Ion-acoustic nonlinear periodic waves in a two-electron-temperature plasma

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We present a comprehensive study of nonlinear periodic waves, namely, the Korteweg–de Vries (KdV) and modified KdV (MKdV) cnoidal waves, and snoidal waves in a two-electron-temperature plasma. In the limiting case, these periodic waves reduce to bounded nonlinear structures, namely, KdV compressive and rarefactive solitons, MKdV compressive and rarefactive solitons, and double layers. The existence regions for these waves in the parameter space (μ_e,σ_e), where μ_e and σ_e are the density and temperature ratios of two electron species, are discussed in detail. The different nonlinear periodic waves and bounded structures have been explained in terms of physical parameters depicting the phase curves. It is found that the frequencies of the MKdV cnoidal and snoidal waves have different amplitude dependence behaviors than that of KdV cnoidal waves. The effect of other parameters on the characteristics of the nonlinear periodic waves are also discussed.

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I. INTRODUCTION

There has been considerable interest in studying the characteristics of nonlinear periodic waves in plasmas [1–12]. The cnoidal waves can be expected to play an important role in the nonlinear transport processes in plasma [3,4]. Nonlinear periodic waves like *dn* waves, where *dn* is the Jacobi elliptic function, are believed to be generated in the defocusing region of the ionospheric plasma [8]. Kauschke and Schlüter [11] have explained the appearance of single-mode drift wave spectra in their previous experiment [10] on the basis of cnoidal waves. Using the plasma model of Lakhin, Mikhailovskii, and Orischenko [13] for drift waves, Kauschke and Schlüter [11] found that the periodic signals observed at the plasma edge [10] can be well described by the cnoidal waves. It may also be noted that the cnoidal waves have been developed in the shallow water medium [14] and in a layered $LiNbO_3$-(SiO film) structure [15]. Nayanov has demonstrated the conversion of cnoidal Rayleigh waves into the solitons [15].

Using the kinetic description, Schamel [1] studied the small-amplitude snoidal ion-acoustic waves. Schamel [2] has also discussed the nonlinear periodic wave solutions for small-amplitude Langmuir waves. Lee and Kan [5] and Yashvir, Bhatnagar, and Sharma [6] have studied the ion-acoustic and ion-cyclotron nonlinear periodic waves in the low-β magnetized plasmas. The stability of the cnoidal waves in a magnetized plasma has been studied by Infeld [7] and Das, Sluijter, and Verheest [12]. In Ref. [12], it has been found that there is an instability of ion-acoustic cnoidal waves when the angle between the direction of perturbation and the external magnetic field exceeds a critical value. Using the reductive perturbation method, Ichikawa [3] and Konno, Mitsuhashi, and Ichikawa [4] have discussed the ion-acoustic cnoidal wave solution of the Korteweg–de Vries (KdV) equation with the positive coefficient of nonlinear term a for a plasma with single electron species, which in the limiting case (modulus $m \to 1$) reduces to a compressive soliton. Roychowdhury, Pakira, and Paul [9] have studied the ion-acoustic cnoidal waves in a weakly relativistic plasma with cold ions and two electron components. They inferred that a cnoidal wave does not exist when the coefficient of the nonlinear term of their KdV equation is negative. However, the present analysis shows that the KdV equation with a negative coefficient of the nonlinear term also gives rise to a cnoidal wave solution, which in the limiting case reduces to the rarefactive soliton solution.

The two-electron-temperature distributions are very common in the laboratory [16,17], as well as in space plasmas [18]. The nonlinear bounded structures, namely, the ion-acoustic compressive and rarefactive solitons and double layers in the two-electron-temperature plasma, have been investigated in detail, theoretically [19–26] as well as experimentally [16,17]. Ion-acoustic solitons and double layers have also been observed in the auroral and magnetospheric plasmas, where the two electron species exist [18,27]. In these observations, the periodic signals also appear frequently, which may be due to nonlinear periodic waves. The aim of this paper is to present a comprehensive study of different nonlinear periodic waves in a two-electron-temperature plasma, along with the limiting bounded nonlinear structures.

The plan of the paper is as follows. Section II is devoted to the basic equations of the system. In Sec. III, we derive the KdV equation. The cnoidal wave solution of the KdV equation is discussed in Sec. IV. The effect of different parameters on the characteristics of the cnoidal waves are discussed in detail. For some parameter values the coefficient of the nonlinear term of the KdV equation vanishes. At this critical curve, we consider the higher order nonlinearity and derive the modified Korteweg–de Vries (MKdV) equation in Sec. V. Its periodic wave solutions are discussed in Sec. VI. In Sec. VII, we summarize the main conclusions.

