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Coupled maps on trees

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We study coupled maps on a Cayley tree, with local (nearest-neighbor) interactions, and with a variety of boundary conditions. The homogeneous state (where every lattice site has the same value) and the node-synchronized state (where sites of a given generation have the same value) are both shown to occur for particular values of the parameters and coupling constants. We study the stability of these states and their domains of attraction. Since the number of sites that become synchronized is much higher compared to that on a regular lattice, control is easier to achieve. A general procedure is given to deduce the eigenvalue spectrum for these states. Perturbations of the synchronized state lead to different spatiotemporal structures. We find that a mean-field-like treatment is valid on this (effectively infinite dimensional) lattice.

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I. INTRODUCTION

Coupled map lattices (CML's) have been explored in a variety of contexts in recent years, particularly as prototypes of spatially extended systems. These are simple models wherein both space and time play a role; furthermore, it is anticipated that the insight gained over the past two decades in studying low-dimensional nonlinear dynamical systems can be profitably exploited in providing an understanding of such complex systems [1].

The phenomenology displayed by coupled maps on regular one- and two-dimensional lattices has been extensively studied by Kaneko [2]. In addition, CML's have been used to model a wide variety of complex phenomena, such as the study of the kinetics of phase ordering processes [3], crystal growth [4], neuronal systems [5], optical fibers [6], and pattern formation [7]. Chaté and Manneville have also used CML's to model spatiotemporal intermittency [8]. A route to a spatiotemporally inhomogeneous state through wavelength doubling bifurcations has also been recently identified [9].

In this paper, we study coupled maps on a Cayley tree. This lattice is embedded in infinite dimensions and thus should give some indication of CML phenomenology in higher dimensions [10]. Although the Cayley tree (sometimes termed the Bethe lattice) is an idealized hierarchical lattice with no immediate physical application, it is convenient for study since there are no closed loops. Furthermore, the Bethe lattice is the simplest sort of branching media model encountered in many physical processes.

Previous studies of coupled map systems (except for a study by Cosenza and Kapral [11] of CML's on a Sierpínski gasket) have largely been carried out with local coupling on regular lattices in one and two dimensions or with global coupling, in which case there is no notion of lattice geometry. Our motivation in choosing the Bethe lattice is twofold. Apart from the mathematical convenience, it is worth considering that in many physical situations the medium supporting dynamics could be nonuniform; in cases like chemical reactions in porous media or on diffusion-limited-aggregation clusters, heterogeneity can lead to hierarchical structures [12]. We note that hierarchical structures have long since been studied in spatiotemporal systems like neural nets, also because of their exponentially higher storage capacity [13].

A related question of some current interest is the control of macroscopically cascaded dynamical systems. The synchronization of a large set of oscillators connected in series [14–17] in a given geometry and with particular boundary conditions is directly related to the problem of whether a similar CML can support a synchronized state [18]. We address this problem and show below that the criterion in CML's for a synchronized (but chaotic) state to be stable is that only *one* Lyapunov exponent be positive and all the rest negative. In both cases, i.e., synchronization of the coupled oscillators or of coupled maps, the essence of the problem lies in the nature of the eigenvalues and eigenvectors of the interaction matrix. In the present work we deal with the situation of asymmetric coupling that is easily obtained in experiments [19].

The plan of this paper is as follows. We define our model and the boundary conditions, and show that the stability of synchronized states depends on the spectrum of eigenvalues of the interaction matrix. This is related to the connectivity matrix for the lattice and has a singular-continuous structure. Different patterns can arise from the secondary instabilities. We note that for coupled piecewise linear maps the study of the interaction matrix gives the whole Lyapunov spectrum, which is related to the correlation length.

In several recent studies of globally coupled maps [2,20], a breakdown of the "law of large numbers" has been observed. We find that for this system, even with local coupling, the mean-field description is valid in the macroscopic limit. This is probably related to the infinite dimensional nature of the lattice.

