Nature of Schott corrections

Reese L. Miller* and R.T. Robiscoe†
Department of Physics, Montana State University, Bozeman, Montana 59717
(Received 10 December 1993; revised manuscript received 2 December 1994)

If a mechanical system with internal degrees of freedom is acted on by an external force, the system's center of mass will accelerate (nonrelativistically) in direct proportion to that force, but constituent parts of the system will *not*. The equations of motion for isolated parts of the system show corrections for internal reaction forces that are similar to the correction of the motion of an accelerated charge by the radiation reaction force, or Schott term, in electrodynamics. By analyzing the motion of a simple mechanical model, constructed to resemble the motion of an accelerated charge in some key details, we find that the mechanical reaction force shares many features in common with the electromagnetic Schott term. For example, the mechanical analogue of the Schott term produces a Larmor-like energy dissipation rate for the system. However, our mechanical analogue does *not* lead to the usual shortcomings of the electromagnetic Schott term, such as the generation of runaway or acausal solutions to the equations of motion. Furthermore, the form of our mechanical analogue indicates a way in which the electromagnetic Schott term can be corrected so as to avoid its usual problems. The lowest order correction results in a modified equation of motion for a charged particle that was suggested by Eliezer [Proc. R. Soc. London Ser. A **194**, 543 (1948)].

PACS number(s): 41.90.+e, 46.10.+z

I. INTRODUCTION: NATURE OF SCHOTT CORRECTIONS

Finding a satisfactory equation of motion for a charged particle in classical electrodynamics has provided a challenge to field theory for quite some time. The "modern" quest began some ninety years ago with the work of Abraham [1,2] and Lorentz [3], and continued with contributions by Schott [4,5], seminal papers by Dirac [6] and analysis by Eliezer [7], a comprehensive book by Rohrlich [8], new approaches by Teitelboim *et al.* [9], review articles by Pearle [10] and Klepikov [11], and current work by Barut and Unal [12], Comay [13], and others. Many hundreds of papers have been published; the references above are but a small sample. After all this work there does not appear to be universal agreement on how a charged particle moves and thus the equation of motion of a charged particle remains a subtle problem, with surprises probably still to come.

Working nonrelativistically, we focus here on the interpretation of the so-called Schott term, which appears as the radiation reaction force in the equation of motion for a particle of charge q and mass m. If the particle moves at velocity $\boldsymbol{v}$, that equation is [14]

$$m\dot{\boldsymbol{v}} = \boldsymbol{F} + m\tau_0\ddot{\boldsymbol{v}}, \text{with } \tau_0 = 2/3\left(q^2/mc^3\right). \qquad (1.1)$$

The dot means d/dt. $\boldsymbol{F}$ is the external force acting on the particle, the Schott term is the one containing $\ddot{\boldsymbol{v}}$, and Eq. (1.1) is usually called the Abraham-Lorentz equation of motion. The Schott term in part accounts for the fact that the charge, while accelerating, radiates electromagnetic energy at a rate given by the Larmor formula

$$\mathcal{R}_L(t) = m\tau_0\dot{\boldsymbol{v}}^2, \qquad (1.2)$$

and that in emitting this radiant energy (and momentum), the charge must show a correction to its otherwise Newtonian motion.

In this paper, we shall calculate the analogue of a Schott correction term for the motion of a simple mechanical system. Besides showing that Schott-like correction terms are not unique to equations of motion in electrodynamics, we shall intercompare the effects of those terms for the mechanical and electrodynamical systems. The $\ddot{\boldsymbol{v}}$ term in electrodynamics, which persists in the relativistic Lorentz-Dirac (LD) version of Eq. (1.1), generates a number of well-known difficulties in analyzing q's motion. (the reader unfamiliar with these difficulties can consult references [14] and [8]); the mechanical analogue is free of these difficulties.

Desirable features of a Schott term are: (1) it can be written in a Newtonian form (no explicit dependence on $\ddot{\boldsymbol{v}}$), (2) any time-nonlocal version of the Schott term shows causal ordering between the applied force and the particle motion, so that preacceleration problems are avoided, (3) runaway solutions do not appear, (4) solutions to standard problems can be reproduced without ambiguity, and (5) the Larmor radiation rate formula remains intact. In analyzing the motion of our simple mechanical system, we find that a correction term appears in the equation of motion, and that this term shares many features in common with the electromagnetic Schott term

*Electronic address: uphrrrm@ruby.oscs.montana.edu
†Electronic address: uphrr@ruby.oscs.montana.edu

in Eq. (1.1). However, our mechanical Schott term does *not* share the pathologies of the electromagnetic term—in fact, it easily qualifies for the desirable features (1)–(5) listed above. The implication is that the electromagnetic Schott term might be successfully modified by drawing on analogies with the mechanical system.

In Sec. II A, we set up and solve the equations of motion for our one-dimensional mechanical system. We find an exact form for the mechanical Schott term $S(t)$, as a reaction force acting in the equation of motion for the "bare" mass m. In Sec. II B, we analyze the nature of $S(t)$, and find that it qualifies for the "desirable features" (1)–(4) listed above. The leading term in $S(t)$ is proportional to $\ddot{v}$ for the particle, as in Eq. (1.1); otherwise, $S(t)$ goes as $\ddot{V}$—where V is the system's center-of-mass velocity. In Sec. II C, we write an energy balance equation for the mechanical system and thereby identify its rate of energy dissipation. This rate is similar to the Larmor radiation rate, Eq. (1.2), but it is based on the *system* acceleration $\dot{V}$ rather than the *particle* acceleration $\dot{v}$. Point (5) on the "desirable change" list is thus addressed. In Sec. III A, we summarize the points of comparison between the mechanical and electromagnetic Schott terms, and we argue that the standard electromagnetic Schott term *could* be corrected for a center-of-mass effect inherent in the acceleration of a charged particle. In lowest order, this correction would not depend on the charge having a finite size. In Sec. III B, we comment on dynamical substitutions that can and have been made to regularize radiation reaction forces, with special emphasis on a substitution first suggested by Eliezer. We find that Eliezer's substitution has merit, but was probably incomplete. In Sec. III C, we discuss elaborations of the mechanical model which might be useful in completing the analogy to the electromagnetic problem of an accelerated charge. Finally, in Sec. IV, we present our conclusions regarding possible corrections to electromagnetic Schott terms.

II. SOLUTION FOR THE MECHANICAL SYSTEM

A. Mechanical system details and equations of motion

Figure 1 depicts the mechanical system we shall analyze. The system moves in one dimension, along the x axis, and it consists of a central particle of "bare" mass m coupled front and back to identical masses μ. The coupling is provided by identical massless springs, each with force constant k, unstressed length ℓ, and damping constant β. An external force F, at most a function of time, acts on m and accelerates the system to the right. The problem of interest is to find an equation of motion for the velocity $v = \dot{x}$ of the particle, m. In what follows, we will often refer to that part of the system exterior to m, namely the μ's plus the springs, as m's "field." Therefore, m's equation of motion will *not* be simply $m\dot{v} = F(t)$, but must be modified by the "field" coupling. Clearly, this field can both store and dissipate energy, and in those regards it shares some similarities with the electromagnetic field of an accelerated charged particle.