II. BASIC EQUATIONS

We consider a fully ionized, collisionless unmagnetized plasma with two electron species having densities n_{ec} and n_{eh} and temperatures T_c and T_h separately in thermal equilibrium. The dynamics of the plasma is described by the following fluid equations:

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nv)=0 , \tag{1}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{\partial \phi}{\partial x} , \tag{2}$$

$$\frac{\partial^2 \phi}{\partial x^2}=n_{ec}+n_{eh}-n , \tag{3}$$

where

$$n_{ec,h}=n_{c,h}\exp(\alpha_{c,h}\phi) \tag{4}$$

is the electron density for cold (hot) electron species, $\alpha_{c,h}=T_{\text{eff}}/T_{c,h}$, $T_{\text{eff}}=T_cT_h/(n_cT_h+n_hT_c)$, and T_c (T_h) is the temperature of the cold (hot) electron species, $n_{c,h}=n_{0c,h}/n_0$. In Eqs. (1)–(4), densities, velocity, space, and potential are normalized by the equilibrium electron plasma density n_0, ion-acoustic velocity $C_s=(T_{\text{eff}}/M)^{1/2}$, the Debye length $\lambda_{D\text{eff}}=(T_{\text{eff}}\epsilon_0/n_0e^2)^{1/2}$, and T_{eff}/e, respectively.

III. DERIVATION OF THE KdV EQUATION

To solve Eqs. (1)–(3), we introduce the stretched coordinates

$$\xi=\epsilon^{1/2}(x-st), \quad \tau=\epsilon^{3/2}t , \tag{5}$$

where ϵ is a small parameter and s is the phase velocity of the wave. We expand the variable quantities about the equilibrium values in powers of ϵ:

$$\begin{aligned} n&=1+\epsilon n^{(1)}+\epsilon^2 n^{(2)}+\epsilon^3 n^{(3)}\cdots ,\\ \phi&=\epsilon\phi^{(1)}+\epsilon^2\phi^{(2)}+\epsilon^3\phi^{(3)}\cdots ,\\ v&=\epsilon v^{(1)}+\epsilon^2 v^{(2)}+\epsilon^3 v^{(3)}\cdots . \end{aligned} \tag{6}$$

Using Eqs. (5) and (6) in Eqs. (1)–(3), we find that the first-order Poisson equation, the continuity equation, and the momentum equation give

$$n^{(1)}=\phi^{(1)} , \tag{7}$$

$$v^{(1)}=\phi^{(1)}+C^{(1)}(\tau) , \tag{8}$$

$$s=1 , \tag{9}$$

where $C^{(1)}(\tau)$ is an integration constant that may depend on τ.

Using first-order solutions, we find that the second-order equations give

$$n^{(2)}=\phi^{(2)}+\frac{\Delta}{2}(\phi^{(1)})^2-\frac{\partial^2\phi^{(1)}}{\partial\xi^2} , \tag{10}$$

$$v^{(2)}=\frac{1}{2}\left[2\phi^{(2)}+\left[\frac{\Delta}{2}-\frac{1}{2}\right](\phi^{(1)})^2-\frac{\partial^2\phi^{(1)}}{\partial\xi^2}+\frac{\partial C^{(1)}}{\partial\tau}\xi\right]+C^{(2)}(\tau) , \tag{11}$$

where

$$\Delta=n_c\alpha_c^2+n_h\alpha_h^2, \quad n_c=1/(\mu_e+1) ,$$
$$n_h=\mu_e n_c, \quad \alpha_h=(\mu_e+1)/(\mu_e+\sigma_e) ,$$
$$\alpha_c=\sigma_e\alpha_h, \quad \mu_e=n_h/n_c, \quad \sigma_e=T_h/T_c .$$

In Eq. (11), $C^{(2)}(\tau)$ is the second integration constant that may depend on τ. The periodic boundary condition implies that

$$\partial C^{(1)}/\partial\tau=0 . \tag{12}$$

Hence $C^{(1)}$ is independent of ξ and τ.

Using Eqs. (8), (11), and (12) in the second-order momentum equation, we obtain the KdV equation

$$\frac{\partial\phi^{(1)}}{\partial\tau}+a\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}+C^{(1)}\frac{\partial\phi^{(1)}}{\partial\xi}+\frac{1}{2}\frac{\partial^3\phi^{(1)}}{\partial\xi^3}=0 , \tag{13}$$

where

$$a=\tfrac{1}{2}(3-\Delta) . \tag{14}$$

For a plasma with single electron species, the KdV equation (13) becomes the same as that obtained by Konno, Mitsuhashi, and Ichikawa [4].

IV. THE CNOIDAL WAVE SOLUTION OF THE KdV EQUATION

For the steady state solution of the KdV equation (13), we consider $\eta=\xi-u\tau$, where u is a constant velocity.

