Growth of long-range correlations after a quench in phase-ordering systems

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(Received 28 September 1994; revised manuscript received 10 February 1995)

We present a general framework for the time-dependent correlation functions in a phase-ordering system after a quench from the disordered phase to or below the critical point and discuss under what conditions the two-time exponents λ or λ_c [characterizing the decay of local autocorrelations, $\langle \phi(\vec{r},0)\phi(\vec{r},t)\rangle \sim L^{-\lambda}$ or $\sim L^{-\lambda_c}$ for quenches to below T_c or to T_c, respectively, where $L(t)$ is the correlation length at time t, and ϕ is the order parameter] are equal to the spatial dimension d in the conserved order parameter case. We present a few cases where exact solutions and numerical simulations suggest $\lambda_c = d$. The same, however, is not true for the exponent λ. We present one example, namely a deterministic conserved model in one dimension, where λ is explicitly less than $d=1$. This led us to study the differences and similarities between stochastic and deterministic models of coarsening. In this paper, we address this general issue with a focus in one dimension, where the two classes of models are discussed in parallel and several analytical and numerical results are derived.

PACS number(s): 64.60.Cn, 64.60.My, 64.70.Md

INTRODUCTION

The kinetics of "phase separation" or "domain coarsening," after a system is rapidly quenched from its homogeneous phase into the multiphase region (two-phase for simple binary mixtures or alloys or Ising model) of its phase diagram, has a long standing history and is a subject of growing experimental and theoretical interest [1]. After the quench, the different broken symmetry phases of the system compete with each other to select the equilibrium state. However, final equilibrium is never achieved in the thermodynamic limit as the relaxation time diverges as a power of the system size. Instead, the system tries to achieve local equilibrium on larger and larger length scales. As a result, domains of equilibrium ordered phases form and grow with time. One important experimental observation is that this nonequilibrium coarsening process exhibits dynamic scaling at the late stages of growth. By that, one means that at late stages the system is characterized by a single length scale $L(t)$ (namely the characteristic linear size of the domains) which grows with time t typically in a power law fashion, $L(t)\sim t^n$ [although, as we will see later that in some deterministic one-dimensional (1D) models, the length scale grows logarithmically with time].

One of the quantities that can be measured by scattering experiments (light, x rays, neutrons, etc.) is the equal-time correlation function $G(r,t)=\langle \phi(\vec{r}',t)\phi(\vec{r}'+\vec{r},t)\rangle$ [where $\phi(\vec{r},t)$ is the order parameter field at time t] or rather its Fourier transform, i.e., the equal-time structure factor $S(\vec{k},t)=\langle \phi(\vec{k},t)\phi(-\vec{k},t)\rangle$. The angular brackets denote averages over random initial conditions and histories of evolution. The observed behavior of the structure factor suggests that at late times it satisfies a dynamical scaling law $S(\vec{k},t)\sim[L(t)]^d F(kL)$ [consequently, the real space equal-time correlation function $G(r,t)$ scales as, $G(r,t)\sim g(r/L(t))$]. Here d denotes the spatial dimension. For several years, the interest in this field had been limited primarily to the determination of the growth exponent n and the scaling function $g(x)$. Then it was pointed out by Fisher and Huse [2] that the two-time correlation function $C(r,t)=\langle \phi(\vec{r}',0)\phi(\vec{r}'+\vec{r},t)\rangle$ (where $t=0$ denotes the initial time right after the quench) also has interesting and nontrivial scaling properties. The function $C(r,t)$ is conjectured to have a scaling form, $C(r,t)\sim L^{-\lambda} f(r/L)$ [and, therefore, its Fourier transform $\tilde{C}(k,t)=\langle \phi(-\vec{k},0)\phi(\vec{k},t)\rangle$, i.e., the two-time structure factor satisfies the scaling form, $\tilde{C}(k,t)\sim L^{d-\lambda}\tilde{f}(kL)$]. This also implies that the autocorrelation function $A(t)=\langle \phi(\vec{r},0)\phi(\vec{r},t)\rangle$ which measures the local correlation with the initial condition, decays as $L^{-\lambda}(t)$. The exponent λ is a new equilibrium exponent. This exponent has recently been measured experimentally for an Ising system using video microscopy [3] and may also be measured via the autocorrelations of the speckle pattern in coherent laser or x-ray scattering [4].

So far we have discussed quenches only into the ordered region of the phase diagram. However, one can also quench the system to the critical point (e.g., in the Ising model, quench to $T=T_c$ from temperatures above T_c). Unlike the quenches to below T_c, in this case, well-ordered domains do not form but the correlation length of the system grows with time as $\xi(t)\sim t^{1/z}$ where z is the usual dynamic exponent. The equal-time correlation function scales as $G_c(r,t)\sim[\xi(t)]^{2-d+\eta} g_c[r/\xi(t)]$. The autocorrelation $A_c(t)$ decays as $[\xi(t)]^{-\lambda_c}$ where λ_c is a "new" nonequilibrium critical exponent [5,6] in the sense

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that it cannot be simply related to the known static or dynamic critical exponents via some scaling relations.

It is now well established that the conservation laws satisfied by the dynamics of phase ordering play an important role in determining the values of the exponents (n,z,λ,λ_c) and the nature of the scaling functions. For scalar order parameters in dimensions $d \geq 2$, it has been argued on general physical grounds that the growth exponent n is $\frac{1}{2}$ for nonconserved [7] and $\frac{1}{3}$ for conserved [8,9] order parameter dynamics. In the past few years, the nonconserved dynamics (which is simpler than the conserved case) has been studied in detail by several authors [1] and a few analytical results have been obtained. A new direction has been the extension to the vector order parameter case for which several interesting results have been obtained by Bray [1] and others. Exact results for the correlation functions and the exponents are known only in a few cases. For example, the nonconserved zero-temperature Glauber dynamics in 1D (where $T_c=0$) can be solved exactly obtaining $n=\frac{1}{2}$ and $\lambda_c=1$ [10]. Another example is the $O(m)$ vector model (where m denotes the dimension of the vector order parameter) in the $m \to \infty$ limit where $n=\frac{1}{2}$, $\lambda=d/2$ [11] and $\lambda_c=(3d-4)/2$ (for $2<d\leq 4$) [6] in the nonconserved case. For the 2D nonconserved stochastic Ising model, Fisher and Huse argued that $\lambda=\frac{5}{4}$ [2], which has been verified by numerical simulations [2] and recently in an experiment by Mason, Pargellis, and Yurke [3] using video microscopy. Fisher and Huse [2] also proposed bounds on λ, namely, $d/2 \leq \lambda \leq d$. The exponents λ and λ_c in dimensions $d \geq 2$ have been determined numerically for the nonconserved kinetic Ising model [5,12]. Apart from these, Mazenko and co-workers [13] have developed an approximate theory for the zero-temperature coarsening which yields nontrivial values of λ which are in good agreement with simulations only for the nonconserved case. Recently, this approximation for two-phase systems has been extended to the coarsening of multiphase systems such as the q-state Potts model and the q dependence of $\lambda(q)$ has been calculated within this approximation [14].

In comparison, relatively less success has been achieved in the conserved order parameter case [1]. The Lifshitz-Slyozov theory [9] predicted the growth exponent $n=\frac{1}{3}$ in the limit of zero volume fraction of the minority phase and also enabled exact calculation of the distribution of the minority domain sizes in that limit. Later, the growth exponent n was established to be $\frac{1}{3}$ for all volume fractions [8]. Apart from these, Tokuyama and Enomoto [15] have recently done a systematic expansion in the low volume fraction of the distribution of minority domain sizes and the equal-time structure factor. However, not much is known about the two-time exponents $[\lambda,\lambda_c]$ or the two-time correlation function. For the conserved $O(m)$ model in the $m \to \infty$ limit, Bray has shown that $\lambda_c=d$ [16]. Numerical simulation of the 2D Ising model with nearest neighbor spin-exchange dynamics (Kawasaki dynamics) has yielded $\lambda_c \approx 2.0=d$ [17]. In a recent paper, we have shown analytically and numerically [18] that $\lambda_c=1$ for a 1D kinetic Ising model with Kawasaki dynamics in the zero-temperature ($T=T_c=0$) limit. These three results seemed to suggest that the exponents λ and λ_c might actually be equal to the spatial dimension d (the upper limit of the Fisher-Huse bound) for the conserved case. We were then able to make a general argument that this result, $\lambda=\lambda_c=d$, was indeed true in any dimension d provided one assumes that the $\vec{k}=\vec{0}$ mode of the two point correlation function scales (we will illustrate this point in detail in Sec. I). However, there may be cases where this assumption may not hold. In fact, all three examples cited above where one can verify this result directly, are quenches to the critical point T_c (for $d=1$, $T_c=0$) where the thermal noise plays an important role. This is in contrast to quenches below T_c where the thermal noise has been argued to be irrelevant [1] in the renormalization group sense and the exponents and scaling functions are characterized by the zero-temperature fixed point. This fact leads to the speculation that maybe the assumption that the $\vec{k}=\vec{0}$ mode scales is valid only for quenches to T_c leading to the general result $\lambda_c=d$ but may break down for quenches to below T_c and the exponent λ may not be equal to d in general. Thus, it would be interesting to find such a counterexample where λ is indeed less than d. In this paper, we present one such example, namely the coarsening in a 1D deterministic conserved Ising model and show that λ is indeed less than one.

Theoretically, there have mainly been two approaches when studying the coarsening process. The first approach is the study of discrete spin models such as the stochastic Ising model. One usually studies the spin flip Metropolis or Glauber dynamics for the nonconserved case and the spin-exchange Kawasaki dynamics for the conserved case. The second approach is to study the overdamped Ginzburg-Landau (GL) or Langevin equations describing the relaxational dynamics of a continuum coarse grained order parameter field. These are called model A (nonconserved case) and model B (conserved case) in the language of Hohenberg and Halperin [19]. These equations, in general, have a noise term corresponding to thermal fluctuations. However, since thermal noise is irrelevant for quenches to below T_c, the noise term can be dropped from the equations (essentially one can assume $T=0$). In that case, the GL equations are completely deterministic, the only randomness being in the initial conditions. This is very different from the kinetic Ising models, which are stochastic by definition (even at $T=0$) and the average of any thermodynamic variable is not only over initial conditions (which is the case for the $T=0$ GL equations) but also over the histories of evolution. The deterministic GL equations are studied either by numerically integrating them (in which case the results are typically very noisy and not very clean) or by making approximations that may or may not be valid in specific cases. It is generally believed that in dimensions $d \geq 2$ and in the long time limit, the stochastic and deterministic models belong to the same universality class. However, in $d=1$ at zero temperature they are known to give very different answers. In the deterministic 1D GL model, the kinks or domain walls in-

teract with each other via exponential forces leading to the logarithmic growth law [20] for domains at late times. In the stochastic 1D models, on the other hand, where the kinks diffuse and annihilate on contact in the nonconserved case [10] and the domains themselves perform random walks and like domains (domains with the same spin) coalesce on contact in the conserved case [18], the growth laws are, respectively, $t^{1/2}$ and $t^{1/3}$ as in higher dimensions. We take the view that the 1D stochastic models are analogous to $T=T_c>0$ in higher d, while the deterministic 1D models correspond to $T=0$ in higher d.

