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Rotating ring-shaped bright solitons

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We show that ring-shaped bright soliton structures can exist in focusing Kerr media. Ring-shaped bright solitons, being initially at rest, demonstrate the inherent tendency to shrink. However, the ring axial rotation makes it possible to stabilize the soliton radius. The analytical results are confirmed by numerical calculations.

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I. INTRODUCTION

Optical solitons have been observed experimentally for the first time as temporal solitons in optical fibers [1] and then as spatial solitons — self-trapped beams [2–7]. As the circular-symmetry beam in a Kerr medium is unstable displaying blow-up and collapse [8], spatial solitons have been observed in bulk media as soliton strips [2,6] or in planar waveguide geometries [3–5,7].

The interest in spatial solitons is motivated by two reasons. Firstly, solitons themselves are perfect, stable self-trapped beams, which are able to interact elastically with each other [9]. So they are very attractive for different types of switching and logic devices. Secondly, spatial solitons give a possibility to create self-induced waveguides for propagation of a weak beam of other wavelength [10–12], or even a beam of comparable power [13]. Another interesting possibility has been discussed recently [14–16]. It was shown that a pair of parallel, weakly overlapping soliton beams forms an induced waveguide for a weak probe beam (of another wavelength) propagating between the solitons, and such a weak beam suppresses the mutual soliton interaction.

Additional advantages for spatial solitons are due to the existence of one more dimension. This was recognized for dark solitons, where the soliton vortices [17–20] and ring-shaped dark solitons [21,22] were discovered. In particular, the soliton vortices exhibit very high stability, which is explained by the presence of a topological charge. Ring-shaped dark solitons are not stationary, they demonstrate a rich dynamics, but their inherent tendency is to diverge, they expand and eventually decay.

In this paper we show that *the ring-shaped bright solitons* also exist. Unlike ring-shaped dark solitons, initially motionless bright ring-shaped solitons shrink. However, these solitons have an additional parameter, namely, *the topological charge*, which may be interpreted simply as axial rotation of a soliton. Such rotation can stabilize the soliton radius. Note, that the similar ring-shaped (so-called cylindrical) solitons (without rotation) were discovered previously in fluid dynamics and they are described by the cylindrical Korteweg deVries equation (see [23] and references therein).

We should emphasize the difference between the analysis presented here and the other problem, namely instability of exact stationary solutions of the circular symmetry (see, e.g., [24–28] and references therein). Ring-shaped bright solitons, considered in this paper, are *intrinsically dynamical* (i.e., expanding or shrinking), and they retain circular symmetry and soliton properties despite this dynamics. Ring-shaped solitons become stationary only at particular values of parameters. By contrast, solutions considered in [24–28] are *stationary* but they exhibit some dynamics due to instability, losing circular symmetry and finally decaying in filaments.

The paper is organized as follows. In Sec. II we introduce the two-dimensional nonlinear Schrödinger (NLS) equation and its conservation laws, and we derive an analytical expression for the dynamics of the soliton radius (Sec. III) and compare these results with numerical computations (Sec. IV). In Sec. V we briefly discuss the possibility of experimental verification of ring-shaped bright solitons and suppression of modulational instability. Finally, Sec. VI summarizes the paper.

II. BASIC EQUATIONS

We start from the (2+1)-dimensional NLS equation, which may be written in the normalized form as follows:

$$i\frac{\partial\psi}{\partial z}+\frac{1}{2}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+|\psi|^2\psi=0, \tag{1}$$

where $\psi(x,y,z)$ is the complex envelope of the electric field, x and y are two transversal coordinates, and z is the coordinate in the propagation direction.

Equation (1) possesses several conservation laws. For our consideration the most important are the energy,

$$E=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\psi|^2dxdy=\int_0^{2\pi}d\varphi\int_0^{\infty}|\psi|^2\rho d\rho, \tag{2}$$

and the Hamiltonian,

$$\begin{aligned}H&=\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\left|\frac{\partial\psi}{\partial x}\right|^2+\left|\frac{\partial\psi}{\partial y}\right|^2-|\psi|^4\right)dxdy\\&=\frac{1}{2}\int_0^{2\pi}d\varphi\int_0^{\infty}\left(\left|\frac{\partial\psi}{\partial\rho}\right|^2+\frac{1}{\rho^2}\left|\frac{\partial\psi}{\partial\varphi}\right|^2-|\psi|^4\right)\rho d\rho. \end{aligned}\tag{3}$$

Note also that if the field ψ has a topological charge, which is described by the multiplier in the form $\exp(im\varphi)$, then the value m is conserved during the propagation. The simple physical reason for this is that the topological charge is quantized, i.e., it may take only integer numbers, $m = 0, \pm 1, \pm 2, \ldots$.