Integrating twice with respect to η, we obtain the so-called energy equation

$$\frac{1}{2}\left[\frac{d\phi^{(1)}}{d\eta}\right]^2+V(\phi^{(1)})=0 , \tag{15}$$

where the Sagdeev potential $V(\phi^{(1)})$ is given by

$$V(\phi^{(1)})=\frac{a}{3}(\phi^{(1)})^3-(u-C^{(1)})(\phi^{(1)})^2+\rho_0\phi^{(1)}-\tfrac{1}{2}E_0^2 , \tag{16}$$

where ρ_0 and E_0 are, respectively, the charge density and electric field when $\phi^{(1)}$ vanishes. If α, β, and γ are the three real zeros of $V(\phi^{(1)})$, these are given by

$$\alpha=(s_1+s_2)+\frac{u_1}{a} , \tag{17}$$

$$\beta=-\tfrac{1}{2}(s_1+s_2)+\frac{u_1}{a}+\frac{\iota\sqrt{3}}{2}(s_1-s_2) , \tag{18}$$

$$\gamma=-\tfrac{1}{2}(s_1+s_2)+\frac{u_1}{a}-\frac{\iota\sqrt{3}}{2}(s_1-s_2) , \tag{19}$$

where $u_1=u-C^{(1)}$,

$$s_{1,2}=[r\pm(q^3+r^2)^{1/2}]^{1/3} ,$$

$$q = \frac{\rho_0}{a} - \frac{u_1^2}{a^2} ,$$

$$r = -\frac{3u_1\rho_0}{2a^2} + \frac{3E_0^2}{4a} + \frac{u_1^3}{a^3} ,$$

with $(q^3 + r^2) \leq 0$.

A cnoidal wave solution of Eq. (15) is given by

$$\phi^{(1)} = \beta + (\alpha - \beta)\text{cn}^2\{D\eta, m\} , \tag{20}$$

where cn is the Jacobi elliptic function. The parameter m, called the modulus, is expressed in terms of α, β, and γ as

$$m^2 = \frac{\alpha - \beta}{\alpha - \gamma} , \tag{21}$$

and D is given by the relation

$$D = \left[\frac{a}{6}(\alpha - \gamma) \right]^{1/2} . \tag{22}$$

The α, β, and γ are such that $\alpha > \beta \geq \gamma$ for $a > 0$, and $\alpha < \beta \leq \gamma$ for $a < 0$.

The amplitude A of the cnoidal wave is given by Eq. (20),

$$A = \pm(\alpha - \beta) . \tag{23}$$

The upper (lower) sign in Eq. (23) corresponds to $a > 0$ (< 0). In the following text the same convention has also been used. The wavelength λ of the cnoidal wave is defined as

$$D\lambda = 2K(m) , \tag{24}$$

where $K(m)$ is the first kind of complete elliptic integral.

The α, β, and γ can also be expressed in terms of modulus m, the amplitude A of the cnoidal wave, by using the conservation condition of particle number density

$$\int_0^\lambda (n-1)d\eta = 0 . \tag{25}$$

To evaluate (25), we assume that $n^{(2)}$ and higher order terms are very much less than $n^{(1)}$.

Therefore, from Eqs. (7), (20), (21), (23), and (25), we obtain

$$\alpha = \pm\frac{A}{m^2}[1 - H(m)] , \tag{26}$$

$$\beta = \pm\frac{A}{m^2}[1 - H(m) - m^2] , \tag{27}$$

$$\gamma = \mp\frac{A}{m^2}H(m) , \tag{28}$$

where $H(m) = E(m)/K(m)$ is the ratio of the complete elliptic integral of the second kind $E(m)$ to that of the first kind $K(m)$. With Eqs. (17)–(19) and (26)–(28), the amplitude of the cnoidal wave can be represented in terms of modulus m as

$$A = \pm 3(u - C^{(1)}) \Big/ \left\{ a \left[\frac{1}{m^2}\{2 - 3H(m)\} - 1 \right] \right\} . \tag{29}$$

The frequency of the cnoidal wave, $\omega = 2\pi V/\lambda$, is given by Eqs. (22), (24), (26), and (28),

$$\omega = \frac{\pi V}{K(m)m} \left[\pm\frac{aA}{6} \right]^{1/2} , \tag{30}$$

where the velocity of the cnoidal wave $V = s + u$, from Eq. (29), is

$$V = 1 + C^{(1)} \pm \frac{aA}{3} \left[\frac{1}{m^2}\{2 - 3H(m)\} - 1 \right] . \tag{31}$$

Roychowdhury, Pakira, and Paul [9] have studied the ion-acoustic cnoidal waves in a weakly relativistic plasma with two electron components. They inferred that a cnoidal wave does not exist in a certain region of parameter space of density and temperature ratios of two electron species. It should be noted that for that region the coefficient of the nonlinear term of their KdV equation is negative. However, the present analysis shows that the KdV equation with a negative coefficient of the nonlinear term also gives rise to cnoidal wave solution, which in the limit $m \to 1$ is reduced to the rarefactive soliton solution.