In order to verify whether λ is equal to or less than d for quenches to below T_c, one could, in principle, integrate the GL equations numerically. However, as mentioned above, the results are typically rather noisy and it is hard to conclude beyond doubt whether $\lambda=d$ or $\lambda<d$. Similarly, the approximate theories [13], that are moderately successful in the case of model A, are known to fail for model B. Under these circumstances, an ideal choice seems to be the 1D case where, although one cannot directly solve the GL equations exactly, it is possible to map the late time evolution (as we show below) into that of a deterministic hard spin model which can be easily simulated yielding very clean numerical results and in some limits it is even possible to extract some exact results.

Thus, apart from addressing the question as to under what conditions the two-time exponents λ or λ_c are equal to d, this raises the interesting general issue regarding the differences and similarities between stochastic and deterministic models of coarsening. In this paper, we address this general issue with a focus in one dimension and discuss the two classes of models in parallel. The spirit of the paper is essentially twofold: the first is to build a general framework for the correlation functions and to derive new analytical and numerical results for stochastic and deterministic models in 1D and the second is to cast the existing known results in line with the general framework.

The paper is organized as follows. In Sec. I, we discuss under what conditions the exponents λ or λ_c are equal to d for the conserved case. As an example where such conditions hold, we rederive the solution for the $O(m)$ model in the $m\to\infty$ limit for the conserved case at T_c and show explicitly that $\lambda_c=d$. In Sec. II A, we reconsider the nonconserved stochastic model in 1D, namely the zero-temperature Glauber model. In Sec. II B, we consider the corresponding nonconserved deterministic model and determine λ numerically. Section III is devoted to the conserved 1D models. In Sec. III A, we consider the stochastic case where the Kawasaki dynamics can be mapped onto a "domain" model in the zero-temperature limit with exponents $n=\frac{1}{3}$ and $\lambda_c=1$ exactly. In Sec. III B, we map the late time dynamics of the deterministic model B onto that of a hard spin model and show numerically that λ is less than $d=1$. In Sec. IV A, we consider the zero volume fraction limit of this conserved deterministic spin model. In this limit, this is equivalent to a deterministic version of an aggregation model $A+A\to A$ in 1D. In Sec. IV B, a stochastic version of this aggregation model is solved exactly. Finally, the paper is concluded with a brief summary and discussion of open issues.

I. GENERAL DISCUSSION

We start out by assuming that the system undergoing coarsening is described by a local order parameter field $\phi(\vec{r},t)$ (for example, it is the local spin for the Ising model) whose Fourier transform is denoted by $\phi(\vec{k},t)$. Note that the $\vec{k}=\vec{0}$ mode $\phi(\vec{0},t)$ is the total order parameter (total magnetization for Ising model) at time t. In both nonconserved and conserved cases, the equal-time structure factor $S(\vec{k},t)=\langle\phi(\vec{k},t)\phi(-\vec{k},t)\rangle$ satisfies scaling at late times,

$$S(\vec{k},t)\approx[L(t)]^d F[kL(t)]\ , \tag{1}$$

where $\approx$ means that we neglect the corrections to scaling at late times. Note that for quenches to T_c, d is replaced by $2-\eta$ in Eq. (1). In the nonconserved case, the peak of the structure factor occurs at $k=0$ and the width of the peak decays as $L^{-1}(t)$. The scaling function $F(x)$ is of order $O(1)$ at $x=0$ indicating that the variance of the total magnetization, $S(\vec{0},t)$ grows as $L^d(t)$. In the conserved case, on the other hand, the $\vec{k}=\vec{0}$ mode $\phi(\vec{0},t)$ is a constant of motion. Clearly then, the scaling function $F(x)$ cannot be nonzero at $x=0$. Because if so, the right-hand side of Eq. (1) for $k=0$ will grow as $L^d(t)$ while the left-hand side remains a constant. Hence, one gets $F(0)=0$ necessarily in the conserved case. However, $S(\vec{0},t)$ is, in general, a nonzero constant set by the initial conditions. How then can one satisfy Eq. (1)? The answer to this question lies in the corrections to the scaling term in Eq. (1) which we had neglected. With the leading order scaling corrections near $k=0$ included, the Eq. (1) reads

$$S(\vec{k},t)=[L(t)]^d F[kL(t)]+F_1[kL(t)]\ , \tag{2}$$

where the correction term $F_1(x)$ is a nonzero constant at $x=0$ that cancels $S(\vec{0},t)$ exactly on the left-hand side of Eq. (2). Note that for $\vec{k}\neq\vec{0}$, $F[kL(t)]$ is nonzero and then in the scaling limit when $L(t)\to\infty$ and $k\to 0$ keeping $kL(t)$ fixed, the correction term is negligible. Thus, for the conserved case, in order to incorporate the $\vec{k}=\vec{0}$ mode in the equal-time scaling, one should always consider a scaling form of Eq. (2) and not just Eq. (1). In fact, there can also be other corrections with positive powers of $L(t)$ that, however, also vanish at $k=0$, so the correction shown in Eq. (2) is, in general, the leading correction only at $k=0$.

Now let us consider the two-time structure function $\tilde{C}(\vec{k},t)=\langle\phi(-\vec{k},0)\phi(\vec{k},t)\rangle$. Once again, $\tilde{C}(\vec{0},t)=S(\vec{0},t)$ and is a constant of motion for the conserved case. By the same argument as in the case of the equal-time structure factor, the $\vec{k}=\vec{0}$ mode is special in the conserved order parameter case and one should include the leading correction to the scaling and, in general, one can write,

$$\tilde{C}(\vec{k},t)=[L(t)]^{d-\lambda} f[kL(t)]+[L(t)]^{\lambda_1} f_1[kL(t)]\ , \tag{3}$$

where $\lambda_1 < d-\lambda$ since the second term is a correction to the first term and is, therefore, of lower order in $L(t)$. However, unlike the equal-time case, two different situations are possible in this case. The first scenario is, as in the equal-time case, $f(0)=0$, $\lambda<d$, $\lambda_1=0$, and $f_1(0)=\tilde{C}(\vec{0},t)$ which is a nonzero constant. In this case, we cannot conclude anything about λ except that it is less than d. A second scenario is, however, possible in the two-time case but not in the equal-time case namely, $\lambda=d$, $f(0)$ is a nonzero constant equal to $\tilde{C}(\vec{0},t)$, and $\lambda_1<0$ so that the correction term is not important at long time for any $\vec{k}$. Thus in this case, the $\vec{k}=\vec{0}$ mode is not special for the two-time structure factor and is included in the single scaling function $f(kL)$ and we say that the $\vec{k}=\vec{0}$ mode also scales. Note that the same argument holds for quenches to the critical point as well and hence for λ_c. In fact, in the three special cases quoted in the Introduction, this latter scenario is valid [18]. However, all these examples are for quenches to T_c. In Sec. III B, we will present an example where the first scenario holds and λ is clearly less than d. In the rest of this section, we will explicitly work out one case where the second scenario holds, namely, the conserved $O(m)$ model in the $m\to\infty$ limit at $T=T_c$. This model has already been solved [11,16] and we will not present any new result but we will derive the solution in a way that will illustrate the above general framework in the most explicit fashion.

The $O(m)$ model is described by an m component order parameter field $\vec{\phi}(\vec{r},t)=[\phi_1(\vec{r},t),\ldots,\phi_m(\vec{r},t)]$ and a coarse grained Landau-Ginzburg free energy functional

$$H(\vec{\phi})=\frac{1}{2}\int d^d\vec{r}\left[(\nabla\vec{\phi})^2+r_0\vec{\phi}^2+\frac{u}{2m}(\vec{\phi}^2)^2\right]. \tag{4}$$

The model-B equation describing the overdamped relaxation with a locally conserved order parameter is in general nonlinear and hard to solve. However, in the large m limit, this equation can be linearized in a self-consistent way and is, therefore, solvable in that limit [11]. In the Fourier space, this linearized equation reads,

$$\frac{\partial\phi(\vec{k},t)}{\partial t}=-k^2[k^2+r_0+uS_0(t)]\phi(\vec{k},t)+\eta(\vec{k},t), \tag{5}$$

where $\eta(\vec{k},t)$ is a Gaussian noise with zero average and a correlator $\langle\eta(\vec{k},t)\eta(\vec{k}',t')\rangle=2Tk^2\delta(\vec{k}+\vec{k}')\delta(t+t')$. Since the different components of $\vec{\phi}$ are completely uncorrelated in the large m limit, we have dropped the subscript and denote any single component by $\phi(\vec{k},t)$. The self-consistency demands that $S_0(t)$ is related to the equal-time structure factor $S(\vec{k},t)$ by

$$S_0(t)=\frac{1}{(2\pi)^d}\int_0^\Lambda S(\vec{k},t)d^d\vec{k}, \tag{6}$$

where Λ is the upper cutoff of the theory. Defining a new variable $k_m(t)$ via the relation, $r_0+uS_0(t)=-d/dt\{2k_m^2(t)t\}$, the two-time correlator $\tilde{C}(\vec{k},t)$ can be written as

$$\tilde{C}(\vec{k},t)=\tilde{C}(\vec{k},0)\exp[-Q(k,t)], \tag{7}$$

where $Q(k,t)=k^4t-2k^2k_m^2t$ and we have used the fact that $\langle\phi(-\vec{k},0)\eta(\vec{k},t)\rangle=0$ for $t\neq0$. Similarly, the structure factor $S(\vec{k},t)$ can be shown to consist of two parts, $S(\vec{k},t)=S_1(\vec{k},t)+S_2(\vec{k},t)$, where $S_1(\vec{k},t)=S(\vec{k},0)\exp[-2Q(k,t)]$ and $S_2(\vec{k},t)$ is given by

