
[5] A. T. Winfree, When Time Breaks Down: The Three Dimensional Dynamics of Electrochemical Waves and Cardiac Arrhythmias (Princeton University Press, Princeton, NJ, 1987).
[6] H. Sakaguchi, S. Shinomoto, and Y. Kuramoto, Prog. Theor. Phys. **77**, 1005 (1987).
[7] H. Sakaguchi, Phys. Lett. A **174**, 289 (1993).
[8] H. Daido, Phys. Rev. Lett. **61**, 231 (1988).
[9] S. H. Strogatz and R. E. Mirollo, Physica D **31**, 1143 (1988).
[10] T. Aoyagi and Y. Kuramoto, Phys. Lett. A **155**, 410 (1991).
[11] M. Bahiana and M. S. O. Massunaga, Phys. Rev. E **49**, R **3558** (1994).
[12] L. L. Bonilla, Phys. Rev. Lett. **60**, 1398 (1988).
[13] G. B. Ermentrout, Physica D **41**, 219 (1990).
[14] P. C. Matthews and S. H. Strogatz, Phys. Rev. Lett. **65**, 1701 (1990).
[15] Y. Oono and S. Puri, Phys. Rev. A **38**, 434 (1988).
[16] Y. Oono and A. Shinozaki, Forma **4**, 75 (1989).
[17] M. Bahiana and Y. Oono, Phys. Rev. A **41**, 6763 (1990).
[18] M. Mondello and N. Goldenfeld, Phys. Rev. A **42**, 5865 (1990).
[19] K. Kitahara, Y. Oono, and D. Jasnow, Mod. Phys. Lett. B **2**, 765 (1988).
[20] J. Hoshen and R. Kopelman, Phys. Rev. B **14**, 3438 (1976).
[21] K. Binder and D. P. Landau, Phys. Rev. B **30**, 1477 (1984).
[22] M. S. S. Challa, D. P. Landau, and K. Binder, Phys. Rev. B **34**, 1841 (1986).
[23] B. Koiller and R. B. Capaz, Phys. Rev. Lett. **74**, 769 (1995).