III. ANALYTICAL APPROACH TO SOLITON DYNAMICS

To obtain the approximate ring-type bright soliton of Eq. (1), we generalize the soliton strip by "bending" it into a ring of the radius r assuming $r\eta \gg 1$,

$$\psi(\rho,\varphi,z) = \eta \operatorname{sech}[\eta(\rho - r)] \exp[i(m\varphi + \Omega\rho + \Gamma z)]. \quad (4)$$

Therefore, the ring-shaped bright soliton is described by the parameters η, Ω, r, m, and Γ. Note that at $z = 0$ the initial values η_0, Ω_0, and r_0 are independent.

Strictly speaking, the ansatz (4) contains singularity at $\rho = 0$ for $m \neq 0$, which does not exist in the exact solution. In order to make finite the integral $\int_0^\infty \frac{m^2}{\rho}|\psi|^2 d\rho$, which arises in the Hamiltonian, we have to improve the ansatz (4), multiplying it by some function $f(x)$, where $f(x) \to 0$ for $x \to 0$ and $f(x) = 1$ for $x > 1$ [for example, $f(x) = \tanh x$].

To analyze the ring-shaped soliton dynamics analytically, we assume that the soliton parameters η, Ω, and r are slowly varying on the propagation distance z. So we can treat the soliton dynamics *adiabatically* neglecting the emission of radiation. Using these propositions, for the approximate solution (4) the energy conservation law yields

$$E \approx 4\pi\eta r, \quad (5)$$

and the Hamiltonian

$$H \approx E\left(\Omega^2 + \frac{m^2}{r^2} - \frac{\eta^2}{3}\right). \quad (6)$$

One can see from the Hamiltonian structure that the topological charge m can significantly affect the ring-shaped soliton dynamics, especially for $m > r\eta$.

Equations (5) and (6) together with the relationship

$$\frac{dr}{dz} = \Omega \quad (7)$$

form the complete set of equations for three unknown functions $\eta(z)$, $\Omega(z)$, and $r(z)$. From this set we obtain the following solution for $r(z)$:

$$r(z) = \sqrt{|C_1(C_2 + z)^2 - (r_0^2\eta_0^2 - 3m^2)/3C_1|}, \quad (8)$$

where the integration constants are $C_1 = \Omega_0^2 + (3m^2 - r_0^2\eta_0^2)/(3r_0^2)$, $C_2 = \Omega_0 r_0/C_1$. The first integration constant may be expressed also as $C_1 = H/E$; the meaning of the second constant will be clarified below. The soliton amplitude and its transverse velocity can be easily found from Eqs. (5) and (7), respectively.

Depending on the initial values η_0, Ω_0, r_0, and m, Eq. (8) describes both the increasing and decreasing soliton radius. To simplify the analytical consideration, first we consider the case $\Omega_0 = 0$ and $m = 0$, i.e., the ring is initially at rest both in the angular and transversal directions. From Eq. (8) we obtain that the soliton radius decreases during the propagation, and eventually becomes equal to zero. So, the initially motionless ring-shaped bright solitons demonstrate the inherent tendency to shrink and collapse. The collapse occurs at

$$z_c = \sqrt{\frac{\eta^2 r_0^2 - 3m^2}{3C_1^2}} - C_2. \quad (9)$$

This effect can be treated, at least qualitatively, as "internal tension" which exists for, e.g., a bright-soliton stripe. Indeed, it is known that the dark soliton stripe is unstable to transverse (snakelike) modulations [29–31], and the ring-shaped dark soliton tends to expand [21,22]. So, the dark soliton stripe can be compared with a compressed spring. The bright soliton stripe is stable to snakelike instability and, being bended to a ring, tends to decrease the ring radius, so it can be compared with a stretched spring.

Otherwise, the tendency of bright ring-shaped solitons to shrink can be considered as a consequence of the overlapping of the ring with itself. At $m = 0$ the whole ring is in-phase and the process of shrinking can be treated as a result of attraction among different parts of the ring. Note that due to two-dimensional geometry the tendency to shrink is relatively strong. For example, the soliton with the parameters $\eta_0 = 1$, $r_0 = 5$, $\Omega_0 = 0$, and $m = 0$ has the focusing length $z_c \approx 8.6$. At the same time, for a pair of the in-phase bright solitons in one dimension with the separation $\Delta = \pm 5$, the interaction semiperiod is found to be $L = 116.5$ (see, e.g., [32]), i.e., such an interaction is much weaker.