$$S_2(\vec{k},t)=2k^2T\exp\{-2Q(k,t)\}\int_0^t\exp[2Q(k,t')]dt'. \tag{8}$$

This second term comes directly from the noise term in Eq. (5) and is, therefore, important for quenches to T_c. For quenches to $T=0$ it vanishes. The variable $k_m(t)$ has to be determined from the self-consistency condition in Eq. (6). For convenience, we reparametrize the theory in terms of (m_0,u) instead of (r_0,u) where m_0 is the equilibrium magnetization and is simply related to r_0 by $r_0=-um_0^2-uT/(2\pi)^d\int_0^\Lambda k^{-2}d^d\vec{k}$. In terms of these new parameters, the self-consistency condition reads

$$\frac{d}{dt}[2k_m^2(t)t]=um_0^2-\frac{u}{(2\pi)^d}\int_0^\Lambda\left[S(\vec{k},t)-\frac{T}{k^2}\right]d^d\vec{k}, \tag{9}$$

where $S(\vec{k},t)$ has to be substituted. Note that the quench to T_c corresponds to putting $m_0=0$ in Eq. (9). In that case, a suitable ansatz for $k_m(t)$ at large t, would be $k_m^4(t)t=a+O(t^{-\alpha})$. As can be checked self-consistently, for $2<d<4$, a is nonzero and $\alpha=(4-d)/4$ and for $d>4$, a is zero and $\alpha=(d-4)/2$ for $4<d<6$ and $\alpha=1$ for $d>6$. Considering a random initial condition (corresponding to a high temperature phase), $S(\vec{k},0)=\tilde{C}(\vec{k},0)=\Delta$, which is a constant of $O(1)$ and writing $x=kt^{1/4}$, it is easy to check that $S_1(\vec{k},t)\approx\Delta\exp\{-2x^4+4\sqrt{a}x^2\}$ and

$$S_2(\vec{k},t)\approx\frac{2T_c}{k^2}\left\{x^4\int_0^1\exp[-2x^4(1-y)+4\sqrt{a}x^2(1-\sqrt{y})]dy\right\}, \tag{10}$$

in the scaling limit $t\to\infty$ and $k\to0$ but keeping x fixed. For $x\neq0$, the second term is going to dominate and the equal-time structure factor scales as $S(\vec{k},t)\approx k^{-2}F_c[k\xi(t)]$ where $\xi(t)\sim t^{1/4}$ indicating $z_c=4$ and the scaling function $F_c(x)$ is given by Eq. (10). However, note that the first term is precisely the correction to the scaling term (at $k=0$) as discussed in Eq. (2) and at $x=0$, $F_c(x)=0$ but $S_1(0)=\Delta$ on par with our general discussion at the beginning of this section. However, the two-time structure factor is given by $\tilde{C}(\vec{k},t)\approx\Delta\exp(-x^4+2\sqrt{a}x^2)$, clearly indicating that $\lambda_c=d$, $f_c(0)=\Delta$, and the correction to scaling is unimportant. This example, therefore, confirms the second scenario discussed earlier where the $k=0$ mode "scales" in the two-time structure factor but not in the equal-time structure factor.

To determine the constant a, we substitute the scaling form of $S(\vec{k},t)$ in Eq. (9). Then the leading term on the right-hand side decays as $t^{-(d-2)/4}$ which is slower than

$t^{-1/2}$ (for $2<d<4$) whereas the left-hand side decays as $t^{-1/2}$. Therefore, for consistency, the coefficient of $t^{-(d-2)/4}$ term on the right-hand side must vanish. This gives a condition

$$\int_0^\infty dx\, x^{d-2}\left\{1-2x^2\int_0^1 \exp[-2x^4(1-y) +4\sqrt{a}\,x^2(1-\sqrt{y})]dy\right\}=0\ , \quad (11)$$

which can be numerically solved to determine a for $2<d<4$. Thus, although the exponents are dimension independent, the scaling functions depend explicitly on the dimension. Note that for $d\geq 4$, the scaling limit is governed by the Gaussian fixed point, i.e., the self-consistent term (arising from the ϕ^4 interaction) becomes irrelevant and the only self-consistent ansatz for a is that it is identically 0, and to match the self-consistency condition in Eq. (9), $k_m(t)$ must vanish as $t^{-(d-2)/8}$ for $4<d<6$ and as $t^{-1/2}$ for $d>6$. In that case, the scaling functions $F_c(x)$ and $f_c(x)$ can be calculated trivially and the same conclusion $\lambda_c=d$ holds corresponding to the second situation of our general discussion.

Another case where one can check explicitly that the second picture holds is the 1D conserved stochastic model [18], which will be discussed in Sec. III A.

II. NONCONSERVED MODELS IN ONE DIMENSION

A. Stochastic models in 1D

A classic example of a stochastic nonconserved model that can be solved exactly in 1D is the zero temperature Glauber model. Several authors have studied this model and all the exponents and the correlation functions are known exactly [10]. A new method of solving this model was presented in Ref. [18] which separates out the deterministic and the stochastic part of the evolution in an explicit manner. We outline this approach below.

We consider an arbitrary Ising spin configuration $\{\phi(x,t);\ \phi(x,t)=\pm 1\}$ at time t on a 1D lattice. The Fourier transform $\phi(k,t)=(1/\sqrt{N})\sum_x \phi(x,t)\exp(ikx)$, where N denotes the number of lattice sites, can be written in a domain wall representation as

$$\phi(k,t)=\frac{1}{i\sqrt{N}\sin(k/2)}\sum_n (-1)^n \exp(ikx_n)\ , \quad (12)$$

where the domains are sequenced as $1,2,\ldots,n,\ldots$ and x_n denotes the location of the domain wall or kink separating the $(n-1)$th and the nth domain. In the zero-temperature Glauber model, these domain walls execute independent random walks and two walls annihilate on contact. The increment in $\phi(k,t)$ in a small time interval Δt can be written as

$$\Delta\phi(k,t)=\frac{1}{i\sqrt{N}\sin(k/2)}\sum_n (-1)^n \exp(ikx_n) \times[\exp(ik\zeta_n)-1]\ , \quad (13)$$

where the $\zeta_n(t)$ are the distances moved by the walls in Δt. They are independent random variables each taking values $+1$, -1, or 0 with probabilities $\frac{1}{2}\Delta t$, $\frac{1}{2}\Delta t$, and $1-\Delta t$, respectively. Taking the limit $\Delta t\to 0$, the equation of motion can then be written as

$$\frac{\partial\phi(k,t)}{\partial t}=-(1-\cos k)\phi(k,t)+\eta(k,t)\ , \quad (14)$$

where the first term on the right-hand side represents the purely deterministic part of the evolution and the second term $\eta(k,t)$ is purely stochastic. It is Gaussian noise with zero average and a correlator $\langle\eta(k',t')\eta(k,t)\rangle = 4\cos^2(k/2)\rho(t)\delta(k+k')\delta(t-t')$, where $\rho(t)=L^{-1}(t)$ is the average density of domain walls at time t. Using the fact, $\langle\phi(-k,0)\eta(k,t)\rangle=0$ for $t\neq 0$, it immediately follows that in the scaling limit ($k\to 0$, $t\to\infty$ but holding $x=kt^{1/2}$ fixed), the two-time correlator is given by $\tilde{C}(k,t)=\Delta\exp(-x^2/2)$ where as usual we have assumed a random initial condition, $\tilde{C}(k,0)=S(k,0)=\Delta$. This immediately gives $\lambda_c=1$. Similarly, the equal-time structure factor can be shown, as in the case of the $O(m)$ model in the $m\to\infty$ limit, to consist of two disjoint pieces, $S(k,t)=S_1(k,t)+S_2(k,t)$ where $S_1(k,t)=\Delta\exp(-x^2)$ and $S_2(k,t)=4\int_0^t \exp[-k^2(t-t')]\rho(t')dt'$. Using, $\rho(t)=L^{-1}(t)\approx t^{-1/2}$ in the scaling limit, we get

$$S_2(k,t)=t^{1/2}\left\{\frac{8}{x}\int_0^x \exp[-(x^2-y^2)]dy\right\}\ . \quad (15)$$

Thus, as in the $O(m)$ case, $S_1(k,t)$ represents the leading correction to scaling and $S_2(k,t)=L(t)F(x)$, with $F(x)$ given by Eq. (15), represents the scaling term. However, note that $F(0)=8$, a nonzero constant as opposed to the conserved case where $F(0)=0$. Thus, once again as in the case of the $O(m)$ model at T_c, the scaling function comes from the stochastic term and the leading correction from the deterministic piece. This is not surprising considering $T_c=0$ in 1D. Thus, in this sense, the zero-temperature Glauber model in 1D corresponds to "critical" coarsening.

B. Nonconserved GL equation in one dimension

In one dimension, the zero-temperature (and, therefore, deterministic) GL equation for nonconserved order parameter field $\{\phi(x,t)\}$ reads,

$$\frac{\partial\phi}{\partial t}=\frac{\partial^2\phi}{\partial x^2}-\frac{\partial V(\phi)}{\partial\phi}\ , \quad (16)$$

where $V(\phi)=(\phi^2-1)^2/2$ is a double well potential with degenerate minima at $\phi=\pm 1$ representing the two ordered phases. The equilibrium profile ($\partial\phi/\partial t=0$) of this equation is given in terms of a "kink" or "domain wall" solution, $\phi(x)=\tanh(x-x_0)$, where x_0 represents the position of the kink. Note that units are chosen so as to set the width of the domain wall to unity. At late times t, Eq. (16) can be mapped onto the deterministic equations of motions of a bunch of interacting kinks and the average length of domains grow logarithmically with time [20–22]. To see in a simple way how the kinks interact at late times leading to a logarithmic growth law, we first

consider, following examples in higher dimensions [1], a bubble of "down" phase in a sea of "up" phase (see Fig. 1). The two ends of the bubble x_0 and x_1 represent a pair of kinks. Clearly, this bubble is going to shrink in time. In higher dimensions, this shrinkage is due purely to surface tension. But in one dimension, the mechanism of shrinking lies in an attractive exponential interaction between the pair of kinks. To see how this comes about, an ansatz for the solution of Eq. (16) (which is appropriate at late times for well-separated kinks) would be,

$$\phi(x,t)\approx\tanh[x-x_1(t)]-\tanh[x-x_0(t)]+1 , \tag{17}$$

for $x_0(t)<x_1(t)$. Substituting this form in Eq. (16) and keeping only the leading order term in an expansion in $\exp[-(x_1-x_0)]$ (i.e., assuming the kink separation x_1-x_0 to be much larger compared to the width of a kink $[\sim O(1)]$ which is valid at late times), we get, $dx_1/dt\approx -a\exp\{-2(x_1-x_0)\}$ and $dx_0/dt \approx a\exp[-2(x_1-x_0)]$ where a is a numerical constant. Thus the length of the bubble, $R(t)=x_1(t)-x_0(t)$ shrinks at late times according to the law,

$$\frac{dR}{dt}\approx -\exp[-2R] , \tag{18}$$

where we have rescaled the time t by the numerical factor $2a$. Thus the bubble shrinks logarithmically with time due to the attractive exponential force between the two kinks. Equation (17) has to be compared to that in higher dimensions $d\geq 2$, where $dR/dt=-(d-1)/R$ and is purely due to surface tension.