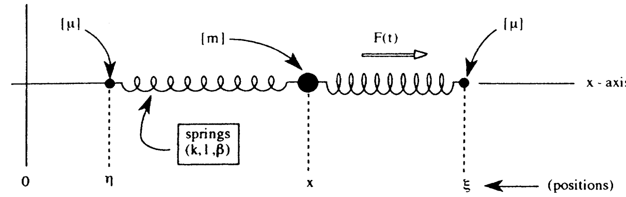

FIG. 1. The mechanical system is confined to the x axis, and consists of a central (bare) mass m coupled front and back to "field" masses μ. The coupling is provided by massless springs, each of force constant k, unstressed length ℓ, and damping constant β. An external force $F(t)$ acts on m and accelerates the system to the right.

On general grounds, we expect two modifications to m's equation of motion. First, the force F acts to accelerate *both* the particle (mass m) *and* the field (mass 2μ), so the net inertial mass $m \to M = m + 2\mu$. One can say that the bare mass m is "dressed" by the field. Second, the applied external force F is not the only force acting on m—acceleration of the field generates a reaction force that alters m's motion. Thus we anticipate an equation of motion of general form

$$M\dot{v} = F(t) + S(t). \qquad (2.1)$$

The dot means d/dt. By contrast with $F(t)$, a force applied to the system by some *external* agent, $S(t)$ is an *internal* force that appears because of the system's intrinsic degrees of freedom (here, the motion of μ relative to m). $S(t)$ is expected to depend on the field parameters (μ, k, β), and should vanish when either $\mu \to 0$ or $k \to \infty$, because then the particle-field coupling vanishes. Also, if the system starts from rest, $S(t)$ should vanish when the force $F(t)$ is zero, for then the system cannot be excited. Finally, $S(t)$ should account for the system's dissipation (via damping constant β) as well as dynamical corrections to the particle velocity $v = \dot{x}$.

If, as in Fig. 1, the instantaneous position of the central particle is $x = x(t)$, and the positions of the μ's are $\xi(t)$ and $\eta(t)$, front and back, the system equations of motion are

$$\mu\ddot{\xi} + 2\beta\mu\dot{\xi} = k\left[\ell - (\xi - x)\right], \qquad (2.2a)$$

$$\mu\ddot{\eta} + 2\beta\mu\dot{\eta} = -k\left[\ell - (x - \eta)\right], \qquad (2.2b)$$

$$m\ddot{x} = F - \mu(\ddot{\xi} + \ddot{\eta}). \qquad (2.2c)$$

The damping constant β has been introduced so as to have dimensions of a frequency. We shall recast Eqs. (2.2) in the center-of-mass (M) coordinates and relative compression

$$\text{CM position: } R_{\rm CM} = (mx + \mu\xi + \mu\eta)/M,$$
$$\text{and } Y = 2x - (\xi + \eta), \qquad (2.3)$$

where $M = m + 2\mu$ is the total system mass. Upon transforming Eqs. (2.2) into these new coordinates, we verify an important fact, namely,

$$M\ddot{R}_{\rm CM} = F. \tag{2.4}$$

This standard result says that the system's center of mass obeys a strict Newtonian equation of motion, no matter how the system's internal degrees of freedom may behave. Also by combining Eqs. (2.2), we find

$$\ddot{Y} + 2\beta\dot{Y} + \omega^2 Y = (2/m)F + 4(M/m)\beta\dot{R}_{\rm CM}. \tag{2.5}$$

Here $\omega^2 = (M/m)\omega_0^2$, is a renormalized natural frequency for the system. We can obtain the equation of motion for the particle in terms of Y by substituting $m = M - 2\mu$ into Eq. (2.2c), rearranging terms and finally using Eq. (2.3). We find

$$M\dot{v} = F + \mu\ddot{Y}. \tag{2.6}$$

The reaction force posited in Eq. (2.1) is thus $S(t) = \mu\ddot{Y}$, and the particle mass m has been "dressed": $m \to M = m + 2\mu$.

Next, we solve Eq. (2.5) for $\ddot{Y}$. Let $Z = \dot{Y}$, and operate through Eq. (2.5) by d/dt. Upon setting $\ddot{R}_{\rm CM} = F/M$, per Eq. (2.4), we find

$$\ddot{Z} + 2\beta\dot{Z} + \omega^2 Z = B(t), \tag{2.7}$$

where $B(t) = (2/m)\dot{F} + (4/m)\beta F$. This is the equation of motion for a damped, driven simple harmonic oscillator. A particular solution to Eq. (2.7) is

$$Z(t) = \int_0^\infty B(t-\tau)J(\tau)d\tau, \tag{2.8}$$

where $J(\tau) = (1/\omega_r)e^{-\beta\tau}\sin\omega_r\tau$. $\omega_r = (\omega^2 - \beta^2)^{1/2}$ is the reduced frequency for the motion; we assume ω_r is real. $J(\tau)$ is the kernel for Eq. (2.7), and causality is guaranteed by taking $\tau = 0$ as the lower limit in the integral. In the present application, we want $\ddot{Y} = \dot{Z}$. After a partial integration, we find

$$\ddot{Y}(t) = \int_0^\infty B(t-\tau)(dJ/d\tau)d\tau. \tag{2.9}$$

Now substitution of $B(t-\tau)$ from Eq. (2.7) and then another partial integration (using the fact that: $\ddot{J} + 2\beta\dot{J} = -\omega^2 J$, for $t > 0$), yields

$$\mu\ddot{Y}(t) = 2\frac{\mu}{m}\left(F(t) - \omega^2\int_0^\infty F(t-\tau)J(\tau)d\tau\right) = S(t). \tag{2.10}$$

Equation (2.10) is one form of the reaction force $S(t) = \mu\ddot{Y}$ appearing in Eqs. (2.6) and (2.1). A form that is sometimes more useful for the analysis to come is

$$S(t) = \mu\ddot{Y}(t) = \frac{d}{dt}p(t), \tag{2.11}$$

where $p(t) = (2\mu/m)\int_0^\infty F(t-\tau)G(\tau)d\tau$, and $G(\tau) = e^{-\beta\tau}[\cos\omega_r\tau + (\beta/\omega_r)\sin\omega_r\tau]$. Equation (2.11) follows from Eq. (2.10) by a partial integration with respect to $dG = -\omega^2 J d\tau$. $G(t)$ satisfies the homogeneous, damped simple harmonic oscillator equation, Eq. (2.7), with initial conditions $G(0+) = 1$, and $\dot{G}(0+) = 0$. At this point, for the mechanical system in Fig. 1, whose equations of motion are given in Eqs. (2.2), we have deduced an equation of motion for the velocity $v = \dot{x}$ of the central particle, m. As anticipated, the particle obeys Eq. (2.1). Now we shall study $S(t)$.