For $m = 1$, $H(m) = 0$, and hence from Eqs. (26)–(28) $\beta = \gamma = 0$, $\alpha = \pm A$, and cn$\to$sech. This situation occurs when ρ_0 and E_0 vanishes. Therefore, for $m = 1$ (i.e., for vanishing ρ_0 and E_0), with (29) or (17) and (18), the cnoidal wave solution (20) is reduced to the soliton solution [19,21]

$$\phi^{(1)} = \phi_m^{(1)}\text{sech}^2(\delta^{-1}\eta) , \tag{32}$$

where the amplitude of the soliton $\phi_m^{(1)} = 3u/a$ and width of the soliton $\delta = (2/u)^{1/2}$. Here, positive (negative) a corresponds to the compressive (rarefactive) soliton. To obtain (32), we have used the condition that for the soliton the perturbation in different quantities vanishes at $\eta = \pm\infty$ and hence have put $C^{(1)} = 0$.

In Fig. 1 we have plotted $a = 0$ in the parameter space

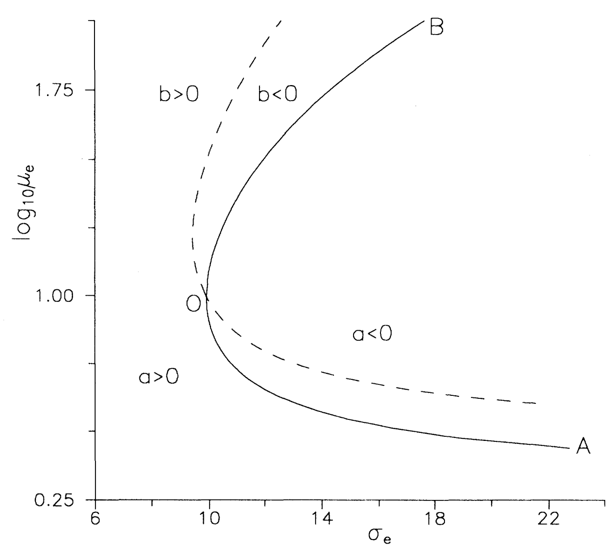

FIG. 1. Plots of $a = 0$ (——) and $b = 0$ (— — —) in the parameter space (μ_e, σ_e).

σ_e and μ_e. It is clear from Fig. 1 that the coefficient of the nonlinear term of the KdV equation a may become positive, zero, or negative, depending on the plasma parameters. From Fig. 1, we see that for σ_e less than a critical value σ_{ec} we always have $a>0$. For the parameter values for which $a>0$ or $a<0$, we have the cnoidal waves, which in the limiting case reduce to compressive and rarefactive solitons, respectively.

In Figs. 2 and 3, we have plotted phase curves using Eq. (15) with a choice of $C^{(1)}=0$, corresponding to $a>0$ and $a<0$, respectively. In Figs. 2 and 3 the values of μ_e and σ_e are chosen so that a is positive and negative, respectively. For $a>0$, when $E_0=\rho_0=0$, the phase curve starts from origin, circling clockwise around the positive ϕ axis, and again stops at origin, entering from the lower side. In the "particle in a potential well analogy," the pseudoparticle starts with zero velocity at $\eta=-\infty$ and gains some positive velocity; after attaining maximum velocity, it again goes to zero velocity. However, a "potential force" reflects the pseudoparticle back toward origin, and the latter gains velocity in the opposite direction and again comes to rest at $\eta=\infty$. In physical space, the value of the potential increases from zero (at $\eta=-\infty$) to a maximum value at $\eta=0$ and then starts decreasing until it vanishes (at $\eta=\infty$), and it represents a compressive soliton. For $a<0$, when $E_0=\rho_0=0$, the phase curve starts from origin, circling clockwise around the negative ϕ-axis, and again stops at origin, entering from the upper side. In physical space, the value of the potential decreases from zero (at $\eta=-\infty$) to a minimum value at $\eta=0$ and then starts increasing until it becomes zero (at $\eta=\infty$), and it represents a rarefactive soliton.

In phase curves Figs. 2 and 3, we see that when $E_0,\rho_0\neq 0$, the phase curves are repeated on the same path and one complete cycle corresponds to a wavelength in the physical space. In the mechanical analogy, whenever the pseudoparticle's velocity becomes zero (i.e., $d\phi/d\eta=0$), the "potential force" (since $-dV(\phi)/d\phi$ does not vanish) reflects it back, and therefore it oscillates between two points. In physical space, the potential oscillates periodically with space between two values. The potential of the cnoidal wave oscillates between two

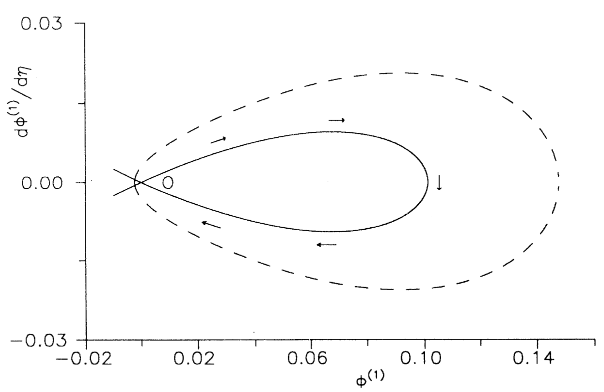

FIG. 2. Phase curve for Eq. (15), corresponding to $a>0$ ($\mu_e=0.5$ and $\sigma_e=4$, i.e., $a=0.888\,89$) with $u=0.03$ and $(|E_0|,\rho_0)=0.003,-0.002$ (— — —) and 0,0 (——).