It is then easy to generalize this solution when the number of domains present is large by superposing the kink solutions as above and the equation of motion of the nth kink is given by [20–22],

$$\frac{dx_n}{dt}\approx\exp[-2(x_{n+1}-x_n)]-\exp[-2(x_n-x_{n-1})] , \tag{19}$$

where we have neglected the interactions beyond nearest neighbors. When two nearby kinks come closer, the corresponding domain between the kinks annihilate and the two adjacent domains coalesce to form a larger domain. In the limit, when the typical separation between domain walls is much larger than the width of a wall (i.e., at late times), the shortest domains collapse instantly compared to the larger domains. The same situation arises when the kinks interact via a long-range force $r^{-\sigma}$ (where r denotes the separation between a pair of kinks) in the limit $\sigma\to\infty$ [23]. In either case, the late time dynamics reduces to that of a kinetic Ising model where at every time step (starting with a random initial configuration of up and down spins), the shortest domain present is found and the spins inside that domain are flipped and the process continues. Thus the system coarsens with the continual removal of shortest domains. This model was originally studied by Nagai and Kawasaki [20] assuming no correlations between the domains. However, Rutenberg and Bray [23] pointed out that this assumption is exact because when the shortest domain collapses and consequently a bigger domain is formed via the merging of three originally adjacent domains, the other domains in the system remain unaffected by this coarsening. Therefore, if in the initial configuration the domains are uncorrelated, no new correlations are generated dynamically. This model is also exactly identical to the "paste all" model studied by Derrida, Godreche, and Yekutieli [24] in the context of "breath figures." The lack of correlations between domains enables an exact calculation of the distribution of domain sizes. Let $n(l,t)dl$ denote the number of domains with lengths in the interval $[l,l+dl]$ and $\tilde{L}(t)$, the shortest length at time t. Then the total number of surviving domains at time t is $N(t)=\int_L^\infty n(l,t)dl$ and the fraction, $f(l,t)=n(l,t)/N(t)$ satisfies a scaling form, $f(l,t)\sim\tilde{L}^{-1}F(l/\tilde{L},t)$ [23,24]. Since the domains are uncorrelated, the evolution of $n(l,t)$ satisfies the equation [24],

$$\begin{aligned} n(l,t+dt)=n(l,t) &+n(\tilde{L},t)d\tilde{L}\Big[-2f(l,t) \\ &+\int_L^\infty dl' f(l',t)f(l-l'-\tilde{L},t) \\ &\times\theta(l-l'-2\tilde{L})\Big] , \end{aligned} \tag{20}$$

where $\theta(x)$ is, as usual, the unit step function. Substituting the scaling form for $f(l,t)$ and using the Laplace transform $\tilde{F}(p,t)=\int_1^\infty dx\,\exp(-px)F(x,t)$, one gets a fixed point distribution (i.e., $d\tilde{F}/d\tilde{L}=0$) of the form, $\tilde{F}(p)=\tanh[E_1(p)/2]$ [20,23], where $E_1(p)=\int_p^\infty dx\,\exp(-x)/x$.

From the knowledge of this fixed point distribution of the domain lengths, Nagai and Kawasaki could calculate the equal-time structure factor $S(k,t)=\langle\phi(-k,t)\phi(k,t)\rangle$ exactly [20]. However, the calculation of the two-time structure $\tilde{C}(k,t)=\langle\phi(-k,0)\phi(k,t)\rangle$ is nontrivial as it requires a knowledge of the history of evolution. However, it is easy to simulate this model to determine the decay of the autocorrelation function. The dynamics is very fast in units of $L(t)$ [where $L(t)$ denotes the average length of domains at time t] and it is possible to reach the late stage of evolution quite fast and obtain clean data. In Fig. 2(a), we have plotted $-\log_{10}[A(t)]$ versus $\log_{10}[L(t)]$ for a 1D lattice with 10^5 sites, where $A(t)=\langle\phi(\vec{r},0)\phi(\vec{r},t)\rangle$ is again the local autocorrelation. Although, this plot seems to be a fairly good straight line to the bare eye, the local slope has an upward curvature as $L(t)$ increases.

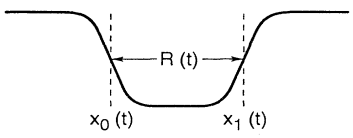

FIG. 1. A bubble of "down" phase in a sea of "up" phase. The two ends of the bubble x_0 and x_1 represent a pair of kinks.

To get a good estimate of λ, we, therefore, calculate the effective exponent λ_{eff} in the following way. The autocorrelation $A(L)$ [since the actual time t never appears in the autocorrelation, we denote $A(t)$ by $A(L)$] decays as $A(L) \approx aL^{-\lambda} + bL^{-\theta} + \cdots$ where the second term denotes the leading correction to scaling and $\cdots$ denotes higher order corrections. We fit the data with the choice $\theta = 1$ [25]. Thus we first define $A'(L) = A(L) - bL^{-1}$. We then define the effective exponent $\lambda_{\text{eff}} = -[1/\log_{10}(c)]\log_{10}[A'(cL)/A'(L)]$. In our simulation, we have chosen $c = 10^{0.2}$. Now the coefficient b is adjusted so that λ_{eff} is independent of L and, therefore, when plotted against L^{-1}, gives a flat line parallel to the L^{-1} axis. By extrapolating this line to $L^{-1} = 0$, one can read off the effective λ from the intercept of this flat line with the vertical axis. In Fig. 2(b), we have plotted λ_{eff} against L^{-1}. By choosing $b \approx = 1.819$, we find that the best fitting straight line is almost horizontal and then from the intercept to the vertical axis, we get $\lambda \approx 0.60 \pm 0.01$ [25]. This result has to be contrasted to the stochastic nonconserved Glauber model (which we call critical coarsening) where $\lambda_c = 1$ exactly. Thus, the difference between these two classes of models shows up not only in the growth laws and equal-time correlations but also in the two-time correlations.

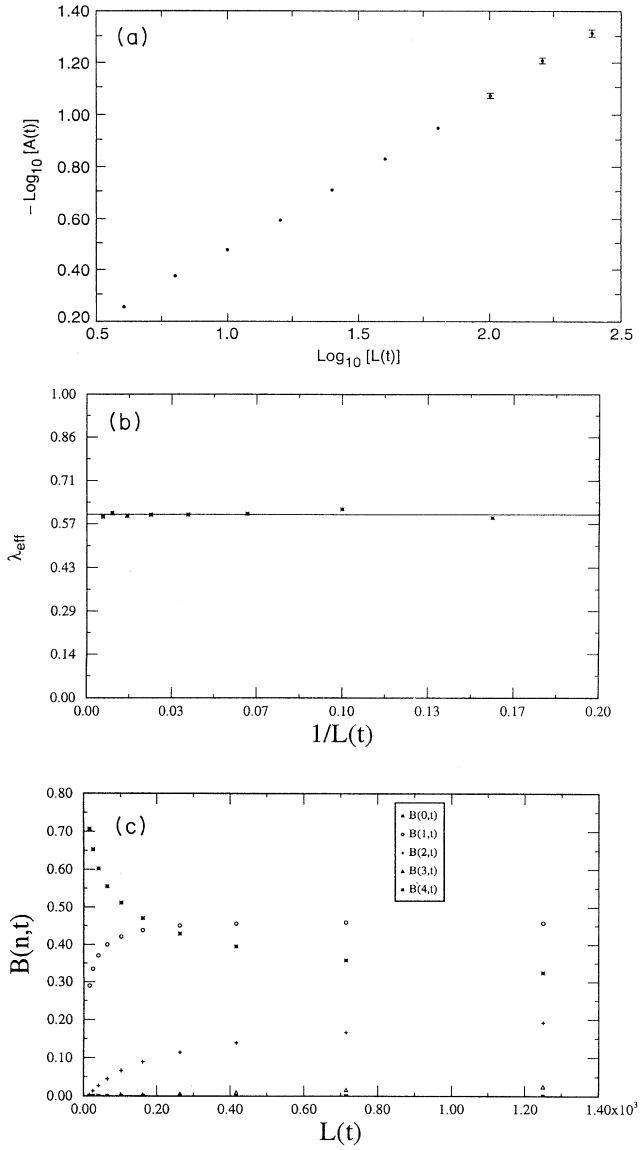

FIG. 2. (a) The autocorrelation function (for the deterministic model A in 1D), namely, $A(t) = \langle s(r,0)s(r,t) \rangle$ (averaged over 70 runs on samples of size 10^5 sites) as a function of $L(t)$ on a log-log plot. (b) The effective exponent $\lambda_{\text{eff}} = -[1/\log_{10}(c)]\log_{10}[A'(cL)/A'(L)]$ plotted against L^{-1} for the nonconserved deterministic model in 1D. Here $c = 10^{0.2}$ and $A'(L) = A(L) - bL^{-1}$. The constant b has been adjusted to be ≈ -1.819 so that the best fitting straight line is parallel to the L^{-1} axis, i.e., λ_{eff} is independent of L^{-1}. The intercept of this line with the vertical axis gives $\lambda \approx 0.60 \pm 0.01$. (c) The functions $B(n,t)$ for $n = 0,1,2,3,4$ against $L(t)$ for the nonconserved deterministic model in 1D (for a sample of 10 000 sites).