B. Role of the Schott term in the equation of motion

Several immediate properties of the Schott correction $S(t)$, just derived for the mechanical system in Fig. 1, are that it does *not* depend on a superacceleration explicitly, and it is nonlocal in time, but causal. These features can be seen by using Eq. (2.10). First, we note that by Eq. (2.4) we can write: $F(t) = M\dot{V}(t)$, where $V = \dot{R}_{\rm CM}$ is the system's center-of-mass velocity. Then Eq. (2.10) becomes

$$S(t) = (2\mu M/m)\left(\dot{V}(t) - \omega^2\int_0^\infty \dot{V}(t-\tau)J(\tau)d\tau\right). \tag{2.12}$$

A complete specification of $S(t)$ requires knowing the CM (center-of-mass) acceleration $\dot{V}(t_0)$ at the time t_0 when the external force acts. $S(t)$ in Eq. (2.12) applies with the tacit assumption [going back to the choice of the particular integrals in Eqs. (2.8) and (2.9)] that $\dot{V}(t < t_0) \equiv 0$, and hence $S(t < t_0) \equiv 0$, i.e., the system is *at rest* prior to application of the external force. This condition poses no limitations on the theory if we can claim that the system *was* at rest at some time in the distant past (even $t_0 \to -\infty$), before the external force acted.

The time-nonlocal character of $S(t)$ is apparent in Eq. (2.12), and also in the more compact expression in Eq. (2.11)

$$S(t) = \frac{d}{dt}p(t), \quad p(t) = (2\mu M/m)\int_0^\infty \dot{V}(t-\tau)G(\tau)d\tau. \tag{2.13}$$

$S(t)$ depends on the values of the force $F = M\dot{V}$ acting at times $t' \le t$; this is causal ordering. The electromagnetic Schott correction, when recast as an integrodifferential equation, also shows a time-nonlocal (but acausal) character [15]. The time-nonlocalization for $S(t)$ extends over a time interval $\Delta t \sim 1/\beta$, which is the characteristic decay time for $G(\tau)$. This interval can be arbitrarily long for systems with little dissipation ($\beta \to 0$).

At this point it is worth noting a significant difference between $S(t)$ for the mechanical system and the electromagnetic Schott correction in Eq. (1.1). The latter depends only on the particle velocity v, while the former depends on the *system* (particle + field) CM velocity V. For the mechanical system, the velocities v and V are related, but they are not the same so long as the particle (mass m) and field (mass 2μ) make individual nonzero contributions to the overall system mass ($M = m + 2\mu$). The relation between the particle velocity v and CM velocity V is found as the first integral of the equation of motion, Eq. (2.1). With $S = dp/dt$, and $p(t)$ per Eq. (2.13)

$$v(t) = V(t) + (2\mu/m) \int_0^\infty \dot{V}(t-\tau) G(\tau)\, d\tau. \quad (2.14a)$$

Thus in the electromagnetic case, one might expect to see CM corrections to the Schott term if the charge and its field make *separate* contributions to the overall mass of the system. We shall return to this point later.

Next, we look at the possibility of runaway solutions for the mechanical system. Suppose the system is at rest at $t < 0$, and then at $t = 0$ it is subjected to an impulsive force $F(t) = MV_0\delta(t)$. The CM will be accelerated from $V = 0$ to $V = V_0$ at $t > 0$, and according to Eq. (2.14a) the central particle will move at $t > 0$ with velocity

$$v(t) = V_0 \left\{1 + (2\mu/m) e^{-\beta t} \left[\cos \omega_r t + (\beta/\omega_r) \sin \omega_r t\right]\right\}. \quad (2.14b)$$

Even for $\beta \to 0$, the motion remains bounded, and exhibits no runaway character. By contrast, in the electromagnetic case, the particle velocity would be increasing at $t > 0$ (and in the force-free regime) by the exponential factor $\exp(t/\tau_0)$. This simple example appears to rule out the occurrence of runaway solutions for the mechanical system, at least as they are known in the analogous electromagnetic situation.

For nonsingular external forces, the Schott correction $S(t)$ can be developed in a Taylor series. First, write $p(t)$ in Eq. (2.13) as

$$p(t) = (2\mu M/m) \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{d}{dt}\right)^n \dot{V}(t) \int_0^\infty \tau^n G(\tau)\, d\tau. \quad (2.15)$$

$G(\tau)$ is defined in Eq. (2.11), and the integral is tabulated [16]

$$\int_0^\infty \tau^n G(\tau) d\tau = \frac{2\beta}{\omega^2} (n!/\omega^n) \left[\sin(n+2)\theta / \sin 2\theta\right], \quad (2.16)$$

where $\sin\theta = \omega_r/\omega$, $\cos\theta = \beta/\omega$. The series for $p(t)$ is now

$$p(t) = MT \sum_{n=0}^{\infty} \left(\frac{\sin(n+2)\theta}{\sin 2\theta}\right) \left(-\frac{1}{\omega}\frac{d}{dt}\right)^n \dot{V}(t), \quad (2.17)$$

where $T = 4\mu\beta/M\omega_0^2$. T is a scale time for the mechanical system; it is analogous to the electromagnetic scale time τ_0 in Eq. (1.1). $S = dp/dt$ becomes

$$S(t) = MT \sum_{n=0}^{\infty} \sigma_n(\theta) \left(-\frac{1}{\omega}\frac{d}{dt}\right)^n \ddot{V}(t), \quad (2.18)$$

where $\sigma_n(\theta) = \sin(n+2)\theta / \sin 2\theta$. The first few expansion coefficients are

$$\begin{aligned} \sigma_0(\theta) &= \sin 2\theta / \sin 2\theta = 1, \\ \sigma_1(\theta) &= \sin 3\theta / \sin 2\theta = -(\omega^2 - 4\beta^2)/2\beta\omega, \\ \sigma_2(\theta) &= \sin 4\theta / \sin 2\theta = -2(\omega^2 - 2\beta^2)/\omega^2, \end{aligned} \quad (2.19)$$

and so the leading terms of the Schott correction are

$$S(t) = MT \left[1 + \left(\frac{\omega^2 - 4\beta^2}{2\beta\omega}\right) \frac{1}{\omega}\frac{d}{dt} - \cdots\right] \ddot{V}. \quad (2.20)$$

To the lowest order derivative: $S \simeq MT\ddot{V}$, and if we ignore the difference between the CM velocity V and the particle velocity $\boldsymbol{v}$ per Eq. (2.14a), then $S \sim MT\ddot{v}$. This crude approximation gives the mechanical analogue of the electromagnetic Schott term appearing in Eq. (1.1). Of course the exact form for the mechanical Schott term in Eq. (2.13) contains much more (and different) detail.