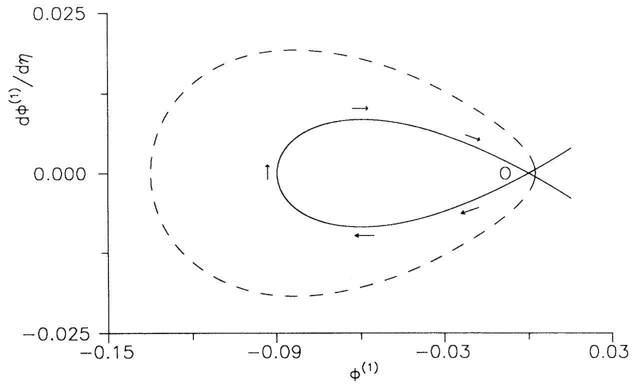

FIG. 3. Phase curve for Eq. (15), corresponding to $a<0$ ($\mu_e=10$ and $\sigma_e=20$; i.e., $a=-1.005\,56$) with $u=0.03$ and $(|E_0|,\rho_0)=0.003,0.002$ (— — —) and 0,0 (——).

upper (lower) values of real zeros of the Sagdeev potential, corresponding to $a>0$ (<0). It should be noted that when a becomes zero or of the order of ϵ, the KdV equation no longer remains valid and one should consider higher order nonlinearity to obtain the MKdV equation.

V. DERIVATION OF THE MKdV EQUATION

The KdV equation (13) is no longer valid when a becomes zero. a vanishes along a curve in the parameter space (σ_e,μ_e). To consider the ion-acoustic waves at this critical curve, higher order nonlinearity must be taken into account to obtain the MKdV equation. Hence we use the modified stretching

$$\xi=\epsilon(x-st),\quad \tau=\epsilon^3 t\ . \tag{33}$$

Using Eqs. (6) and (33) in Eqs. (1)–(3), we find that the first-order equations give the same solutions as in Eqs. (7)–(9).

Using first-order solutions, from the second-order continuity equation, momentum equation, and Poisson equation, we obtain

$$\{a\phi^{(1)}+C^{(1)}\}\frac{\partial\phi^{(1)}}{\partial\xi}=0\ . \tag{34}$$

Since $\phi^{(1)}$ is not constant with respect to ξ, this equation is identically satisfied only if

$$C^{(1)}=0 \quad\text{and}\quad a=0\ . \tag{35}$$

Using first-order solutions with $C^{(1)}=0$ and the second-order momentum equation, we have

$$v^{(2)}=\phi^{(2)}+\tfrac{1}{2}(\phi^{(1)})^2+C^{(3)}\ , \tag{36}$$

where $C^{(3)}$ is an integration constant. With first- and second-order solutions, the next higher order equations give the MKdV equation

$$\frac{\partial\phi^{(1)}}{\partial\tau}+b(\phi^{(1)})^2\frac{\partial\phi^{(1)}}{\partial\xi}+C^{(3)}\frac{\partial\phi^{(1)}}{\partial\xi}+\frac{1}{2}\frac{\partial^3\phi^{(1)}}{\partial\xi^3}=0\ , \tag{37}$$

where

$$b=\tfrac{1}{4}\{15-\delta_1\}\ ,$$
$$\delta_1=n_c\alpha_c^3+n_h\alpha_h^3\ . \tag{38}$$

VI. PERIODIC WAVE SOLUTIONS OF THE MKdV EQUATION

For the steady state solution of the MKdV equation (37), we consider $\eta=\xi-u\tau$, where u is a constant velocity. Integrating with respect to η, we obtain

$$-(u-C^{(3)})\phi^{(1)}+\frac{b}{3}(\phi^{(1)})^3+\frac{1}{2}\frac{d^2\phi^{(1)}}{d\eta^2}=C\ . \tag{39}$$

The MKdV equation is an even function of $\phi^{(1)}$ and is satisfied for both $+\phi^{(1)}$ and $-\phi^{(1)}$; therefore, constant C becomes zero. Integrating once again with respect to η, we obtain the Sagdeev potential energy equation

$$\frac{1}{2}\left[\frac{d\phi^{(1)}}{d\eta}\right]^2+V(\phi^{(1)})=0\ , \tag{40}$$

where the Sagdeev potential $V(\phi^{(1)})$ is given by

$$V(\phi^{(1)})=\frac{b}{6}(\phi^{(1)})^4-[u-C^{(3)}](\phi^{(1)})^2-\tfrac{1}{2}E_0^2\ . \tag{41}$$