Recently, Bray, Derrida, and Godreche [26] have computed exactly another nontrivial nonequilibrium exponent of the deterministic model. Defining $B(0,t)$ as the fraction of spins that have not flipped at all up to time t ("dry part" as in Ref. [24]), they found that $B(0,t)$ decays at long times as $B(0,t) \sim [L(t)]^{-\beta}$ where $\beta \approx 0.174$ exactly for the 1D deterministic model discussed above and $\beta_c \approx 0.74$ numerically for the 1D Glauber model at zero temperature. The decay of $B(0,t)$ with different exponents shows further differences between stochastic and deterministic models in 1D. In fact, one can, in general, define $B(n,t)$ as the probability that a spin has flipped exactly n times up to time t. It then follows immediately that the autocorrelation $A(t)$ is given by, $A(t) = \sum_{n=0}^{\infty}(-1)^n B(n,t)$. Due to the cancellation between even and odd n, the autocorrelation $A(t)$ decays faster (with a larger exponent namely, $\lambda \approx 0.60$ for the deterministic case and $\lambda_c = 1$ for the stochastic case) than $B(0,t)$. The calculation of $B(0,t)$ in Ref. [24] for the deterministic case can be extended to calculate the asymptotic decay of $B(n,t)$ and we find that $B(n,t)$ for all n decay with the same exponent $\beta \approx 0.174$. However, this does not enable us to calculate the exponent λ exactly since to calculate the autocorrelation $A(t)$, we need to know $B(n,t)$ even at early times. For $n \geq 1$, $B(n,t)$ grows at early times, then remains constant for long times and then finally starts to decay. However, for a finite lattice, this asymptotic regime of decay is hard to see. To illustrate this point, we have plotted in Fig. 2(c), the functions $B(n,t)$ for $n = 0,1,2,3,4$ against $L(t)$ for a 1D lattice with 10 000 sites and up to times when the number of domains left is 10. As can be seen from the figure, the time at which $B(n,t)$ for $n \geq 1$ starts to decay is longer for larger n. The value of $A(t)$ (up to the time shown in the figure) obtained by keeping in terms up to $n = 4$, $A(t) = \sum_n (-1)^n B(n,t)$ agrees well with the direct calculation of $A(t)$.

III. CONSERVED MODELS IN ONE DIMENSION

A. 1D Stochastic conserved model

An example of the stochastic 1D conserved model is the nearest neighbor spin-exchange (Kawasaki) kinetic Is-

ing model. In this model, a nearest neighbor spin pair is chosen at random and their spins are exchanged according to the usual Metropolis algorithm. Now, in one dimension the ordered phases coexist only at $T=0$ and, therefore, coarsening can occur only at $T=0$. However, at $T=0$ of the Kawasaki dynamics (where spin pairs are exchanged only if the move does not increase the energy), the domains stop growing indefinitely as the system gets trapped in a metastable excited state with isolated domain walls [27]. Thus in 1D Kawasaki dynamics, coarsening can occur only in the limit $T\to 0$ and on a special time scale. This case has been discussed in detail in Ref. [18] and here we just outline the results.

A new time scale $\tau = t\exp(-4J/k_B T)$ was defined in [18], with the real time $t\to\infty$ and $T\to 0$ but keeping τ fixed (here J denotes the exchange coupling of the Ising model and k_B, the Boltzmann's constant) and it was shown that on this special time scale the 1D Kawasaki dynamics is equivalent to a "domain" model of Cornell, Kaski, and Stinchecombe [28], where each domain as a whole performs a random walk with a rate proportional to $1/L$ where L is the length of the domain. Coarsening occurs via the merging of these diffusing domains and the average length of domains grow as $L(\tau)\sim\tau^{1/3}$ [18,28] as in the higher dimensional cases. By proceeding in a similar way as in the nonconserved zero-temperature Glauber model [i.e., writing $\phi(k,t)$ in a domain wall representation and separating out the deterministic and stochastic part of the evolution], we argued that the two-time exponent $\lambda_c = 1$ exactly for this model. This result along with the scaling of the equal-time and the two-time correlation functions was verified numerically [18].

This model thus represents another case where the exponent λ_c is equal to the spatial dimension (in this case $d=1$).

B. Conserved GL equation in one dimension

In the dynamics of model B, the local conservation of the order parameter field $\phi(x,t)$ implies a continuity equation, $\partial\phi/\partial t + \partial j/\partial x = 0$ where the current density $j(x)$ is derived from a chemical potential $j(x) = -\partial\mu/\partial x$. The chemical potential $\mu(x)$ is given by

$$\mu(x) = \frac{\partial^2\phi}{\partial x^2} - \frac{\partial V(\phi)}{\partial\phi} , \tag{21}$$

where $V(\phi)$ is the usual double well potential as defined in case of model A in the preceding section. Thus the chemical potential $\mu(x)$ in model B plays the same role as $\partial\phi/\partial t$ in model A. The coarsening occurs via the bulk diffusion of the "up" phase from a region of higher chemical potential to a region of lower chemical potential. Here we have dropped the noise term thus confining ourselves to the deterministic zero temperature dynamics. One again, as was achieved in the nonconserved case, the goal is to map the late time dynamics of this model onto that of a kinetic Ising model.

We proceed as in the nonconserved case by first considering a small bubble of down phase confined between two kinks $x_1(t)$ and $x_2(t)$ as shown in Fig. 3(a) and let $R(t) = x_2(t) - x_1(t)$. Let $x_0(t)$ denote the location of the nearest kink to the left of $x_1(t)$ and $L_1(t) = x_1(t) - x_0(t)$ [see Fig. 3(a)]. Similarly, $x_3(t)$ is the location of the nearest kink to the right of $x_2(t)$ and $L_2(t) = x_3(t) - x_2(t)$. We consider the situation when $R(t)$ is much bigger than the width of a kink [$\sim O(1)$] but much smaller than the lengths of other nearby domains, e.g., $R(t) \ll L_1(t)$. Now, the chemical potential $\mu(x)$ satisfies a diffusion equation $d^2\mu/dx^2 = 0$, in the bulk of the up phases, i.e., in the regime I ($x_0 \le x \le x_1$) and regime II ($x_2 \le x \le x_3$). Let us first focus on regime II. In this regime, one has to solve the diffusion equation with the boundary conditions, $\mu(x_2) = \mu_2$ and $\mu(x_3) = \mu_3$. Now, proceeding exactly as in the nonconserved case, i.e., considering the superposition of the kink solutions as a suitable late time ansatz for $\phi(x,t)$ and substituting in Eq. (20), we get $\mu_2 \sim \exp[-R(t)]$ to leading order in the exponential interactions and $|\mu_3| \ll |\mu_2|$, since the domains on either side of the kink at x_3 are much larger than $R(t)$. Then, the solution of the diffusion equation satisfying these boundary conditions is simply

$$\mu(x) \approx \mu_2 \frac{(x_3 - x)}{L_2} . \tag{22}$$

Thus a space independent current, $j(x) = -\partial\mu/\partial x = \mu_2/L_2$ flows, which causes the short down bubble with length $R(t)$ to shrink. Consequently, the next down domain to the right (which is already longer) grows by the diffusion of down matter from the shorter to the longer bubble. Now, the same thing happens on the left of the bubble of radius $R(t)$, i.e., in regime I. The only difference is that in this regime, the current is given by $j = -\mu_1/L_1$, where $\mu_1 = \mu(x = x_1)$. Noting however that $\mu_1 \approx \mu_2 \sim \exp[-2R(t)]$, since the bubble with radius $R(t)$ is the shortest one, so we find that the shortest bubble shrinks with currents (of down phase) flowing from it to the nearest left or to the nearest right bubble of identical phase and the ratio of the left to the right current is given

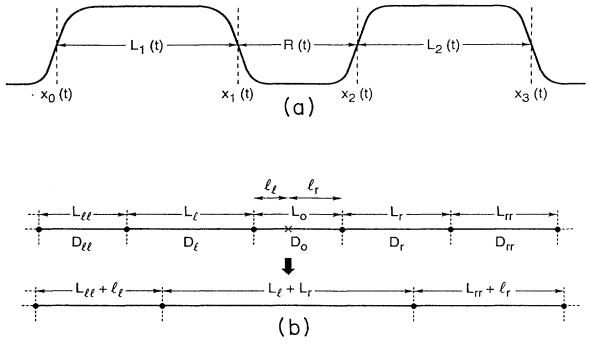

FIG. 3. (a) A small down bubble of length $R(t)$ confined between a pair of larger up bubbles of lengths $L_1(t)$ and $L_2(t)$, respectively. x_0, x_1, x_2, and x_3 denote the locations of the kinks. (b) The schematic representation of the dynamics of the kinetic Ising model derived from the deterministic GL equation in 1D for the conserved case (model B). The shortest domain D_0 has length L_0 which is broken into two pats of lengths l_l and l_r according to the ratio $l_l : l_r = L_r : L_l$.

by $j_l:j_r=L_2:L_1$.

This then completes our mapping of the model-B dynamics to an equivalent kinetic Ising model at late times. The dynamics of this Ising model is as follows. One starts with a random initial configuration of up and down domains. Then at every time step, the shortest domain present in the system is found. Let this domain be denoted by D_0 and its length by L_0. Let the two nearest domains to the left of D_0 be denoted by D_l (with length L) and D_{ll} (with length L_{ll}), respectively [see Fig. 3(b)]. Similarly the two nearest domains on the right of D_0 are denoted by D_r (with length L_r) and D_{rr} (with length L_{rr}). Now the shortest domain is broken into two parts of lengths l_l (the left half) and l_r (the right half) according to the ratio $l_l:l_r=L_r:L_l$. Then, the shortest domain D_0 is removed, the domains D_l and D_r are coalesced together to form a single domain of length L_l+L_r and the lengths of the domains D_{ll} and D_{rr} are increased to $L_{ll}+l_l$ and $L_{rr}+l_r$, respectively. Then at the next time step, one again finds the current shortest domain and the process continues.

One of the main differences from the nonconserved case discussed in the preceding section is that this dynamics introduces correlations between neighboring domain lengths, so we have been unable to solve analytically as in the nonconserved case. At late times and in the scaling limit, the exact ratio of the two parts (l_l/l_r) of the shortest domain is not important (as has been checked numerically) and one can break it into two equal halves. In that case, one can write down a mean field equation (i.e., ignoring correlations between neighboring domains although it is clear that the correlations are important even for the "equal halves" case) in the same way as in the nonconserved case. Using the name notations as in the preceding section and ignoring correlations between domains, we obtain

$$n(l,t+dt)=n(l,t)+n(\tilde{L},t)d\tilde{L}\left[-4f(l,t)+2f\left[l-\frac{\tilde{L}}{2},t\right]+\int_{\tilde{l}}^{\infty}dl'f(l',t)f(l-l',t)\theta(l-l'-\tilde{l})\right], \tag{23}$$

where $\tilde{L}(t)$ is the shortest length at time t and $d\tilde{L}=\tilde{L}(t+dt)-\tilde{L}(t)$. Assuming a scaling form $f(l,t)=\tilde{L}^{-1}F(l/\tilde{L},t)$ and using the Laplace transform $\tilde{F}(p,t)=\int_1^{\infty}dx\,\exp(-px)F(x,t)$, a straightforward algebra shows that the fixed point distribution $\tilde{F}(p)$ satisfies the equation

$$\frac{d\tilde{F}(p)}{dp}=\frac{1}{2p}\tilde{F}^2(p)-\frac{(1-e^{-p/2})}{p}-\frac{e^{-p}}{2p}, \tag{24}$$

where we have used the consistency condition $\lim_{t\to\infty}F(1,t)=\frac{1}{2}$. This first order equation falls in the class of generalized Ricatti equation [29] and cannot in general be solved in the closed form. However, a closed form solution would not give us much insight since the domain correlations are neglected in the mean field theory. Furthermore, it does not give any information regarding the two-time correlations.

