Microcirculations in turbulent flows

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(Received 2 March 1995)

Velocity circulations over infinitesimal fluid contours (microcirculations) are considered. A statistical evolution of microcirculations is studied (in particular, initial tendencies) with various initial orientations of fluid contours relative to the vorticity field. The obtained exact results suggest another direction for numerical experiments in turbulence.

PACS number(s): 47.27.−i

In well developed three-dimensional (3D) turbulent flows, the major nonlinear effect of vortex stretching is statistically balanced with viscous dissipation [1,2]. It was predicted [2–4] that balance between these two effects takes place not only globally but also conditionally for any fixed magnitude of vorticity ω and that contributions of other terms in the vorticity balance are $\sim \mathrm{Re}^{-1/2}$ (Re is the Reynolds number). This prediction was confirmed recently by direct numerical simulations [5], which revealed that rates of vortex stretching and dissipation increase exponentially with ω. It was also argued [5] that local imbalance of these two major opposing effects leads to the formation and destruction of twisted vortex strings, observed experimentally [6]. A detailed theory of these events can drastically reduce the effective number of degrees of freedom for numerical simulations of turbulent flows (from the classical estimate [7] $N \sim \mathrm{Re}^{9/4}$ to the number $N_s \sim \mathrm{Re}^{9/10}$ associated with vortex strings [8]). The present paper constitutes a simple step toward the understanding of dynamics and statistics of vortex filaments and corresponding microcirculations in turbulent flows.

Consider the following equations for the 3D vorticity field in viscous incompressible fluid:

$$\frac{d\omega_i}{dt} = \frac{\partial v_i}{\partial x_j}\omega_j + \nu\Delta\omega_i, \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + v_j\frac{\partial}{\partial x_j}, \quad \frac{\partial v_i}{\partial x_i} = 0 \ . \tag{1}$$

Here ω_i is vorticity, v_i is velocity, ν is kinematic viscosity, and we have a summation over the repeated indexes from 1 to 3. The first term on the right-hand side of (1) represents the effect of vortex stretching, which is absent for 2D flow.

The intensity of the vortex filament is determined by the velocity circulation over a contour encircling this filament:

$$\oint_C v_i \delta l_i = \int_S \omega_i \delta s_i \ . \tag{2}$$

Here δs_i is an oriented element of cross section S, limited by contour C. The evolution of an infinitesimal fluid surface element s_i (we omit δ) is determined by the equation

$$\frac{ds_i}{dt} = -\frac{\partial v_j}{\partial x_i}s_j \ . \tag{3}$$

Indeed, for a linear infinitesimal element r_i we have

$$\frac{dr_i}{dt} = \frac{\partial v_i}{\partial x_j}r_j \ , \quad r_i = \frac{\partial x_i}{\partial a_j}r_{j0} \ , \tag{4}$$

where $x_i(t,\mathbf{a})$ is the trajectory of the fluid particle with the initial position $\mathbf{a}$ and r_{i0} is the initial element. Conservation of volume [incompressibility (1)] and the first part of Eqs. (4) give

$$\frac{d}{dt}(r_i s_i) = 0, \quad r_i\left[\frac{ds_i}{dt} + \frac{\partial v_j}{\partial x_i}s_j\right] = 0 \ . \tag{5}$$

Equation (3) follows from the second part of Eqs. (5), if we remember that r_i is initially arbitrary (see the similar derivation in Ref. [9]). We can also consider the surface element as the vector product of two linear elements $s_i = \epsilon_{ijk} l_j \rho_k$, where ϵ_{ijk} is the unit antisymmetric tensor. Conservation of volumes (5) and (3) follows from (4) and general equalities:

$$\epsilon_{ijk} x_{il} x_{jm} x_{kn} = J\epsilon_{lmn}, \quad J \equiv \det\{x_{ik}\} \ , \tag{6}$$

$$v_{mi}\epsilon_{mjk} + v_{mj}\epsilon_{mki} + v_{mk}\epsilon_{mij} = v_{mm}\epsilon_{ijk} \ , \tag{7}$$

where in our case, $x_{ij} = \partial x_i/\partial a_k$, $v_{mi} = \partial v_m/\partial x_i$, $J=1$, and $v_{mm}=0$. The solution of Eq. (3) can be presented in terms of inverse trajectory $\mathbf{a}(t,\mathbf{x})$: $s_i = (\partial a_j/\partial x_i)s_{j0}$.