A. Case (i): *b* is positive

Equation (40) can be written as

$$\frac{3}{b}\left[\frac{d\phi^{(1)}}{d\eta}\right]^2=f(\phi^{(1)})\ , \tag{42}$$

where

$$f(\phi^{(1)})=[(\phi^{(1)})^2-\alpha_2][\alpha_1-(\phi^{(1)})^2]\ ,$$

with

$$\alpha_{1,2}=\frac{3(u-C^{(3)})}{b}\pm\left[\frac{9(u-C^{(3)})^2}{b^2}+\frac{3E_0^2}{b}\right]^{1/2}\ . \tag{43}$$

The solution of Eq. (42) is given by

$$\phi^{(1)}=\sqrt{\alpha_1}\,\mathrm{cn}(D\eta,m)\ , \tag{44}$$

where

$$m^2=\frac{\alpha_1}{\alpha_1-\alpha_2},\quad D=\left[\frac{b(\alpha_1-\alpha_2)}{3}\right]^{1/2}\ , \tag{45}$$

with $\alpha_1>0$ and $\alpha_2\le 0$.

The integration constants α_1 and α_2 can also be represented in terms of modulus m and the amplitude A of the cnoidal wave

$$\alpha_1=\frac{A^2}{4},\quad \alpha_2=\frac{A^2}{4m^2}(m^2-1)\ . \tag{46}$$

Using Eqs. (43) and (46), we can express the amplitude in terms of modulus m as

$$A=\left\{\frac{24(u-C^{(3)})}{b\left[2-\frac{1}{m^2}\right]}\right\}^{1/2}\ . \tag{47}$$

The frequency ω of the cnoidal wave is given by Eq. (24), using Eqs. (45) and (46), as

$$\omega=\frac{\pi VA}{2mK(m)}\left[\frac{b}{3}\right]^{1/2}\ , \tag{48}$$

where the velocity of the cnoidal wave $V=s+u$, from Eq. (47), is

$$V=1+C^{(3)}+\frac{b\{2-(1/m^2)\}}{24}A^2\ . \tag{49}$$

For $m=1$, $\alpha_1=A^2/4$, $\alpha_2=0$, and cn$\rightarrow$sech. This situation occurs when E_0 vanishes. Therefore, for $m=1$ (i.e., for vanishing E_0), using (43) or (46) and (47), the cnoidal wave solution (44) is reduced to the MKdV soliton solution as

$$\phi^{(1)}=\phi_m^{(1)}\mathrm{sech}(\eta/\delta)\ , \tag{50}$$

where amplitude

$$\phi_m^{(1)}=\pm\left[\frac{6u}{b}\right]^{1/2}\ , \tag{51}$$

and width

$$\delta=\left[\frac{3}{b\alpha_1}\right]^{1/2}=\left[\frac{1}{2u}\right]^{1/2}\ . \tag{52}$$

To obtain Eq. (50), we have used the condition that for the soliton the perturbation in different quantities vanish at $\eta=\pm\infty$, and hence we have put $C^{(3)}=0$.

The plus and minus signs in Eq. (51) represent the compressive and rarefactive solitons, respectively. Equation (51) shows that compressive and rarefactive MKdV solitons can coexist in the two-electron-temperature plasma. It should be noted that the coexistence of compressive and rarefactive MKdV solitons has been experimentally verified in a plasma with negative ions [28]. No experiment has been reported so far on the MKdV solitons in a two-electron-temperature plasma. However, we expect that for a certain range of parameters, for which $a=0$ and b is positive, compressive and rarefactive MKdV solitons can coexist in a plasma with two electron species.

B. Case (ii): *b* is negative

In this case, Eq. (40) can be written as

$$\frac{3}{(-b)}\left[\frac{d\phi^{(1)}}{d\eta}\right]^2=[\alpha_3-(\phi^{(1)})^2][\alpha_4-(\phi^{(1)})^2]\ , \tag{53}$$

with

$$\alpha_{3,4}=\frac{3(u-C^{(3)})}{b}\mp\left[\frac{9(u-C^{(3)})^2}{b^2}+\frac{3E_0^2}{b}\right]^{1/2}\ . \tag{54}$$

The solution of Eq. (53) is given by

$$\phi^{(1)}=\sqrt{\alpha_3}\,\mathrm{sn}(D\eta,m)\ , \tag{55}$$

where

$$D=\left[-\frac{b\alpha_4}{3}\right]^{1/2}, \tag{56}$$

$$m^2=\frac{\alpha_3}{\alpha_4},\quad \text{with } \alpha_3\le\alpha_4\ . \tag{57}$$

The integration constants α_3 and α_4 can be expressed in terms of modulus m and the amplitude A of the snoidal wave

$$\alpha_3=\frac{A^2}{4},\quad \alpha_4=\frac{A^2}{4m^2}\ . \tag{58}$$

Using Eqs. (54) and (58), we can represent the amplitude A of the snoidal wave in terms of modulus m as

$$A=\left\{\frac{24(u-C^{(3)})}{b\left[1+\frac{1}{m^2}\right]}\right\}^{1/2}. \tag{59}$$