If $\Omega_0 \neq 0$, the ring is initially shrinking or expanding. For $\Omega_0 < 0$ such modulation decreases only the collapse length, while for $\Omega_0 > 0$ there are two different scenarios. We have found that there exists a threshold value of the initial transversal velocity Ω_{th}. For $0 < \Omega_0 < \Omega_{\text{th}}$ the ring-shaped soliton initially expands, then its radius reaches the maximum value at the distance

$$z_s = -C_2, \quad (10)$$

the soliton amplitude is minimal at the turning point, and then the soliton begins to shrink. Equation (10) indicates a simple physical meaning of the second integration constant C_2. The threshold velocity is given by

$$\Omega_{\text{th}} = \left(\frac{\eta_0^2 r_0^2 - 3m^2}{3r_0^2}\right)^{1/2}. \quad (11)$$

For $\Omega_0 > \Omega_{\text{th}}$ the soliton expands monotonically and subsequently decays. For instance, at $\eta_0 = 1$, $r_0 = 5$, and $m = 0$ the threshold velocity $\Omega_{\text{th}} \approx 0.58$.

And, at last, for the nonzero topological charge $m \neq 0$ the additional "centrifugal force" appears, which tends to expand the ring. As a result, for the same r_0 the threshold value $\Omega_{\rm th}$ decreases. On the other hand, at $\Omega_0 = 0$ there is relationship between r_0 and m, which corresponds to the stationary solution — the soliton radius and other parameters are constant during the propagation. Due to discreteness of m such states are possible only for some certain values of r_0. For example, at $\eta_0 = 1$, $m = 3$ the theory predicts the stationary state at $r_{\rm st} \approx 4.9$.

IV. NUMERICAL SIMULATIONS

To prove the predictions of the analytical approach discussed above, we perform numerical simulations using the split-step method with the fast Fourier transform on a diffraction step and a rectangular numerical grid. We choose $\eta_0 = 1$ in all calculations, as it may be rescaled by a simple normalization. First, we demonstrate the validity of the adiabatic approximation. In particular, this approximation leads to the approximate relation $A(z)\,W(z) \approx 1$ for any z, where A is the ring amplitude and W is the ring width (width of the stripe). We use this proposition in the approximate solution (4), as the same variable η denotes the ring amplitude $A = \eta$ and ring width $W = 1/\eta$. Figure 1 shows the dynamics of the amplitude A, radius r, and product $A\,W$ for the initially motionless soliton with $r_0 = 5$, $\Omega_0 = 0$, and $m = 0$, where A, r, and W are extracted from numerical solution of Eq. (1). It is clearly seen that both soliton ring amplitude and radius change several times, and the product $AW \approx 1$ with very good precision. So, the adiabatic approximation can be applied. This test has been used in all numerical calculations.

We observe a very good quantitative agreement between analytical and numerical data in a wide region of the parameters Ω_0 and m. Figure 2 shows the dependences of the soliton radius on z for $\eta_0 = 1$, $r_0 = 5$, $m = 0$, and several values of Ω_0. Note that the results agree rather well even for small $r \sim 1$, i.e., just before breakdown. To study the soliton dynamics near the threshold value of Ω_0, we extend the propagation distance up to $z = 25$ and again observe a good agreement (see Fig. 3).