However, as in the nonconserved case, this Ising model can be simulated and it is possible to reach the late times quite fast since the dynamics does not involve the real time directly. In Figs. 4(a) and 4(b), we have shown the scaling of the equal-time and the two-time correlation functions, respectively. Note that the corrections to scaling are quite apparent at early times. Figure 4(c) shows a plot of the $-\log_{10}[A(t)]$ versus $\log_{10}L(t)$ where $A(t)$ is the autocorrelation $A(t)=\langle\phi(x,0)\phi(x,t)\rangle$ and $L(t)$ is the average length of domains at time t. Once again, as in the nonconserved case [Fig. 2(a)], we notice an upward curvature in the local slope as $L(t)$ increases. We, therefore, proceed to calculate the effective exponent $\lambda_{\rm eff}$ as in the nonconserved case. There is, however, an important difference from the nonconserved case. In the conserved case, we expect the leading correction to the scaling in the autocorrelation $A(L)$ [once again we denote $A(t)$ by $A(L)$ since only L appears in the expression for A] to decay as L^{-d} (following the discussion of Sec. I). Thus for the conserved case, we fit to $A(L)\approx aL^{-\lambda}+bL^{-d}+\cdots$, where $\cdots$ denotes the higher order terms in $1/L$. In this case, we first define $A'(L)=A(L)-bL^{-d}$ and then calculate the effective exponent, $\lambda_{\rm eff}=-[1/\log_{10}(c)]\log_{10}[A'(cL)/A'(L)]$ with $c=10^{0.2}$. Once again, the coefficient b is adjusted so that $\lambda_{\rm eff}$ when plotted against L^{-1} gives a straight line parallel to the L^{-1} axis. With the choice of $b\approx-0.984$, we find in Fig. 4(d) that the best fitting straight line is almost parallel to the horizontal axis. From the intercept of this line with the vertical axis, we get $\lambda\approx0.67\pm0.02$.

Thus, this constitutes a clear example of a deterministic conserved model where $\lambda<d$, in contrast to the other known conserved cases, all corresponding to quenches to T_c of stochastic models where we have found $\lambda_c=d$. Thus, according to our general discussion of Sec. I, the deterministic conserved model falls in the "first" category where, as in the equal-time structure scaling, the two-time structure factor is also dominated by corrections to scaling at $k=0$.

IV. MAPPING TO THE AGGREGATION MODELS IN 1D

A. Deterministic aggregation model

In this section, we consider the deterministic conserved model in 1D as discussed in Sec. III B but in the limit when the volume fraction of one of the phases (e.g., the down phase) is small. In this limit, the dynamics of this model can be mapped onto that of a deterministic aggregation model as we show below. When the volume fraction ϕ_0 of the minority down phase is small, a typical domain of the majority up phase (with volume fraction

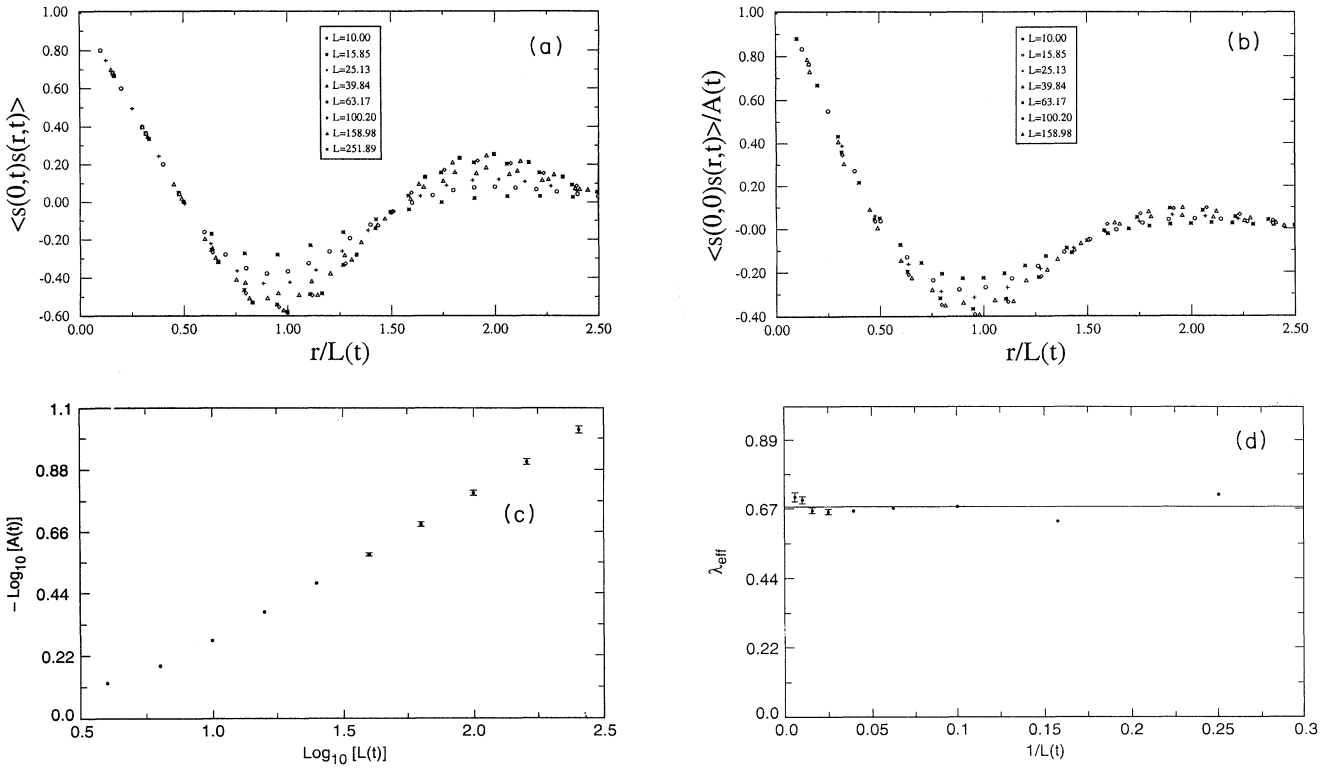

FIG. 4. (a) The equal-time correlation function (for the deterministic model B in 1D), namely, $\langle s(0,t)s(r,t)\rangle$ (averaged over 70 runs on samples of size 10^5 sites) as a function of the scaled distance $r/L(t)$ at eight different times (shown by symbols) when L is 10.00, 15.85, 25.13, 39.84, 63.17, 100.20, 158.98, and 251.89. (b) The two-time correlation function (for the deterministic model B in 1D), namely, $\langle s(0,0)s(r,t)\rangle$ [normalized by the autocorrelation $A(t)$ and averaged over 70 runs as in Fig. 4(a)] as a function of the scaled distance $r/L(t)$ when L is 10.00, 15.85, 25.13, 39.84, 63.17, 100.20, and 158.98. (c) The autocorrelation function (for the deterministic model B in 1D), namely, $A(t)=\langle s(r,0)s(r,t)\rangle$ (averaged over 70 runs on samples of size 10^5 sites) as a function of $L(t)$ on a log-log plot. (d) The effective exponent $\lambda_{\rm eff}=-[1/\log_{10}(c)]\log_{10}[A'(cL)/A'(L)]$ plotted against L^{-1} for the conserved deterministic model in 1D. Here $c=10^{0.2}$ and $A'(L)=A(L)-bL^{-d}$ with $d=1$. The constant b has been adjusted to be ≈ -0.984 so that the best fitting straight line is parallel to the L^{-1} axis, i.e., $\lambda_{\rm eff}$ is independent of L^{-1}. The intercept of this line with the vertical axis gives $\lambda\approx 0.67\pm 0.01$.

$1-\phi_0$) is much longer compared to that of the minority phase. Since the interaction between a pair of kinks falls off exponentially with their separation, it is clear that in the limit $\phi_0\to 0$, the dynamics of the minority phase domains occurs exponentially faster than that of the majority phase. Thus in this limit, one can view a typical configuration as consisting of a set of particles (each particle representing a majority domain) separated by the intervals of the minority phase and the mass of a particle is equal to the length of the corresponding majority domain. The dynamics then consists of finding the pair of closest particles (i.e., finding the smallest of the minority domains) and replacing them by a single particle (with mass equal to the sum of the masses of the two "parent" particles) midway between them. We note that without this "aggregation" procedure, this model is geometrically identical to the "cut-in-two" model discussed by Derrida, Godreche, and Yekutieli [24] in the context of breath figures.

In the limit of zero volume fraction of the minority phase, the distribution of the sizes of the minority bubbles was calculated exactly in $d=3$ by Lifshitz and Slyozov [9] and later this calculation was extended to higher dimensions [30] as well as to $d=2$ [31]. However $d=1$ is very special where the distribution of the lengths of the minority domains is the same as the distribution of intervals in the cut-in-two model discussed above. The distribution function $P(l,\tilde{L})$ denoting the fraction of intervals of length l when the shortest interval is $\tilde{L}$, scales as $P(l,\tilde{L})\sim\tilde{L}^{-1}R(l/\tilde{L})$ and was studied in detail by Derrida, Godreche, and Yekutieli [24]. Unfortunately, due to the correlations introduced between neighboring intervals as one cuts the shortest interval into two halves and pastes them to its neighbors, the scaling function $R(x)$

cannot be calculated exactly as in the nonconserved "paste-all" model where there are no correlations between neighboring domains. One can calculate $R(x)$ within a mean field theory (i.e., neglecting correlations between neighbors) which however is very different from the actual scaling function obtained by numerical simulation [24].