To deal with finite quantities, let

$$\sigma_i \equiv \frac{s_i}{s_0}, \quad \sigma_i^2|_{t=0} = 1, \quad \gamma \equiv \omega_i\sigma_i \ . \tag{8}$$

The initial area of surface element s_0 is assumed, for simplicity, to be the same for all points. The γ field (8) is associated with microcirculation of velocity and local intensity of the corresponding vortex filament.

Taking into account that s_0 is constant, we get from (3) and (8)

$$\frac{d\sigma_i}{dt} = -\frac{\partial v_j}{\partial x_i}\sigma_j \ . \tag{9}$$

For the evolution of γ from (1), (8), and (9) we have

$$\frac{d\gamma}{dt} = \nu\sigma_i\Delta\omega_i \ . \tag{10}$$

In 2D incompressible flow, the area of the surface element is invariant of motion ($s \equiv s_0$) and γ coincides with vorticity ω, which has only one component. In 3D flow, γ plays a role similar to vorticity in 2D, which is the inviscid invariant of motion (Kelvin's theorem, applied to the infinitesimal contour).

Integration of (10) over a simple-connected fluid surface gives

$$\frac{d}{dt}\oint_C v_i \delta l_i = \nu \int_S \Delta\omega_i \delta s_i = \nu \oint_C \Delta v_i \delta l_i = -\nu\epsilon_{ijk}\oint_C \frac{\partial \omega_k}{\partial x_j}\delta l_i \ . \quad (11)$$

Let us note that (11) has an important application to the case of free-surface turbulent flows when the contour is totally or partially on a free surface [10]. Moreover, when the contour is totally on a free surface, (11) is correct even for a nonbarotropic fluid [10].

For a statistical description of the γ field in turbulent flow, we consider three different initial conditions for fluid contours. In case I, σ_i is initially independent of the vorticity field and statistically homogeneous and isotropic. In case II, σ_i is initially oriented along the vorticity field ($\sigma_{i0} = \omega_{i0}\omega_0^{-1}$). Thus, case I corresponds to arbitrarily chosen fluid contours, while case II corresponds to the dynamics of vortex filaments (at least initially). We will also consider a more general case, case III, described below.

For the evolution of the n-order moment of γ in statistically homogeneous turbulent flow, we have from (10)

$$\frac{\partial}{\partial t}\langle \gamma^n \rangle = n\nu \langle \gamma^{n-1}\sigma_i \Delta\omega_i \rangle \ . \quad (12)$$

Here $\langle\ \rangle$ means statistical averaging and we used the definition of the total time derivative, incompressibility (1), and homogeneity of the flow.

In case I, odd moments are zero at any time:

$$\langle \gamma^{2m+1} \rangle \equiv 0, \quad m = 0,1,2,\ldots \ . \quad (13)$$

Indeed, initially the moment (13) is proportional to the odd moment of σ_i (recall that in case I σ_i is initially independent of vorticity). One-point moments of a statistically homogeneous and isotropic vector field can be expressed only in terms of products of the unit tensor δ_{ij}. Thus all odd moments are zero. This proves that (13) is correct initially. The time derivative of this moment, according to (12), is also initially proportional to an odd moment of σ_i and, therefore, is zero. Similarly, by using (1), (9), (10), (12), and the initial independence of σ_i from the vorticity field (and thus from the velocity field), we see that high-order time derivatives are also zero initially. This proves (13) at any time.

For even moments in case I we can write

$$\langle \gamma^{2m} \rangle = \langle \omega^{2m}\overline{\sigma^{2m}\mu^{2m}} \rangle, \quad \mu \equiv \frac{\omega_i\sigma_i}{\omega\sigma} \ . \quad (14)$$

Here the overbar means conditional averaging with fixed ω_i (see details about such averaging in Ref. [3]). By definition (8), initially $\sigma = 1$. Isotropy of σ_i distribution gives

$$\overline{\mu^{2m}} = \tfrac{1}{2}\int_{-1}^{1}\mu^{2m}d\mu = \frac{1}{2m+1} \ . \quad (15)$$

Thus, initially,

$$\langle \gamma^{2m} \rangle = \frac{1}{2m+1}\langle \omega^{2m} \rangle, \quad m = 1,2,\ldots \ . \quad (16)$$