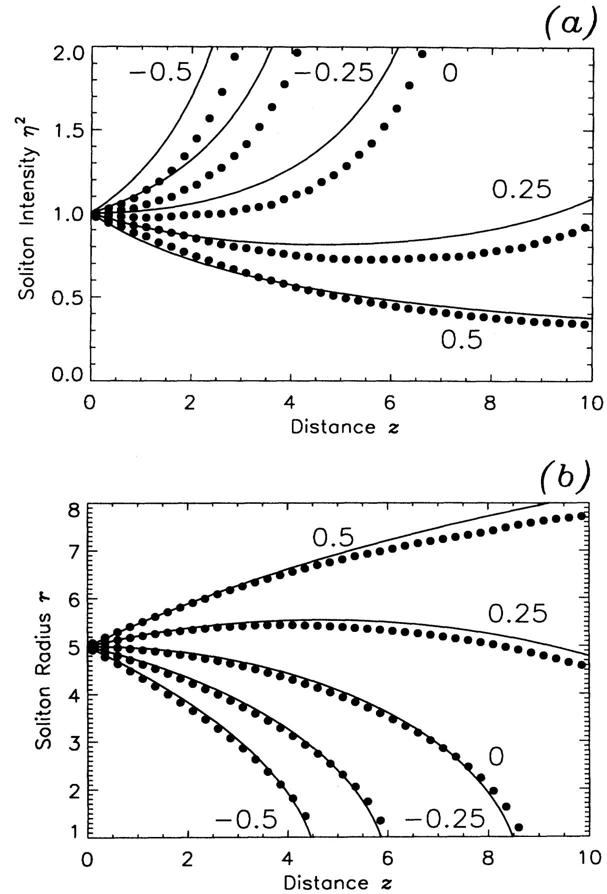

FIG. 2. Dynamics of ring-shaped soliton for several Ω_0, at $r_0 = 5$, $\eta_0 = 1$, and $m = 0$. (a) Soliton intensity $I(z) = \eta^2(z)$ and (b) soliton radius. The solid lines are according to Eq. (8), the dotted lines correspond to a numerical solution. The values of Ω_0 are written near the curves.

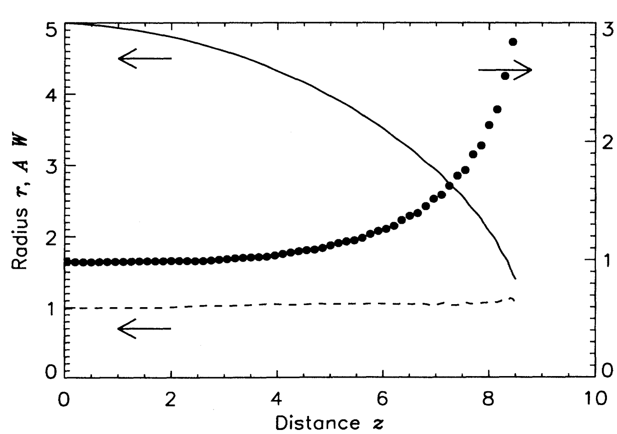

FIG. 1. Numerical data for the ring radius r (solid line), amplitude A (dotted line), and amplitude-width product $A\,W$ (dashed line). Initial values are $\eta_0 = 1$, $r_0 = 5$, $\Omega_0 = 0$, and topological charge $m = 0$.

Also, our analytical solution agrees well with the numerical results for the rotating solitons (see Fig. 4). In particular, for $\eta_0 = 1$, $\Omega_0 = 0$, and $r_0 = 5$ we observe the soliton ring radius stabilization at $m = 3$, as it is predicted by the theory. Strictly speaking, such a state is unstable to deviation of the initial ring radius r_0 from the value $r_{\rm st} \approx 4.9$, however such an instability develops relatively slow, and the ring radius does not change significantly at the distances $z \sim 10$.

The obtained agreement between the analytical and numerical results is much better than that for ring-shaped dark solitons [22]. This may be explained by the following reasons. Solution (4) approximates rather well the exact solution for bright solitons, while for dark ones there is a discrepancy as the background intensity differs significantly inside and outside the dark soliton. One more reason is that the NLS equation for dark solitons in the small-amplitude approximation becomes a KdV-type equation [21]. It is known that under the action of perturbations (even weak) a KdV soliton develops a tail, while for a NLS soliton this effect does not occur. Indeed, for dark solitons we observed strong perturbations of the background outside the soliton [22].

On the other hand, such a good agreement between the analytical calculations and the theory shows that, although the considered waves are not solitons in the exact sense [as Eq. (1) is not integrable by the inverse scattering transform, and, moreover, the soliton amplitude and width are not constant even in the absence of perturbations], their behavior is very similar to that known for