Since the external force is $F = M\dot{V}$, we can write the leading term in Eq. (2.20) as $S \simeq T\dot{F}$. The particle equation of motion, Eq. (2.1), is then

$$M\dot{\boldsymbol{v}} = [1 + T(d/dt)]\, F(t). \quad (2.21)$$

This result, which is "exact" for the mechanical system to lowest order derivatives, recalls an equation of motion that has been postulated by Eliezer [17], in an attempt to eliminate such difficulties as runaway solutions and preacceleration problems for the electromagnetic case. Here, Eliezer's equation arises naturally in the analysis of our mechanical system, and in fact it does eliminate runaways and preacceleration problems, as we have noted above.

Equation (2.21) has an easy solution for simple harmonic motion. If the binding force for the particle is $F = -M\nu^2 x$, then (2.21) gives

$$\ddot{x} + \nu^2 T\dot{x} + \nu^2 x = 0, \quad (2.22)$$

with solutions for the central particle's position

$$x(t) = e^{-(\Gamma/2)t} \exp(\pm i\nu_r t), \quad (2.23)$$

where $\Gamma = \nu^2 T$, $\nu_r = \nu[1 - (\nu T/2)^2]^{1/2}$. If the scale time $T = 4\mu\beta/M\omega_0^2 > 0$ (i.e., the "field" mass μ and damping constant β are nonzero), and if we set $\nu = \omega_0$, then the oscillator amplitude is damped at a rate $\Gamma/2 = (2\mu/M)\beta$, and its frequency is shifted downward to $\nu_r = [\omega_0^2 - (\Gamma/2)^2]^{1/2}$. This behavior reproduces the standard result [18] for a radiation-damped oscillator, except for a numerical factor in the frequency shift ($\nu - \nu_r$).

In this section, we have analyzed how the Schott term $S(t)$ for the mechanical system contributes to the equation of motion for the central (bare particle) mass. Our findings are that $S(t)$

(1) is Newtonian (if the system was at rest in the distant past);

(2) is nonlocal in time, but causal (no preacceleration problems);

(3) depends on the system's CM velocity, rather than the particle velocity;

(4) shows no apparent runaway solutions for the particle's motion;
(5) reduces (in a crude approximation) to the electromagnetic Schott term;
(6) leads naturally to Eliezer's postulated equation of motion (2.21);
(7) reproduces the behavior of a radiation-damped oscillator.

We shall now study how $S(t)$ contributes to the energy balance in the mechanical system. In particular, we are interested in what the "radiation rate," i.e., the energy dissipation rate, turns out to be.

C. Role of the Schott term in the system energy balance

To derive a general energy balance equation for the mechanical system in Fig. 1, we start from the equation of motion in (2.1), put in $S = dp/dt$ from Eq. (2.13), and multiply both sides of the equation by the central particle velocity v. As a consequence

$$\frac{d}{dt}\left(\frac{1}{2}Mv^2\right) = \frac{dW}{dt} + v\frac{dp}{dt}, \tag{2.24}$$

where $dW/dt = Fv$ is the rate at which the external force F does work on the particle. Next, we write $v\dot{p} = d(vp)/dt - \dot{v}p$, and for $\dot{v}$ here we use Eq. (2.14): $\dot{v} = \dot{V} + \dot{p}/M$, where p is defined in Eq. (2.13). On rearranging terms, we find

$$\frac{dW}{dt} = \frac{d}{dt}K_{\rm CM} + \dot{V}p, \tag{2.25}$$

where $K_{\rm CM} = MV^2/2$ is the kinetic energy of the system CM. The term in $\dot{V}p$ in (2.25) is present because of the Schott correction; it will provide us with the system's dissipation rate, as well as an additional correction to $K_{\rm CM}$.

By use of Eq. (2.17) for $p(t)$, we can write the $\dot{V}p$ term of (2.25) in the following form, with $\sigma_n(\theta)$ defined in Eq. (2.18):

$$\dot{V}p = MT\left\{\dot{V}^2 + \sum_{n=1}^{\infty}\sigma_n(\theta)(-1/\omega)^n\left[\dot{V}\left(\frac{d}{dt}\right)^n\dot{V}\right]\right\}. \tag{2.26}$$

The first term on the right-hand side (rhs) is a Larmor-like dissipation rate. We write it as

$$\mathcal{R}(t) = MT\dot{V}(t)^2. \tag{2.27}$$

As for the second term on the rhs in (2.26), we use the identity

$$Q(d/dt)^nQ = \frac{d}{dt}\sum_{\lambda=0}^{n-1}(-)^\lambda\left[(d/dt)^\lambda Q\right]\left[(d/dt)^{n-1-\lambda}Q\right], \tag{2.28}$$

and—after a minor bit of algebra—we find that (2.26) becomes

$$\dot{V}p = \mathcal{R}(t) - \frac{d}{dt}\mathcal{E}(t),$$

where

$$\mathcal{E}(t) = (MT/\omega)\sum_{n=0}^{\infty}\sigma_{n+1}(\theta)\left[\sum_{\lambda=0}^{n}(-1)^\lambda\dot{V}^{(\lambda)}\dot{V}^{(n-\lambda)}\right]. \tag{2.29}$$

Here $Q^{(k)}$ denotes $(1/\omega)^k(-d/dt)^kQ$. In this way, the Schott energy rate correction $\dot{V}p$ splits into a dissipation rate $\mathcal{R}$ and an energy correction $\mathcal{E}$. The energy balance equation (2.25) is

$$\frac{dW}{dt} = \frac{d}{dt}\left[K_{\rm CM} - \mathcal{E}(t)\right] + \mathcal{R}(t). \tag{2.30}$$

This is the energy equation we will use in what follows.

That $\mathcal{R}(t)$ on the rhs of Eq. (2.30) is a true dissipation rate for the mechanical system that can be seen by first noting it is positive definite, and then integrating Eq. (2.30) over a time interval $t_1 \le t \le t_2$. This operation yields

$$W(t_2) - W(t_1) = \left[K_{\rm CM} - \mathcal{E}(t)\right]\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\mathcal{R}(t)\,dt. \tag{2.31}$$

In the absence of $\mathcal{R}$, the energy $(W - K_{\rm CM} + \mathcal{E})$ is conserved. As it is, a nonzero $\mathcal{R}$ produces an *irrecoverable* energy loss on the rhs of (2.31). As such, $\mathcal{R}$ must be the system's energy dissipation rate. Note that, with the time scale $T = 4\mu\beta/M\omega_0^2$ in (2.17), Eq. (2.27) specifies $\mathcal{R} = (4\mu\beta/\omega_0^2)\dot{V}^2$, so—in an accelerating system—$\mathcal{R}$ vanishes only when the "field" damping rate $\beta \to 0$ (or when the field itself vanishes, $\mu \to 0$).