For the time derivative from (12) we have

$$\frac{\partial}{\partial t}\langle \gamma^{2m} \rangle = 2m\nu \langle \omega^{2m-1}\Delta\omega_i \overline{\sigma_i\sigma^{2m-1}\mu^{2m-1}} \rangle \ . \quad (17)$$

Initially $\sigma = 1$, and, by using isotropy, we can write

$$\overline{\sigma_i\mu^{2m-1}} = A\omega_i\omega^{-1} \ , \quad (18)$$

where scalar A may depend only on the magnitude of vorticity ω, and ω_i is the only distinguished vector in conditional averaging (compare with Ref. [3]). Multiplication of (18) by $\omega_i\omega^{-1}$, with the use of (15), gives $A = 1/(2m+1)$. Thus initially

$$\frac{\partial}{\partial t}\langle \gamma^{2m} \rangle = \frac{2m\nu}{2m+1}\langle \omega^{2m-2}\omega_i\Delta\omega_i \rangle. \quad (19)$$

Simple algebra, with the use of homogeneity, gives

$$\frac{\partial}{\partial t}\langle \gamma^{2m} \rangle = -\frac{2m\nu}{2m+1}\left\langle \omega^{2m-2}\left[\left[\frac{\partial\omega_i}{\partial x_j}\right]^2 + 2(m-1)\left[\frac{\partial\omega}{\partial x_j}\right]^2\right]\right\rangle, \quad (m = 1,2,\ldots) \ . \quad (20)$$

Let us note that all even moments of the γ field decay initially even if vorticity increases because of vortex stretching. Numerical experiments can give us a detailed evolution of the γ field.

For the fluid surface elements in homogeneous turbulent flow we have from (9)

$$\frac{\partial}{\partial t}\langle \sigma^2 \rangle = -2\left\langle \frac{\partial v_i}{\partial x_j}\sigma_i\sigma_j \right\rangle, \quad \frac{\partial}{\partial t}\langle \sigma \rangle = -\left\langle \frac{\partial v_i}{\partial x_j}\frac{\sigma_i\sigma_j}{\sigma} \right\rangle \ . \quad (21)$$

In case I at $t = 0$,

$$\langle \sigma_i\sigma_j \rangle = \langle \sigma_i\sigma_j\sigma^{-1} \rangle = \tfrac{1}{3}\delta_{ij} \ . \quad (22)$$

Having in mind incompressibility and initial statistical independence of σ_i from turbulence, we see that the right-hand sides of Eqs. (21) are zero at $t = 0$. Time differentiation of the first part of Eq. (21) gives

$$\frac{\partial^2}{\partial t^2}\langle \sigma^2 \rangle = -2\left\langle \frac{\partial}{\partial t}\left[\frac{\partial v_i}{\partial x_j}\right]\sigma_i\sigma_j \right\rangle - 2\left\langle \frac{\partial v_i}{\partial x_j}\frac{\partial(\sigma_i\sigma_j)}{\partial t} \right\rangle \ . \quad (23)$$

The first term in (23) is zero at $t=0$ by the same argument that was used above. The second term in (23) is evaluated with the use of (9), (22), homogeneity, and incompressibility. A similar procedure is applied to the second part of Eqs. (21). At $t=0$ we get

$$\frac{\partial^2}{\partial t^2}\langle\sigma^2\rangle=\frac{2}{3}\left\langle\left[\frac{\partial v_i}{\partial x_j}\right]^2\right\rangle,\quad \frac{\partial^2}{\partial t^2}\langle\sigma\rangle=\frac{4}{15}\left\langle\left[\frac{\partial v_i}{\partial x_j}\right]^2\right\rangle . \tag{24}$$

Thus, in case I surface elements are spreading, at least initially. So far we have not assumed isotropy of turbulence. For isotropic turbulence, it was shown by a different approach [11,12] that surface elements, distributed initially independently of turbulence, will not shrink in a finite time: $\langle\sigma^2\rangle\geq 1$ in our notation. A detailed time dependence of $\langle\sigma\rangle$ and $\langle\sigma^2\rangle$ can be obtained numerically from Eq. (9) with a given realization of the velocity field.