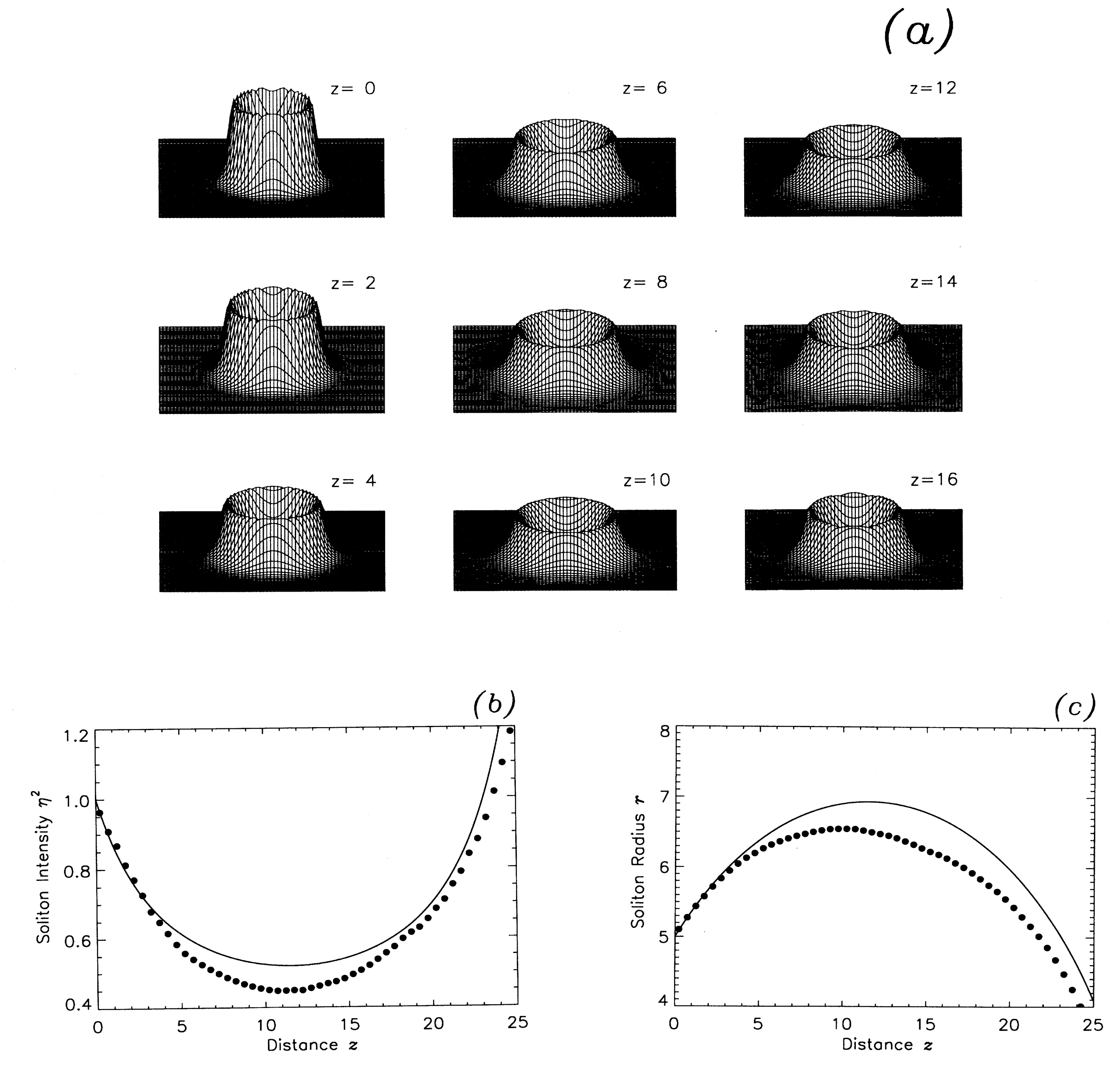

FIG. 3. Dynamics of the initially diverging soliton, $\eta_0 = 1$, $\Omega_0 = 0.4$ ($\Omega_0 < \Omega_{\text{th}}$), $r_0 = 5$, and $m = 0$. (a) 2D plot, (b),(c) comparison of numerical and analytical results for the soliton intensity and radius.