It is worth remarking that the (LD) equation of motion for a point charge (q, m) leads to the same sort of partitioning as Eq. (2.30). The (LD) equation is

$$m\dot{v}^\kappa = (q/c)F^{\kappa\lambda}v_\lambda + m\tau_0\left[\ddot{v}^\kappa + \frac{1}{c^2}\left(\dot{v}^\alpha\dot{v}_\alpha\right)v^\kappa\right], \tag{2.32}$$

where v^κ is m's four velocity, the dot means differentiation with respect to m's proper time, $F^{\kappa\lambda}$ is an *external* field, τ_0 is the scale time in Eq. (1.1), indices κ run from 0 to 3, and the metric $\eta_{\kappa\lambda} = (1, -1, -1, -1)$ on the diagonal. The Schott term is apparent; together with the radiant term in $1/c^2$, it forms the radiation reaction for the accelerating charge. By isolating the timelike component, $\kappa = 0$, of (2.32) and converting to lab time t, it is a straightforward task to show

$$\frac{dW}{dt} = \frac{d}{dt}\left[K - \mathcal{E}(t)\right] + \mathcal{R}_L(t), \tag{2.33}$$

$dW/dt = q\boldsymbol{E}\cdot\boldsymbol{v}$ is the rate at which the external electric field $\boldsymbol{E}$ does work on the particle, $K = (\gamma - 1)mc^2$ is the particle's relativistic kinetic energy, $\mathcal{R}_L = -m\tau_0(\dot{v}^\alpha\dot{v}_\alpha)$ is the relativistic Larmor rate, and $\mathcal{E} = mc^2\tau_0\gamma(d\gamma/dt)$

is a Schott correction to the kinetic energy. We see that, in both the nonrelativistic (2.30) and relativistic (2.33) cases, the radiation term in the equation of motion results in a kinetic energy correction as well as a rate of energy dissipation.

In deriving the energy balance in Eq. (2.30), we could have focused on the Schott reaction force $S(t)$ alone, calculating the rate at which it does work on the central particle. It is instructive to do this, in order to achieve a different view of how the energy partitioning occurs in (2.30). The Schott rate of work is

$$vS(t) = v\dot{p} = \frac{d}{dt}(vp) - \dot{v}p. \tag{2.34}$$

Use: $\dot{v} = \dot{V} + \dot{p}/M$, which is the equation of motion [Eq. (2.1)], incorporating $p(t)$ of Eq. (2.13). Then use Eq. (2.29) for $\dot{V}p$. The result is

$$vS(t) = \frac{d}{dt}\left[K - (K_{\rm CM} - \mathcal{E})\right] - \mathcal{R}(t), \tag{2.35}$$

where $K = Mv^2/2$, $K_{\rm CM} = MV^2/2$, $\mathcal{E}$ is defined in (2.29), and $\mathcal{R}$ appears in (2.27). When (2.35) is put into (2.24), we get the same energy balance as we have achieved in Eq. (2.30). From this, we see that the Schott reaction force not only generates the energy dissipation rate $\mathcal{R}$ and kinetic energy correction $\mathcal{E}$, but it also shifts the kinetic energy from the particle coordinates (v) to CM coordinates (V).

We close this section by looking at how the mechanical system's energy dissipation rate, $\mathcal{R}$ in Eq. (2.27), *differs* from the Larmor radiation rate $\mathcal{R}_L$ in Eq. (1.2). The particle masses cancel, and we are comparing the mechanical rate (for one-dimensional (1D) motion)

$$\mathcal{R}(t) = \gamma\dot{V}^2,\ \gamma = 4\beta\mu^2/k, \tag{2.36a}$$

with the electromagnetic rate

$$\mathcal{R}_L(t) = \gamma_L\dot{v}^2,\ \gamma_L = 2q^2/3c^3. \tag{2.36b}$$

Apart from the different physical constants appearing in the coefficients γ and γ_L, the salient difference between the rates in Eqs. (2.36) is that $\mathcal{R}$ depends on the acceleration of the CM, $\dot{V}$, while $\mathcal{R}_L$ depends on the particle's acceleration $\dot{\boldsymbol{v}}$. For the mechanical system, the velocities $\boldsymbol{v}$ and V are related by Eq. (2.14). By use of the expansion in (2.17), the coefficients σ_n in (2.18), and the derivatives $Q^{(n)} = (-1/\omega)^n(d/dt)^nQ$, we can write

$$v(t) = V(t) + \sum_{n=0}^{\infty}\sigma_n(\theta)\dot{V}^{(n)}. \tag{2.37}$$

This series can be inverted to give V in terms of v and its derivatives. To first order in ωT, we find

$$V(t) = v(t) - T\sum_{n=0}^{\infty}\sigma_n(\theta)\dot{v}^{(n)}, \tag{2.38}$$

from which we can calculate (still to first order in ωT)

$$\dot{V}^2 = \left(1 - T\frac{d}{dt}\right)\dot{v}^2 - 2T\sum_{n=1}^{\infty}\sigma_n(\theta)\dot{v}\dot{v}^{(n)}. \tag{2.39}$$

The series remaining here can be reduced by use of (2.28), but we shall ignore it and retain just the first two terms on the rhs of (2.39). These are enough to differentiate between $\mathcal{R}$ and $\mathcal{R}_L$.

Suppose we replace the standard Larmor rate $\mathcal{R}_L$ in (2.36b) by its mechanical equivalent, i.e., we *define* the electromagnetic radiation rate as

$$\mathcal{R}(t) = \gamma_L\dot{V}^2 \simeq \left(1 - \tau_0\frac{d}{dt}\right)\mathcal{R}_L(t). \tag{2.40}$$

This result follows from (2.39), by neglecting derivatives of v that are third and higher order, and by setting the time scales equal: $T = \tau_0$. The term in τ_0 in Eq. (2.40), present because of the difference in the mechanical and electromagnetic rates in Eqs. (2.36), tends to damp any sudden changes in the standard Larmor rate $\mathcal{R}_L$. For example, suppose the particle is subjected to an accelerative pulse of duration Δt, of the form

$$\dot{v} = a\exp\left[-\left(\frac{t}{2\Delta t}\right)^2\right], \tag{2.41}$$

where a is a constant. Then the rate in Eq. (2.40) is

$$\mathcal{R}(t) \simeq \left[1 + \frac{\tau_0}{\Delta t}(t/\Delta t)\right]\mathcal{R}_L(t). \tag{2.42}$$

For $t < 0$, when $\mathcal{R}_L$ is increasing, the modified rate $\mathcal{R}$ is depleted; for $t > 0$, when $\mathcal{R}_L$ is decreasing, $\mathcal{R}$ is augmented. The effect is pronounced when the acceleration time Δt is comparable to the (electromagnetic) scale time τ_0.