In one dimension, the distribution of the lengths of the majority phase is also of interest. In terms of the above aggregation model, this is identical to the distribution of mass, $\tilde{P}(m,\tilde{L})$, i.e., the fraction of particles having mass m when the shortest interval is $\tilde{L}$. This distribution also scales as $\tilde{P}(m,\tilde{L})\sim\tilde{L}^{-1}S(m/\tilde{L})$. The introduction of this new variable m in the cut-in-two model also introduces correlations between the mass of a particle and the lengths of the neighboring intervals since a large mass is typically neighbored by long intervals. Consequently, exact calculation of the scaling function $S(x)$ is also difficult. However, as in the case of length distribution, one can write down a mean field theory ignoring all correlations. Let $n(l,m,\tilde{L})dl\,dm$ denote the number of intervals of length between $[l,l+dl]$ followed by a mass between $[m,m+dm]$ at a time when the shortest interval is of length $\tilde{L}$. Assuming no correlations, it is easy to write down an evolution equation for $n(l,m,\tilde{L})$ as was done earlier in Secs. II and III. Assuming a two variable scaling, $n(l,m,\tilde{l})\sim\tilde{L}^{-2}Y(l/\tilde{L},m/\tilde{L})$, it is straightforward algebra that shows that the function $Y(x,y)$ satisfies the following equation:

$$x\frac{\partial Y}{\partial x}+y\frac{\partial Y}{\partial y}+Y(x,y)+Y(x-\tfrac{1}{2},y)+\int_0^\infty\int_0^\infty Y(1,y_1)\times Y(x-\tfrac{1}{2},y_2)\delta(y-y_1-y_2)dy_1dy_2=0\ , \quad (25)$$

where the function $Y(x,y)$ is identically 0 for $x<1$. One can solve Eq. (25) by introducing the double Laplace transform,

$$\tilde{Y}(p,q)=\int_0^\infty dy\,\exp(-qy)\int_1^\infty dx\exp(-px)Y(x,y)\ .$$

Using Lagrange's method of auxiliary equations [32], we find

$$\tilde{Y}(p,q)=\int_q^\infty\frac{dt}{t}g(1,t)\exp\left[-\frac{pt}{q}\right]+\int_q^\infty\frac{du}{u}\left[1-\{1+g(1,u)\}\exp\left[-\frac{up}{2q}\right]\right]\ , \quad (26)$$

where the function $g(1,t)=\int_0^\infty Y(1,y)\exp(-ty)dy$. Note that for $q=0$ [after making a change of variable $t\to pt/q$ in Eq. (26)], i.e., by integrating over the mass variable, Eq. (26) reduces to the Eq. (21) of Derrida, Godreche, and Vekutieli [24] with the choice $g(1,0)=1$ as demanded by the fact that the fixed point distribution is characterized by a finite coverage, i.e., the first moment of the scaling function $R(x)$ is finite. For calculating the scaling function $S(y)$ for the mass distribution, we need to integrate over the length, i.e., to put $p=0$ in Eq. (25). Then, the Laplace transform of $S(y)$, i.e., $\tilde{Y}(0,q)=\int_0^\infty S(y)\exp(-qy)dy$ is given in terms of the unknown function, $g(1,q)$,

$$\tilde{Y}(0,q)=\int_q^\infty\frac{dt}{t}g(1,t)\exp\left[-\int_q^\infty\frac{du}{u}g(1,u)\right]\ . \quad (27)$$

However, contrary to the case of the length distribution where we need to know the value of the function $g(1,q)$ only at a particular point, i.e., at $q=0$ (and which is fixed to be one by the finite coverage constraint as mentioned above), for the mass distribution we need to know the full function $g(1,q)$ for which there is no *a priori* consistency condition. Thus there exists a continuum of such solutions each characterized by a function $g(1,q)$ [as opposed to the case of length distribution, where each solution is characterized by a parameter $g(1,0)$]. Which one of these solutions is picked up by most initial conditions is not clear *a priori*.

In any case, the mean field theory is not accurate as it neglects correlations. We have, therefore, performed a direct simulation of the aggregation model to determine the mass distribution. In Fig. 5, we have plotted the scaled mass distribution versus $x=m/L$ where L is the average length of the domains which is proportional to the length of the shortest interval.

A somewhat related one dimensional deterministic model where such distributions can be computed exactly is Burger's equation [33]. The exact solution of the velocity field satisfying Burger's equation consists of a series of shocks each having a strength (or mass) (i.e., the discontinuity in the velocity field across a shock) and an advance velocity that differs from shock to shock. When two shocks collide, they coalesce into one and their strengths are summed and also the momentum (i.e., strength×velocity) is conserved in a collision. In this model, the distribution of the shock strengths as well as the distribution of the length of the separation between two adjacent shocks can be computed exactly [33]. Further connections between the aggregation models and

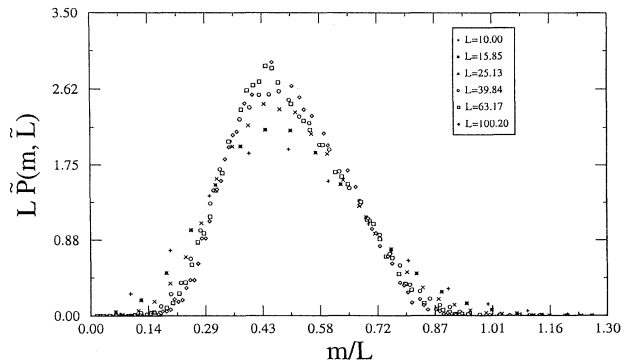

FIG. 5. The mass distribution function $\tilde{P}(m,\tilde{L})$ [scaled by the average length $L(t)$] plotted against the scaled mass m/L at six different times when L is 10.00, 15.85, 25.13, 39.84, 63.17, and 100.20 (averaged over 10 runs on samples of size 10^6) for the deterministic aggregation model in 1D.

Burger's equation have been discussed by Hayakawa, Yamamoto, and Takayasu [34].

B. Stochastic aggregation model

In this section, we consider a stochastic model of aggregation and solve for the distribution of the length of the intervals and the mass exactly. In this model, one considers a number of massive particles performing random walk on a one-dimensional lattice. We consider a continuous time dynamics where in a small time interval Δt, every particle hops to its nearest neighbor on the right with probability $\frac{1}{2}\Delta t$, to the left with probability $\frac{1}{2}\Delta t$ or stays back in its present location with probability $(1-\Delta t)$. If more than one particle happens to be at the same site, they coagulate instantaneously into a single particle whose mass is equal to the sum of the masses of the coagulating particles. One is then interested in calculating the distribution of the separation between the particles $P(l,t)$ and the distribution of masses $\tilde{P}(m,t)$ at time t.

Note that the discrete time version of this model is identical to the growth model introduced by Takayasu, Nishikawa, and Tasaki [35] except for the fact that in the Takayasu model, there is an additional injection of one particle at every site at every time step. This injection procedure was introduced to achieve a nontrivial steady state ($t\to\infty$), where the distribution $P(m,\infty)$ has a power law tail. This is because, without this additional injection procedure, the steady state is trivial where all the particles coagulate into a single one whose mass tends to infinity. The statics [35] and the dynamics [36] of this model with injection has been solved exactly. Also, without the additional variable mass, our model is exactly the same as the stochastic $A+A\to A$ reaction model. The length distribution for the $A+A\to A$ model has been calculated by ben-Avraham, Burschka, and Doering [37].

Let $M_i(t)$ denote the mass at the lattice site i at time t. The dynamics is represented by the stochastic equation

$$M_i(t+\Delta t)=\sum_j w_{ij}M_j(t)\ , \tag{28}$$

where w_{ij} are random variables that take values 0 or 1 according to the following rules. First of all, w_{ij} satisfy the constraint $\sum_i w_{ij}=1$. Second, $w_{ij}=0$ with probability 1 if $j\neq i-1$, $i+1$ or i. w_{ij} for $j=i-1$, $i+1$ is 1 with probability $\frac{1}{2}\Delta t$ and 0 with probability $1-\frac{1}{2}\Delta t$. Finally, w_{ii} is 1 with probability $1-\Delta t$ and 0 with probability Δt. We define the r-body generating function $Z(r,\rho,t)=\langle\exp[-\rho\sum_{j=1}^{i+r-1}M_j(t)]\rangle$ where $\langle\ \rangle$ represents the average over the joint probability distribution of masses at the r consecutive sites. Note that $Z(r,\rho,t)$ is independent of i due to translational invariance. Using Eq. (28) and the constraint, $\sum_i w_{ij}=1$, we get

$$Z(r,\rho,t+\Delta t)=\left\langle\exp\left[-\rho\left\{w_{i,i-1}M_{i-1}(t)+(1-w_{i-1,i})M_i(t)+\sum_{j=i+1}^{i+r-2}M_j(t)\right.\right.\right.$$
$$\left.\left.\left.+(1-w_{i+r,i+r-1})M_{i+r-1}(t)+w_{i+r-1,i+r}M_{i+r}(t)\right\}\right]\right\rangle\ , \tag{29}$$

where $\langle\ \rangle$ denote an average over the M_j at time t as well as over the set W: $[w_{i,i-1},\ w_{i-1,i},\ w_{i+r,i+r-1},\ w_{i+r-1,i+r}]$. The members of the set W are independently distributed [independent of each other and also of the $M_j(t)$]. Each member of W can be either 1 with probability $\frac{1}{2}\Delta t$ or 0 with probability $1-\frac{1}{2}\Delta t$. We first perform the average over this set W in Eq. (29). Note that out of 16 possible configurations of W, only five configurations contribute to $O(1)$ or $O(\Delta t)$ on the right-hand side of Eq. (29). Out of these five [0,0,0,0] contributes to $O(1)$ and [1,0,0,0], [0,1,0,0], [0,0,1,0], and [0,0,0,1] contribute to $O(\Delta t)$. The other 11 configurations contribute only to $O(\Delta t^2)$ or higher. Neglecting terms of $O(\Delta t^2)$ and taking the limit $\Delta t\to 0$, we find that the function Z evolves according to the equation,

$$\frac{\partial Z(r,\rho,t)}{\partial t}=[Z(r+1,\rho,t)+Z(r-1,\rho,t)-2Z(r,\rho,t)]\ , \tag{30}$$

with the boundary condition $Z(0,\rho,t)=1$ for all t. Note that this equation does not contain ρ explicitly and the solution depends on ρ only through the initial condition. This equation is identical to that satisfied by the two-point equal-time spin-spin correlation function $G(r,t)=\langle s(0,t)s(r,t)\rangle$ in the zero-temperature Glauber model [10]. Making a Laplace transform following Glauber and inverting it back, the solution can be written as

$$Z(r,\rho,t)=1+e^{-2t}\sum_{n=1}^{\infty}[Z(n,\rho,0)-1]\times[I_{r-n}(2t)-I_{r+n}(2t)]\ , \tag{31}$$

where $I_n(t)$ is the modified Bessel function of the first kind. Note that by definition, $Z(1,\rho,t)$ is the Laplace transform of the single site mass distribution function, i.e., $\tilde{Q}(m,t)$.