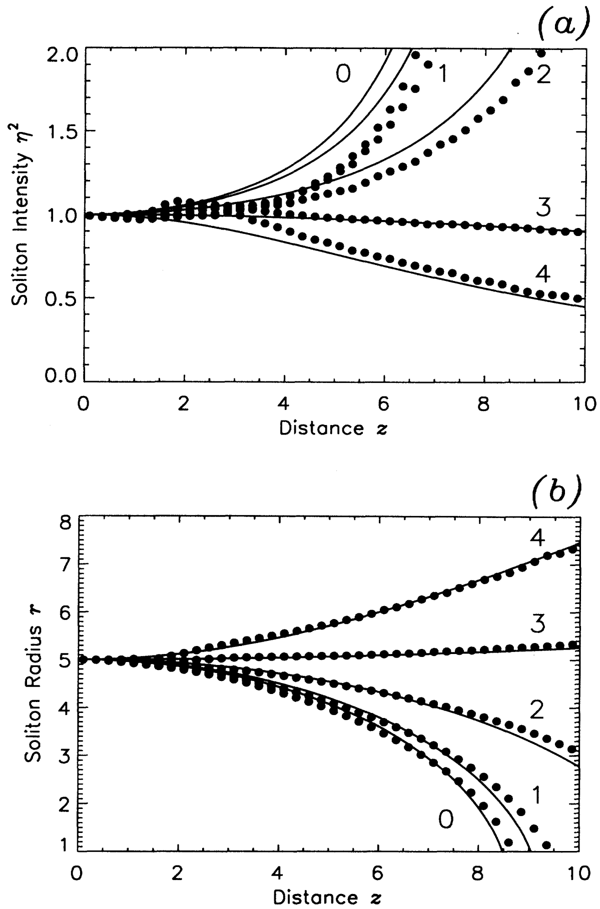

FIG. 4. Bright soliton dynamics vs the topological charge m, $\eta_0 = 1$, $\Omega_0 = 0$, and $r_0 = 5$. (a) Intensity, (b) radius. The values of m are written near the curves.

solitons. In other words, emission of radiation is small and the ring structures remain solitary waves despite the inherent dynamics.

V. POSSIBILITIES OF EXPERIMENTAL OBSERVATION OF RING-SHAPED BRIGHT SOLITONS

First of all, experimental verification of the ring-shaped solitons predicted here requires the creation of a ring-shaped beam. For solitons without topological charge, it can be done by using a ring amplitude mask, while the additional phase mask can create the topological charge. Although experimental realizations deal usually with pulsed beams, this may be sufficient only for the soliton parameters near the threshold Ω_{th}.

The main difficulties are expected due to the occurrence of modulational instability. In our numerical experiments we do not observe this effect. Indeed, since the ring radius and ring intensity are determined as the radius and intensity of the maximum of the ring, a good agreement between analytical and numerical data evidences the absence of filamentation for $r > 1$. The ring stability is also clearly seen in Fig. 3.

It can be explained by the relatively slow development of the modulational instability. Indeed, even near the maximum growth rate the propagation distance $z = 15 - 20$ is required for initial modulation with amplitude 10^{-2} to develop significantly; due to smaller effective intensity of the ring this distance is even more. Additionally, only discrete frequencies $\omega_n = n/r$ can be excited on the ring, and the maximum growth rate corresponds to ω_7 for $r = 5$. However, decay on seven filaments goes through modulation of both the ring amplitude and width, which makes this process even slower.

From our calculations we can draw the conclusion that there exists an optimal value of the initial soliton radius $r_0 \sim 5$ when the internal soliton dynamics can be clearly observed at relatively short distances $z \sim 10$, while on such spatial scales modulational instability does not develop significantly. However, in real experiments the modulation may be essential, especially due to deviation of the ring mask center from the center of the incident beam or due to piecelike structure of the phase mask.

In experiments with soliton stripes in bulk media (see, e.g., [2,6]) modulation instability is usually suppressed by means of the two-beam interference technique, where the second beam, identical to the first one, copropagates simultaneously through a nonlinear medium at a slight angle in the longitudinal dimension. Possibly, this technique may be applied to ring-shaped solitons as well. To do this, two identical ring-shaped beams are to be passed through the phase masks of opposite "polarities," and the resulting interference structure will be stable to modulation instability.

VI. CONCLUSION

We have described ring-shaped bright solitons which can exist in a bulk Kerr medium. We have derived the soliton dynamics from conservation laws and show that these solitons form a four-parameter family, i.e., the soliton dynamics is determined by the initial soliton amplitude η_0, radius r_0, transversal velocity Ω_0, and topological charge m. Being initially at rest, a ring-shaped soliton demonstrates the inherent tendency to shrink. However, in the presence of transverse and angular velocities the soliton dynamics becomes more complicated. In particular, there is a threshold value of the transversal velocity Ω_{th} which separates the scenarios of expanding or shrinking solitons. Also, the angular rotation can stabilize the soliton radius. Analytical results are confirmed by numerical calculations. Very good agreement between numerical and analytical results shows that radiation of linear waves is small and soliton dynamics can be treated adiabatically.

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