III. THE MECHANICAL SYSTEM AND ELECTROMAGNETIC ANALOGIES

A. Plausible corrections to the electromagnetic Schott term

In Sec. I we have suggested analogies between the Schott terms that appear for our mechanical system and for the electromagnetic problem of an accelerated charge, and in Sec. II we have analyzed some dynamical details that support these analogies. The formal similarity between the two problems rests on a comparison of the reaction force and dissipation rate for the mechanical system, viz.

$$S(t) = MT\ddot{V} + \cdots,\ \mathcal{R}(t) = MT\dot{V}^2, \tag{3.1}$$

[from Eqs. (2.20) and (2.27)], with the same quantities for the electromagnetic problem [Eqs. (1.1) and (1.2)]

$$s(t) = m\tau_0\ddot{v} + \cdots,\ \mathcal{R}_L(t) = m\tau_0\dot{v}^2. \tag{3.2}$$

The masses M and m in Eqs. (3.1) and (3.2) are analogous in that both are supposed to consist of a "bare" mass plus a contribution from the attendant field. The time scales $T = 4\beta/M\omega_0^2$ for the mechanical system [Eq. (2.17)], and $\tau_0 = 2q^2/3mc^3$ for the electromagnetic problem [Eq. (1.1)], depend on very different physical constants, but they play similar roles in the dynamics of the two systems. For example, for simple harmonic motion at natural frequency ω_0, the damping rate in the mechanical system is $\Gamma = \omega_0^2 T$, per Eq. (2.23), while the electromagnetic damping rate is $\omega_0^2\tau_0$. If the masses and time scales in Eqs. (3.1) and (3.2) are similar in these ways, what remains is the major *difference* that the mechanical quantities depend on the system *center-of-mass* (CM) velocity V, while the electromagnetic quantities depend on the *particle* velocity v.

In passing, we note that $S(t)$ in Eq. (3.1) is just the leading term in a series [see Eq. (2.18)] whose terms do not vanish unless the mechanical counterpart of the entire "field" is turned off, i.e., unless the masses $\mu \to 0$, and/or spring constant $k \to \infty$. In Eq. (3.2), $s(t)$ is the leading term in a series whose higher order terms vanish when the size of the charged particle shrinks to zero; the field need *not* be turned off.

In the mechanical system, there is clearly a difference between the system CM velocity V, and the central particle velocity, v—see Eq. (2.38). A natural question for the electromagnetic problem is, can we make a similar distinction between a CM velocity and a particle velocity? If so, the electromagnetic quantities in Eq. (3.2) might be corrected to

$$s(t) \to m\tau_0\ddot{V},\ \mathcal{R}_L(t) \to m\tau_0\dot{V}^2. \tag{3.3}$$

This correction scheme would have the desirable features listed in Sec. I, namely, that the corrected electromagnetic Schott term $s(t)$ would (1) be Newtonian with no dependence on superacceleration, (2) be causal, (3) avoid runaways, (4) provide standard solutions, and (5) give a Larmor-like radiation rate. Such a list of advantages suggests that we seriously consider the $v \to V$ correction for the electromagnetic problem.

In this regard, the following exercise is instructive. Consider a charged particle (q, m), at rest at the origin for $t < 0$. At time $t = 0$, the particle is given an impulse such that it travels at constant velocity v along the x axis for $t > 0$. The situation is pictured in Fig. 2. The sudden acceleration at $t = 0$ generates a radiation pulse which travels radially outward from the origin in a spherical shell of radius $R = ct$ and thickness $c\,\delta t$, where δt is the duration of the impulse. By this time, the "bare" mass m has traveled to position $x = vt$. Assuming the motion of (q, m) is nonrelativistic, we want to find the CM of this system. Let the charge q be a spherical shell of radius b, so that the rest mass of its field is $\mu_f = (1/c^2)q^2/2b$. Before the impulse, the total system mass is

$$M = m + \mu_f,\ \text{with}\ \mu_f = q^2/2c^2b. \tag{3.4}$$

At time $t > 0$, the radiation pulse and the static field beyond R remain centered at the origin, while the bare mass m and a *portion* of q's field have traveled to position

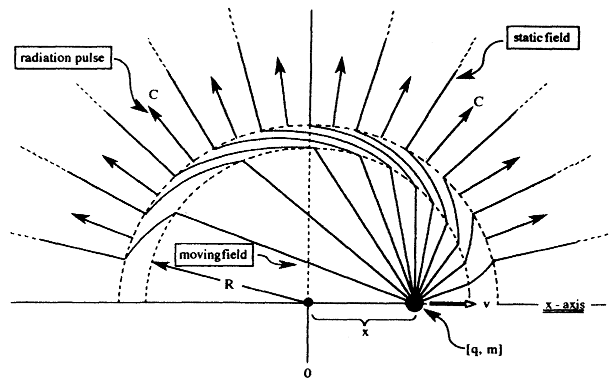

FIG. 2. A particle of charge q and (bare) mass m is initially at rest at the origin. At time $t = 0$, the particle is given an impulse such that at $t > 0$ it travels at constant velocity $\boldsymbol{v}$ along the x axis. The sudden acceleration at $t = 0$ generates a pulse of radiation that travels radially outward from the origin in a spherical shell of radius $R = ct$. The center of mass of the system (particle plus field) lies to the *left* of m's instantaneous position $x = vt$.

x. At a given time t, when $x = vt$ and $R = ct$, the system CM is at position

$$R_{\text{CM}} = (m + \mu_i)x/M, \tag{3.5}$$

where $\mu_i = \mu_f - (q^2/2c^2R)$ is the field mass inside the radiation shell. Evidently, after the impulse, we find the CM at position

$$R_{\text{CM}} = \left[1 - \left(q^2/2Mc^3\right)\frac{1}{t}\right]x, \tag{3.6}$$

where we have put in $R = ct$ for the radiation shell. This result is *independent* of the charge size b, and it clearly shows that the CM is not coincident with the position x of the bare mass m and its *bound* field μ_i. Now in (3.6), we put $x = vt$ and $R_{\text{CM}} = Vt$, where V is the CM velocity. In the second term on the rhs, we encounter a factor v/t, which at short times will be of order $\dot{v}$, the acceleration experienced by the charge at its initial impulse. Then Eq. (3.6) yields

$$V \simeq v - \left(q^2/2Mc^3\right)\dot{v}, \tag{3.7}$$

for the CM velocity V just at the end of the acceleration period. This result shows that the system CM velocity is *distinct* from the particle velocity v. The distinction *does* not depend on a finite charge size, but only on the fact that the total system mass $M = m + \mu_f$ is finite, even though m (particle) and μ_f (field) may be separately divergent.