Now consider case II, when initially $\sigma_i=\omega_i\omega^{-1}$. All moments of the γ field are now positive at $t=0$:

$$\langle\gamma^n\rangle=\langle\omega^n\rangle,\quad n=1,2,\ldots . \tag{25}$$

Equation (12) gives

$$\frac{\partial}{\partial t}\langle\gamma^n\rangle=n\nu\langle\omega^{n-2}\omega_i\Delta\omega_i\rangle . \tag{26}$$

With the use of homogeneity,

$$\frac{\partial}{\partial t}\langle\gamma^n\rangle=-n\nu\left\langle\omega^{n-2}\left[\left[\frac{\partial\omega_i}{\partial x_j}\right]^2+(n-2)\left[\frac{\partial\omega}{\partial x_i}\right]^2\right]\right\rangle . \tag{27}$$

For $n=1$, Eq. (27) can be written in the form

$$\frac{\partial}{\partial t}\langle\gamma\rangle=-\frac{\nu}{2}\left\langle\frac{1}{\omega}\left[\frac{\omega_i}{\omega}\frac{\partial\omega_j}{\partial x_k}-\frac{\omega_j}{\omega}\frac{\partial\omega_i}{\partial x_k}\right]^2\right\rangle . \tag{28}$$

Thus all moments of the γ field are decaying initially. Let us note that for even moments the relative rate of decay

$$\lambda_{2m}=-\langle\gamma^{2m}\rangle^{-1}\frac{\partial}{\partial t}\langle\gamma^{2m}\rangle \tag{29}$$

at $t=0$ is the same for cases I and II.

For surface elements in case II we have at $t=0$, from (21),

$$\frac{\partial}{\partial t}\langle\sigma^2\rangle=-2\langle\alpha(\boldsymbol{\omega})\rangle,\quad \frac{\partial}{\partial t}\langle\sigma\rangle=-\langle\alpha(\boldsymbol{\omega})\rangle . \tag{30}$$

Here $\alpha(\boldsymbol{\omega})$ is the conditionally averaged deformation rate with fixed vorticity [3]. For isotropic turbulence, α depends only on the magnitude of vorticity ω, and numerical experiments [5] show that $\alpha(\omega)>0$ for all ω. Thus, in case II, surface elements are shrinking initially. This is expected because vortex stretching in incompressible fluid requires reduction of the cross section.

In a more general situation (case III) we consider simultaneously three elements $\sigma_i^\alpha(\alpha=1,2,3)$, which are initially orthogonal:

$$\sigma_i^\alpha\sigma_i^\beta=\delta_{\alpha\beta}\ \text{ at } t=0 . \tag{31}$$

The vector γ field is defined by

$$\gamma^\alpha=\omega_i\sigma_i^\alpha . \tag{32}$$

From (9), (10), and (32) we have equations

$$\frac{d\sigma_i^\alpha}{dt}=-\frac{\partial v_j}{\partial x_i}\sigma_j^\alpha,\quad \frac{d\gamma^\alpha}{dt}=\nu\sigma_i^\alpha\Delta\omega_i . \tag{33}$$

We can assume that

$$\mathrm{Det}\{\sigma_i^\alpha\}\neq 0 . \tag{34}$$

This means that surface elements, which are initially orthogonal (31), remain independent in the sense that they do not have the same line of intersection. A violation of (34) is associated with a singularity in the velocity gradient. It is known [13] that for the Navier-Stokes equation the Hansdorff dimension of the space-time set of possible singularities is no more than 1. Under condition (34), we can decompose the vorticity field in terms of the vector γ field:

$$\omega_i=\sum_\alpha c_i^\alpha\gamma^\alpha,\quad c_i^\alpha\sigma_i^\beta=\delta_{\alpha\beta},\quad c_i^\alpha=(\sigma_i^\alpha)^{-1} . \tag{35}$$

This formulation of hydrodynamics in terms of inviscid invariants of motion γ^α deserves a more detailed study in the future. Numerical experiments can substantially extend this statistical analysis of microcirculations.

This work is supported by the U.S. Department of Energy under Grant No. DE-FG03-91-ER14188 with Dr. Oscar P. Manley as program manager and by the Office of Naval Research under Grants No. ONR-N00014-92-J-1610 and No. ONR-14-94-1-0040 with Dr. Edwin P. Rood as program manager.

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