The frequency of the snoidal wave $\omega=2\pi V/\lambda$ is given by Eq. (24), using Eqs. (56) and (58), as

$$\omega=\frac{\pi VA}{2mK(m)}\left[-\frac{b}{3}\right]^{1/2}, \tag{60}$$

where the velocity of the snoidal wave $V=s+u$, from Eq. (59), is

$$V=1+C^{(3)}+\frac{b\{1+(1/m^2)\}}{24}A^2\ . \tag{61}$$

From Eq. (53), we see that, for the existence of a nonlinear structure, the term in square brackets in Eq. (54) should be positive. This implies that there is a critical value of $|E_0|$, $|E_0|_{\mathrm{cr}}$, above which no nonlinear structure is possible. The value of $|E_0|_{\mathrm{cr}}$ is given by

$$|E_0|_{\mathrm{cr}}=\left[-\frac{3(u-C^{(3)})^2}{b}\right]^{1/2}. \tag{62}$$

For the extremum values of E_0, we have $\alpha_3=\alpha_4$ (i.e., $m=1$), and we obtain the double layer solution. Therefore, for extremum values of E_0, using (54) or (58) and (59), the snoidal wave solution (55) is reduced to the double layer solution as

$$\phi^{(1)}=\pm\phi_m^{(1)}\tanh(\eta/\delta)\ , \tag{63}$$

where

$$\text{amplitude}=2\phi_m^{(1)}=2\left[\frac{3u}{b}\right]^{1/2} \tag{64}$$

and width $d=2\delta$ is given by

$$d=\left[-\frac{3}{b\alpha_4}\right]^{1/2}=\frac{2}{\phi_m^{(1)}}\left[-\frac{3}{b}\right]^{1/2}. \tag{65}$$

The double layer solution of the MKdV equation similar to Eq. (63) has been discussed by Torvén for a general state of electron distribution [29]. The plus (minus) sign in (63) corresponds to the double layer moving toward the high (low) potential side and corresponds to $E_0=-E_{0\mathrm{cr}}$ $(E_{0\mathrm{cr}})$.

From the above discussion, it is implied that the MKdV equation gives rise to a cnoidal (snoidal) wave solution corresponding to the positive (negative) coefficient of the cubic nonlinear term b, which in the limit $m\to 1$ reduces to the MKdV soliton (double layer) solution. The curve $b=0$ is shown in Fig. 1, in the parameter space (σ_e,μ_e). The curve $b=0$ intersects the curve $a=0$ at a point O. We see that for μ_e less than a critical value μ_{ec}, we always have $b>0$ corresponding to curve $a=0$. It is clear from Fig. 1 that the lower (upper) branch of the curve $a=0$ [i.e., AO (OB)] corresponds to positive (negative) b and hence corresponds to cnoidal (snoidal) waves.

In Figs. 4 and 5, we have plotted phase curves using Eq. (40) with a choice of $C^{(3)}=0$, corresponding to $b>0$ and $b<0$, respectively. In Figs. 4 and 5, the values of μ_e and σ_e are so chosen that $a=0$ and b is positive and negative, respectively. For $b>0$, when $E_0=0$, the phase curve shows two symmetric contours. For the right (left) side contour, the phase curve starts from origin, circling clockwise around the positive (negative) ϕ axis, and again stops at origin, entering from the lower side (upper side). In physical space, the right (left) side contour corresponds to the compressive (rarefactive) MKdV soliton and can be explained as before. For $|E_0|$ greater than zero, the phase curve is repeated and we have the MKdV cnoidal waves. Whenever the pseudoparticle's velocity becomes zero, it still "feels" a "restoring force" [since $-dV(\phi)/d\phi$ does not vanish] and therefore oscillates between two points, $\pm\sqrt{(\alpha_1)}$, and the phase curve is symmetric about both of the axes. In physical space, the potential of the MKdV cnoidal wave oscillates between two values $\pm\sqrt{(\alpha_1)}$.

For $b<0$, when $|E_0|=|E_0|_{\mathrm{cr}}$, in mechanical analogy, corresponding to the upper (lower) curve, the pseudopar-

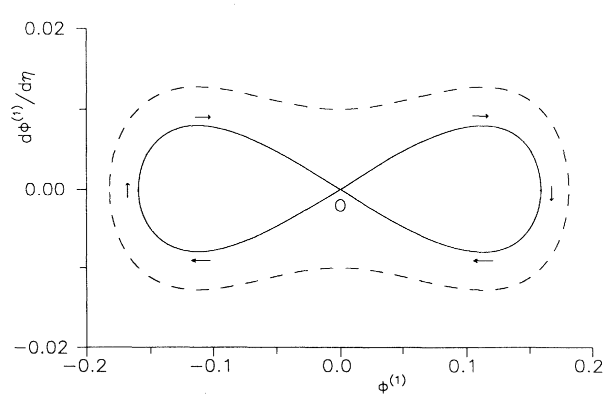

FIG. 4. Phase curve for Eq. (40), corresponding to $a=0$ and $b>0$ ($\mu_e=3$ and $\sigma_e=18.797\,96$; i.e., $a=0$ and $b=1.183\,504$) with $u=0.005$ and $|E_0|=0.01$ (— — —) and 0 (——).