Equation (3.7) not only suggests a distinction between the CM and particle velocities for the electromagnetic problem, but it also bears a strong resemblance to the velocity relationship established for the mechanical system. By Eq. (2.38), to leading terms, that relation is

$$V \simeq v - T\dot{v}, \tag{3.8}$$

where T is the mechanical time scale. Evidently, (3.7) and (3.8) become essentially indistinguishable if we identify T with the electromagnetic time scale $\tau_0 = 2q^2/3Mc^3$. This analogy has the interesting consequence that if, for the electromagnetic problem, we take the *particle* $v \to$ *system* V corrections seriously [as suggested in Eq. (3.3)], then the electromagnetic Schott term of Eq. (3.2) will be changed to a series

$$s(t) = M\tau_0\ddot{v} + M\tau_0 \sum_{n=1}^{\infty} \Omega_n (d/dt)^n \ddot{v}, \tag{3.9}$$

where the additional terms do *not* depend on charge structure, but only on the difference between the CM and particle velocities for the electromagnetic system. With appropriate choice of coefficients Ω_n in Eq. (3.9), the electromagnetic Schott term might be made to display the desirable features (1)–(5) listed below Eq. (3.3).

At this point, the analogies between the mechanical and electromagnetic system become less precise; in fact, as yet we have no obvious way of finding the coefficients Ω_n in terms of electromagnetic parameters. The mechanical model needs elaboration to more nearly match the electromagnetic problem; we shall discuss model improvements in Sec. III C. But first, having established the plausibility of a CM correction to the electromagnetic Schott term, we discuss some of the ramifications of such a correction, particularly as used by Eliezer.

B. Eliezer's substitution for radiation reaction

We have noted that the approximate equation of motion for the mechanical system can be written as in Eq. (2.21)

$$M\dot{v} = F + T\,(d/dt)\,F, \tag{3.10}$$

where M is the total system mass (particle plus field), v is the particle velocity, T is the system time scale, and $F = F(t)$ is the external force. The term in $T\dot{F}$ is the "radiation reaction" for the motion. The electromagnetic counterpart to (3.10) is Eq. (1.1), which we can state for 1D motion as

$$M\dot{v} = F + \tau_0\,(d/dt)\,M\dot{v}. \tag{3.11}$$

Here we have denoted the mass by M, to emphasize the fact that it is a *system* mass, presumably composed of both particle and field contributions. Now if (3.10) is postulated to be the correct form for the equation of motion for an accelerated charge, then the Abraham-Lorentz Eq. (3.11) is corrected by simply replacing the *particle* acceleration $\dot{v}$ in the radiation reaction term by the *system* acceleration F/M. To leading order, this substitution in the equation of motion is sufficient to accommodate the electromagnetic CM correction we have just discussed.

The $\dot{v} \to F/M$ substitution was first suggested by Eliezer [17]; he realized that the corrected electromagnetic Schott term would show the sort of desirable features (1)–(5) listed below Eq. (3.3). Furthermore, Eliezer generalized the correction to the relativistic case in the following way. The Lorentz-Dirac Eq. (2.32) is

$$M\dot{v}^\kappa = (q/c)\,F^{\kappa\lambda}v_\lambda + f^\kappa_{RR},$$

where

$$f^\kappa_{RR}(\text{Dirac}) = \tau_0 \left[\frac{d}{d\tau}\,(M\dot{v}^\kappa) + \frac{1}{c^2}\dot{v}^\alpha\,(M\dot{v}_\alpha)\,v^\kappa\right]. \tag{3.12}$$

f^κ_{RR} is the radiation reaction four force. Eliezer corrected it by substituting $\dot{v}^\kappa \to (q/Mc)F^{\kappa\lambda}v_\lambda$ for each of the terms containing M in the Dirac version of f^κ_{RR}, thus obtaining

$$f^\kappa_{RR}(\text{Eliezer}) = (q\tau_0/c)\left[\frac{d}{d\tau}\,(F^{\kappa\lambda}v_\lambda) + \frac{1}{c^2}\left(\dot{v}^\alpha F_{\alpha\lambda}v^\lambda\right)v^\kappa\right]. \tag{3.13}$$

With the radiation reaction force in (3.13), which is Minkowskian, Eliezer's resulting equation of motion is covariant, and it reduces to the form of Eq. (3.10) in the nonrelativistic limit. Runaway solutions are avoided immediately because the reaction force vanishes when the external field $F^{\kappa\lambda}$ is zero. The corrected equation of motion has several other desirable features, but it predicts at least one peculiar solution to a standard problem: the total cross section for the scattering of light by a charged particle turns out to be a universal constant, independent of the incident light frequency ν. The conventional result is that this cross section falls off at high frequencies as $1/\nu^2$ [19]. On these grounds, we might conclude that Eliezer's $\dot{v} \to F/M$ substitution has merit, but does not provide a complete cure for the ills of standard radiation reaction terms.

Why Eliezer's $\dot{v} \to F/M$ substitution may be incomplete is suggested by looking at the exact (nonrelativistic) equation of motion for the mechanical system. From Eqs. (2.1), (2.13), and (2.17), we have

$$M\dot{v} = F(t) + T\,(d/dt)\,\mathcal{F}(t), \tag{3.14}$$

where, in the reaction term, we have defined a force $\mathcal{F}$ such that

$$\mathcal{F}(t) = p(t)/T = (2\mu/mT)\int_0^\infty F(t-\tau)G(\tau)d\tau = \sum_{n=0}^{\infty}\sigma_n(\theta)\left(-\frac{1}{\omega}\frac{d}{dt}\right)^n F(t). \tag{3.15}$$

The coefficients $\sigma_n(\theta)$ are defined in Eq. (2.18), and the angle θ represents mechanical system parameters per Eq. (2.16). The leading terms of $\mathcal{F}$ are

$$\mathcal{F}(t) = F(t) + \left(\frac{\omega^2 - 4\beta^2}{2\beta\omega}\right)\frac{1}{\omega}\dot{F}(t) - \cdots. \tag{3.16}$$

Only the first term was used in (3.10) to discuss the approximate equation of motion. However, *all* the terms in

$\mathcal{F}$ are required to accurately represent the effects of a CM correction to the reaction force; replacing $\dot{v}$ by $F/M = \dot{V}$ is not enough. Comparing (3.11) with (3.14), instead of (3.10), suggests that Eliezer's substitution should have been

$$\dot{v} \to \mathcal{F}/M = \left(\frac{2\mu}{m}\right)\frac{1}{MT}\int_0^\infty F(t-\tau)G(\tau)d\tau = \sum_{n=0}^{\infty}\sigma_n(\theta)\left(-\frac{1}{\omega}\frac{d}{dt}\right)^n F(t)/M, \quad (3.17)$$

at least from the standpoint of the mechanical model. Thus, $\dot{v}$ in the reaction force is replaced by an infinite series of terms in F/M and all its derivatives.