We first calculate the distribution of the separation between particles, i.e., $P(l,t)$. Let $E(l,t)$ denote the probability that a segment of length l is empty. Then clearly, the function $P(l,t)$ (normalized to unity) is related to $E(l,t)$ via

$$P(l,t)=\frac{1}{c(t)}[E(l+2,t)+E(l,t)-2E(l+1,t)]\ . \tag{32}$$

The function $E(l,t)$ is just the ρ independent term in the expansion of $Z(l,\rho,t)$, i.e., $\lim_{\rho\to\infty}Z(l,\rho,t)$ and is easily seen from Eq. (31) to be

$$E(l,t)=\left[1-e^{-2t}\left\{I_0(2t)+I_l(2t)+2\sum_{n=1}^{l-1}I_n(2t)\right\}\right]+e^{-2t}\sum_{n=1}^{\infty}g(n)[I_{n-1}(2t)-I_{n+l}(2t)]\ , \tag{33}$$

where $g(n)=\lim_{\rho\to\infty}Z(n,\rho,0)$ is the probability of occurrence of an empty sector of length n in the initial condition. While the first term in Eq. (33) is independent of the initial condition, the second term depends explicitly on the initial condition. However, as long as all positive moments of $g(n)$ exist [i.e., $\sum_{n=1}^{\infty}n^p g(n)$ is finite for any positive integer p], the second term decays in time faster than the first term and, therefore, drops out in the scaling limit when $t\to\infty$. Thus, for all initial conditions, that satisfy this condition, the expression for $E(l,t)$ is universal and is given by the first term of Eq. (33). As specific examples, one could consider two different initial conditions that satisfy this condition: (i) when each site is occupied by a single particle of mass unity for which $Z(n,\rho,0)=\exp(-\rho n)$ and, therefore, $g(n)=0$ identically and (ii) when every site is either occupied by a single particle of unit mass with probability $\frac{1}{2}$ or empty with probability $\frac{1}{2}$ in which case $Z(n,\rho,0)=([1+e^{(-\rho)}]/2)^{-n}$ and, therefore, $g(n)=2^{-n}$. Thus the concentration of particles decays universally as $c(t)=1-E(1,t)\approx 1/\sqrt{\pi t}$ and consequently the average separation between particles grows as $\langle L\rangle(t)=1/c(t)\approx\sqrt{\pi t}$ at long times, a result that was already derived by ben Avraham, Burschka, and Doering for the $A+A\to A$ model by using a different technique [37]. Using the asymptotic expansion of the Bessel function, we find from Eqs. (33) and (32) that in the scaling limit, i.e., $t\to\infty$, $l\to\infty$ but keeping $x=l/\langle L\rangle(t)$ fixed, $P(l,t)\approx\langle L\rangle^{-1}R(l/\langle L\rangle)$ where the universal scaling function $R(x)$ is given by

$$R(x)=(\pi x/2)\exp\left[-\frac{\pi x^2}{4}\right]\ . \tag{34}$$

This result was also derived by ben-Avraham, Burschka, and Doering [37] for the $A+A\to A$ model by a different method.

We now turn to the calculation of the single site mass distribution function $\tilde{Q}(m,t)$. By definition, $Z(1,\rho,t)=\sum_{m=0}^{\infty}e^{-\rho m}\tilde{Q}(m,t)$, where $\tilde{Q}(0,t)$ denotes the concentration of vacancies at time t. Let us first write $Z(n,\rho,0)=\sum_{m=0}^{\infty}f(m,n)e^{-\rho m}$ where $f(m,n)$ denotes the probability that at $t=0$ a block of n consecutive sites will contain a total mass m. For example, for the two initial conditions discussed in the previous paragraph, $f(m,n)=\delta_{m,n}$ for case (i) and $f(m,n)=\binom{n}{m}2^{-n}$ for case (ii). Then, from Eq. (29) we find (using properties of Bessel functions) that for $m>0$,

$$\tilde{Q}(m,t)=\sum_{n=1}^{\infty}f(m,n)\frac{e^{-2t}}{t}nI_n(2t) \tag{35}$$

and $\tilde{Q}(0,t)=1-E(1,t)$ as discussed in the preceding paragraph. The fraction of particles having mass m at time t, i.e., $\tilde{P}(m,t)$ (normalized to unity) is given by $\tilde{P}(m,t)=\tilde{Q}(m,t)/c(t)$. Now, let us assume that the initial condition satisfy the following two conditions, namely, for any positive integer p (a) the sum $\sum_{n=1}^{\infty}n^p f(m,n)$ exists for all m and (b) $\sum_{n=1}^{\infty}n^p f(m,n)\approx[\sum_{n=1}^{\infty}nf(m,n)]^p$ to leading order for large m. Then, using the asymptotic expansion of the Bessel function and taking the limits $t\to\infty$, $m\to\infty$ but keeping $x=m^*(m)/\langle L\rangle(t)$ fixed where $m^*(m)=\sum_{n=1}^{\infty}nf(m,n)$, it is not difficult to see that, $\tilde{P}(m,t)\approx\langle L\rangle^{-1}S(m^*(m)/\langle L\rangle)$ where the scaling function $S(x)$ is given by

$$S(x)=\frac{\pi x}{2}\exp\left[-\frac{\pi x^2}{4}\right]\ . \tag{36}$$

Thus, provided the initial conditions satisfy the two conditions (a) and (b) as mentioned above, this scaling function $S(x)$ is universal. It is easy to verify that the two specific examples of initial conditions (i) and (ii) discussed in the preceding paragraph satisfy these two conditions and $m^*(m)=m$ in case (i) and $m^*(m)=4m+2$ in case (ii).

It is amusing to note that the distributions of both the intervals and masses are described by the same scaling function. Note that from the function $Z(r,\rho,t)$, one can also compute the probability $E(r,m,t)$ that a segment of length r contains a total mass m by calculating the coefficient of $\exp(-\rho m)$ in the expansion of $Z(r,\rho,t)$. This is useful in demonstrating that the corrections between neighboring sites play an important role in this model also as in the case of the deterministic model. To see this explicitly, we first calculate $P(2,m,t)=\exp(-2t)[I_{m-2}(2t)-I_{m+2}(2t)]$ and then the quantity, $U(m,t)=\sum_{m_1=0}^{m}P(1,m_1,t)P(1,m-m_1,t)$ which is given by

$$U(m,t)=\frac{e^{-4t}}{t^2}\sum_{m_1=0}^{m}m_1(m-m_1)I_{m_1}(2t)I_{m-m_1}(2t)\ . \tag{37}$$

Had there been no correlations between neighboring sites, $P(2,m,t)$ would have been equal to $U(m,t)$. However, once again using the asymptotics of Bessel function, it can be demonstrated that in the scaling limit, $P(2,m,t)$ decays as t^{-1} whereas $U(m,t)$ decays faster as $t^{-3/2}$ indicating that correlations are important. In fact, around a large mass at a given site, there will always be a neighboring depletion regime as in the deterministic case.

Before concluding this section, we note that the stochastic $A+A\to A$ model also describes the coarsening of the q-state Potts model in 1D (at $T=0$) in the limit $q\to\infty$ (which also describes the coarsening of soap bubbles) [14]. This can be understood in the following way. Representing a domain wall between two Potts phases by a particle A on a 1D line, the dynamics of the zero-temperature Potts model is equivalent to that of a generalized reaction diffusion model where the particles diffuse, annihilate, and coagulate according to the following rules: Each particle undergoes diffusion until two of

them meet in which case they either annihilate, i.e., $A+A\rightarrow 0$ (which corresponds to the merging of two domains of the same phase in the Potts model) with probability $1/(q-1)$, or coagulate, i.e., $A+A\rightarrow A$ (when two domains of different phases meet in the Potts model) with probability $(q-2)/(q-1)$. For $q=2$, the particles only annihilate and thus one has the zero-temperature Glauber model, whereas for $q\rightarrow\infty$, they only coagulate. Thus the two models $A+A\rightarrow 0$ and $A+A\rightarrow A$ are the two solvable limits of the Potts model. For the latter case, i.e., in the $q\rightarrow\infty$ limit, it has been argued [14] recently that the two-time exponent $\lambda=d$ in all dimensions. In one dimension, the coarsening of the q-state Potts model can be mapped onto that of an Ising model with magnetization $m=2/q-1$ and, therefore, following the same method as discussed in Sec. II A, it is easy to see that $\lambda_c=1$ for all q in 1D. This, therefore, constitutes another example where $\lambda_c=d$.

SUMMARY AND CONCLUSIONS

In this paper, we have studied mainly the two-time correlation function in a phase ordering system after a quench to or below the critical point. We have built a general framework for the equal-time and the two-time structure factors and discussed under what conditions the two-time exponents λ or λ_c are equal to the spatial dimension d in the conserved order parameter case. We have presented a few cases where exact solutions and numerical simulations suggest that $\lambda_c=d$. All these cases however correspond to quenches to $T=T_c$ where the stochastic noise plays an important role. This raised the question whether the same result, i.e., $\lambda=d$ is true for deterministic models as well, which correspond to quenches to below T_c. However, we have found one example, namely the deterministic model B in 1D where λ is explicitly less than $d=1$. Thus we conclude that while $\lambda_c=d$ may be true in all dimensions, the exponent λ, in general, can be less than d. In most parts of this paper, we have confined ourselves to $d=1$ where analytical results are possible to obtain and also the numerical simulation is easier. We have presented a detailed analytical and numerical study of the similarities and differences between stochastic and deterministic models in 1D.

While we have found a specific example where $\lambda<d$, so far we have not found any counterexample to the conjecture that $\lambda_c=d$ is generally true. This, therefore, remains an open question and either a general proof of this result or even a counterexample would be interesting. Although the exponent λ has been measured experimentally [3] for the nonconserved Ising model in 2D, there has not been any experiment so far measuring λ or λ_c for the conserved case. Any such effort, therefore, would also be very interesting.

We have mainly confined ourselves to the short-range models in 1D. Recently, however, there has been much activity [23] in the one-dimensional long-range models where the interaction between kinks decays as a power law instead of exponentially as in the case of short-range models. It would be interesting to study how the exponents λ or λ_c change in that case.

ACKNOWLEDGMENTS

We thank A. J. Bray for very useful discussions.

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