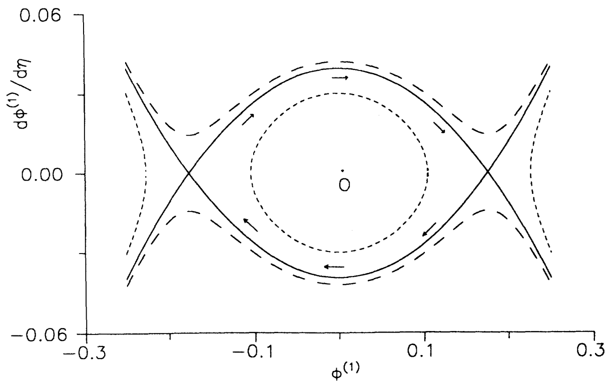

FIG. 5. Phase curve for Eq. (40), corresponding to $a=0$ and $b<0$ ($\mu_e=95$ and $\sigma_e=17.29321$; i.e., $a=0$ and $b=-4.819473$) with $u=-0.05$ and $|E_0|=0.03<|E_0|_{cr}$ (- - -), $0.0394485=|E_0|_{cr}$ (——), and $0.042>|E_0|_{cr}$ (— — —).

ticle starts from zero velocity in the positive (negative) direction, at $\eta=-\infty$; its velocity increases; it attains maximum value; then it starts to decrease; and the pseudoparticle comes to rest at $\eta=\infty$. It may be noted that for this case, the "potential" is such that there is no reflection of the pseudoparticle; i.e., the velocity of the pseudoparticle $d\phi/d\eta$ does not change sign. In physical space, the potential is monotonically increasing or decreasing, i.e., a double layer. Phase curves show that for the extremum value of E_0^2, the system can support two types of double layers. In physical space, the upper (lower) phase curve corresponds to the double layer moving toward the high (low) potential side of the double layer. Both types of ion-acoustic double layers moving toward the high-potential side as well as the low-potential side have been observed [30,31]. For $|E_0|$ less than the critical value, we have the nonlinear periodic waves, namely, the snoidal waves, and the phase curve can be explained as discussed before. The pseudoparticle oscillates between two points $\pm\sqrt{\alpha_3}$, and the phase curve is symmetric about both of the axes. In physical space, the potential of the snoidal wave oscillates between two values $\pm\sqrt{\alpha_3}$ and is antisymmetric with respect to η. For $|E_0|>|E_0|_{cr}$, the phase curve does not cross the ϕ axis and we do not have any nonlinear structure for $|E_0|>|E_0|_{cr}$.

From Eqs. (43) and (44), we see that as $|E_0|$ increases, the amplitude of the MKdV cnoidal wave increases. From Eqs. (43) and (44), we also note that the value of $|E_0|$ is, however, restricted to a limit such that the amplitude of the nonlinear wave is small enough so that the perturbation theory is applicable. From Eqs. (54) and (55), we see that as $|E_0|$ decreases, the amplitude of the snoidal wave decreases.

From Eqs. (30), (48), and (60), we see that the frequencies of the MKdV cnoidal and snoidal waves have different amplitude dependence behaviors than that of the KdV cnoidal waves. Although we have not applied the present theory to some specific situations, we think that this study may be useful to explain the periodic signals and bounded nonlinear structures, namely, compressive and rarefactive solitons and double layers, in a plasma, where two electron species exist; for example, in the auroral regions.

VII. CONCLUSIONS

Our main conclusions are as follows:

(i) We have presented a comprehensive study of nonlinear periodic waves, namely, the Korteweg–de Vries and modified KdV cnoidal waves, and snoidal waves in a two-electron-temperature plasma. In the limiting case, these periodic waves reduce to bounded nonlinear structures, namely, KdV compressive and rarefactive solitons, MKdV compressive and rarefactive solitons, and double layers.

(ii) The existence regions for these waves, in terms of plasma parameters, has been discussed in detail.

(iii) The frequencies of the different nonlinear periodic waves are functions of the amplitude. It is found that the frequencies of the MKdV cnoidal and snoidal waves have different amplitude dependence behaviors than that of KdV cnoidal waves.

(iv) It is found that corresponding to the MKdV equation with negative coefficient of cubic nonlinear term, there is a critical value of $|E_0|$, $|E_0|_{cr}$, above which no nonlinear structure is possible.

(v) The present theory is a more general one in the sense that it can explain the characteristics of nonlinear periodic waves, compressive and rarefactive KdV and MKdV solitons, and double layers in a two-electron-temperature plasma.

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