The argument that the substitution $\dot{v} \to F/M$ in the reaction force is an incomplete correction for the *mechanical* system is persuasive, and all the details of the complete correction in Eq. (3.17) are in place—we know the kernel $G(\tau)$ from Eq. (2.11), and the series coefficients $\sigma_n(\theta)$ from Eq. (2.18). Our claim that a similar situation prevails for the *electromagnetic* problem is reasonable, but as yet we do not have the electromagnetic counterparts of $G(\tau)$ or the coefficients σ_n, except for $\sigma_0 = 1$. If the mechanical system is to serve as a guide for construction of the electromagnetic versions of $G(\tau)$ and the σ_n, then that system needs some elaboration. Although the equation of motion and dissipative corrections for the mechanical system and the electromagnetic problem are formally similar in many ways, it is not yet clear how to map the mechanical "field," as specified by the spring frequency $\omega_0 = (k/\mu)^{1/2}$ and damping constant β, onto the actual field of a charge q.

C. Elaboration of the mechanical model

While it is true that the overall dynamics of the present mechanical model are quite similar to those of an accelerated charge (at least similar enough to suggest corrections to electromagnetic Schott terms), it is also true that the extended, three-dimensional field of a charged particle cannot be completely modeled by a localized, one-dimensional spring and mass system. The question is, how can our mechanical model be improved?

We can relate the force constant k and damping constant β for the spring system in Fig. 1 with aspects of the field of a moving charge q, in the following heuristic fashion. Suppose q undergoes a simple harmonic motion at frequency ω_0. Then, in the way β was used in Eq. (2.2a), energy will be lost at a rate $\sim \beta\mu\dot{\xi}^2$, and this energy loss supplies the Larmor radiation rate $\sim (q^2/c^3)\ddot{\xi}^2$. If we set these rates equal, and—for an oscillation amplitude $\Delta\ell$—put in average values $\dot{\xi}^2 \sim (\omega_0\Delta\ell)^2$, and $\ddot{\xi}^2 \sim \left(\omega_0^2\Delta\ell\right)^2$, then

$$\beta \sim \left(q^2/\mu c^3\right)\omega_0^2, \quad (3.18)$$

to within numerical factors. The oscillation frequency in (3.18) is related to the spring constant k by $\omega_0^2 = k/\mu$. Suppose the spring has length ℓ, and consider the electromagnetic field energy stored in a spherical shell of thickness $\Delta\ell$ at distance ℓ from q

$$U(\ell, \ell+\Delta\ell) = \frac{q^2}{2}\left(\frac{1}{\ell} - \frac{1}{\ell+\Delta\ell}\right). \quad (3.19)$$

When q moves by $\Delta\ell$ in the direction of ℓ, this field energy will be replaced (at the site) by $U(\ell-\Delta\ell, \ell)$, roughly speaking, and so the local field energy changes by $\Delta U \simeq \left(q^2/\ell\right)\left(\Delta\ell/\ell\right)^2$. This energy is recovered when q moves in the opposite direction. If ΔU is identified with the spring's energy storage $\sim k(\Delta\ell)^2$, then

$$k \sim q^2/c^3 \text{ and } \beta \sim q^4/\mu^2c^3\ell^3, \quad (3.20)$$

to within numerical factors. These estimates imply that the simple mechanical system in Fig. 1 has a "field" that accurately represents the field of a charged particle over a limited region of space (specifically, a spherical shell of radius = spring length ℓ, and thickness $\Delta\ell \ll \ell$).

An alternate way of looking at the limitation just discussed is to say that springs at a single frequency $\omega_0 = (k/\mu)^{1/2}$ can represent only a limited portion of the frequency spectrum available to the field of an accelerated charge. An evident improvement in the mechanical model would then be to couple the central (bare) mass m in Fig. 1 to an *assembly* of springs of differing force constants k and attached masses μ. Thus, the spring natural frequency ω_0 could be varied, in order to better accommodate the field frequency spectrum. At the same time, the effective length $\ell^3 \sim q^2/k$ at which the spring system describes the actual field would be extended to a range of lengths.

Another improvement in the mechanical model would be to invest each spring with a nonzero mass density, so as to achieve a finite propagation speed connecting the motions of the central mass m with the "field" masses μ. This refinement would complicate the system equations of motion, but would also provide a velocity parameter that could be adjusted to fit energy transfer rates in the actual field of a charged particle.

Finally, a geometrical improvement in the mechanical model would be to couple the central (bare) mass m to a three-dimensional array of springs and field masses μ, rather than the one-dimensional configuration shown in Fig. 1. For springs of a given force constant k, the mass μ presumably would be distributed uniformly over a spherical shell of radius ℓ, such that $\ell^3 \sim q^2/k$, in order to preserve the relations in Eq. (3.20). The three-dimensional model could then be accelerated in any direction, while showing equivalent reaction forces. However, other than accommodating three-dimensional motion, the 1D to 3D transition for the mechanical system is not expected to change its basic operating parameters by more than numerical factors.

IV. SUMMARY AND CONCLUSIONS: CORRECTIONS TO SCHOTT TERMS

Our choice of the mechanical system we have analyzed in Sec. II, and have used in Sec. III to infer corrections to electromagnetic Schott terms, was made with two immediate goals in mind: first, to display a fully documented Schott correction in a system much simpler than that of an accelerated charge, and second, to model at least some key features of the accelerated charge problem. The system in Fig. 1 is very nearly the simplest model meeting these requirements: it produces an exact expression for the Schott corrections, and it provides mechanisms for both energy storage and dissipation during a forced motion—features shared by the fields of an accelerated charge. In Sec. II C, we have derived an energy balance equation for the mechanical system, and have identified the rate at which the system dissipates energy. We have shown that the work done by the Schott force $S(t)$ generates both the energy dissipation in the system, and a correction to the system's kinetic energy. This is analogous to the LD equation, where work done by the radiation reaction force splits into the Larmor radiation rate and a kinetic energy correction. Finally, we have compared our mechanical dissipation rate (defined by the *system* CM acceleration $\dot{V}$) with the electromagnetic dissipation rate (defined by the *particle* acceleration $\dot{v}$), and have found that these rates can be significantly different at early times. In Sec. III, we have discussed how the $\dot{v} \to \dot{V}$ correction might be used to modify the electromagnetic Schott term, and we have also discussed several improvements that could be made to the mechanical model of Fig. 1, to make it more closely resemble the problem of describing the behavior of an accelerated charge. The springs could be given a mass density, so as to establish an energy transport velocity in the mechanical system, and the spring array could be extended from one to three dimensions, to accommodate arbitrary motions of the central mass. A more important improvement would be to couple the central mass to an assembly of springs of different force constants and "field" masses; this would allow a more elaborate description of the field modes available to a charged particle. Even without these improvements, however, the formal analogies between the motions of the two systems can be supported by relating the available physical parameters.

In conclusion, we have presented arguments suggesting how and why the (nonrelativistic) Schott term in conventional electrodynamics should be corrected. The arguments rely on a plausible analogy with a simple mechanical model, and to leading order the corrections are of a form first suggested by Eliezer [17]. Possible higher order corrections await refinements of the model.

ACKNOWLEDGMENTS

The authors wish to thank the Montana State University physics department for its continuing support. We are also grateful to Dr. Lee Lindblom for commenting on this paper.

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