II. MODEL

We study CML's on a Cayley tree with coordination number 3, which we construct in stages as follows. At the first level, there are three branches, each of which splits in two and then further bifurcates, and so on. Each site thus has one parent and two daughters, except for the origin, which has no parent site and three daughters, and boundary sites, which have no daughters. Each site on the lattice is assigned a variable x, which evolves according to a deterministic rule depending on its own value and the value of the nearest neighbors. The evolution rule is taken to be the following:

$$x(i,t+1) = h_0 f(x(i,t)) + h_p f(x(i_p,t)) + h_d \sum_{j \in i_d} f(x(j,t)) , \tag{1}$$

where f is the function that determines how the lattice variables evolve. A common choice for f is the logistic map $f(x) = \mu x(1-x)$. The notation above is as follows: i_p is the parent of site i, and i_d are the daughters. h_0, h_p, and h_d are coupling constants and are taken such that $h_0 + h_p + 2h_d = 1$. Thus the evolution is contained in the same phase space as that of a single map, e.g. [0,1] in the case of a logistic map. We also assume that the couplings are positive, though most of our results do not explicitly require this.

The evolution rules for the origin and for boundary sites are somewhat different since the former does not have a parent and the latter do not have daughters. For the origin,

$$x(0,t+1) = o_0 f(x(0,t)) + o' \sum_{j \in 0_d} f(x(j,t)) , \tag{2}$$

and for the boundary sites,

$$x(k,t+1) = b_0 f(x(k,t)) + b' f(x(k_p,t)) , \tag{3}$$

with different choices for o_0, b_0, o', b' to ensure that the dynamics remains in the same phase space.

Each point on the Bethe lattice can be indexed by a string $a = \{a_1, a_2; \ldots, a_i\}_i$, where i denotes generation; a_1 can take the single value $a_1 = 0$, a_2 can be 0, 1, or 2; and for $j > 2$, the a_j's take two values, $a_j = 0$ or 1. With this notation, it is possible to assign a unique number $n(A) = n(a_1, \ldots, a_i)$ to each lattice point.

$$n(a_1, a_2, \cdots, a_i) = g(i-1) + a_i + 2a_{i-1} + 2^2 a_{i-2} + \cdots + 2^{i-2} a_2 + 1 , \tag{4}$$

where

$$g(i) = 1 + 3(2^{i-1} - 1) \tag{5}$$

is the total number of sites at the ith generation and g(0)=0. (See Fig. 1.) We can thus formally write

$$X(t+1) = IF(X(t)) , \tag{6}$$

where $F(X(t))$ is a column matrix, $X(t)$ is an array of variable values assigned to the lattice points arranged in ascending order of $n(A)$, and I is the interaction matrix. For example, for the Cayley tree with three generations, i.e., ten sites, the equation above reduces to

$$X(t+1) = \begin{pmatrix} o_0 & o' & o' & o' & 0 & 0 & 0 & 0 & 0 & 0 \\ h_p & h_0 & 0 & 0 & h_d & h_d & 0 & 0 & 0 & 0 \\ h_p & 0 & h_0 & 0 & 0 & 0 & h_d & h_d & 0 & 0 \\ h_p & 0 & 0 & h_0 & 0 & 0 & 0 & 0 & h_d & h_d \\ 0 & b' & 0 & 0 & b_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b' & 0 & 0 & 0 & b_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b' & 0 & 0 & 0 & b_0 & 0 & 0 & 0 \\ 0 & 0 & b' & 0 & 0 & 0 & 0 & b_0 & 0 & 0 \\ 0 & 0 & 0 & b' & 0 & 0 & 0 & 0 & b_0 & 0 \\ 0 & 0 & 0 & b' & 0 & 0 & 0 & 0 & 0 & b_0 \end{pmatrix} F(X(t)) , \tag{7}$$

where

$$F(X(t)) = (f(x(1,t)), f(x(2,t)), f(x(3,t)), \ldots, f(x(g(k),t)))^T , \tag{8}$$

where T denotes transpose. Note that the interaction matrix I is such that the evolution can be written as

$$x(l,t+1) = \sum_j I_{lj} f(x(j,t)) , \tag{9}$$

where the sites have been labeled by the unique number $n(A)$.

III. COHERENT PATTERNS AND THEIR STABILITY

Inspection of the symmetries of the lattice allow for the determination of allowed coherent patterns. It is easy to verify that if one starts with the pattern in which all the points at each generation i are assigned the same value $z_i(t)$ at time t, the nature of the pattern cannot change in time since points at the same generation have equivalent

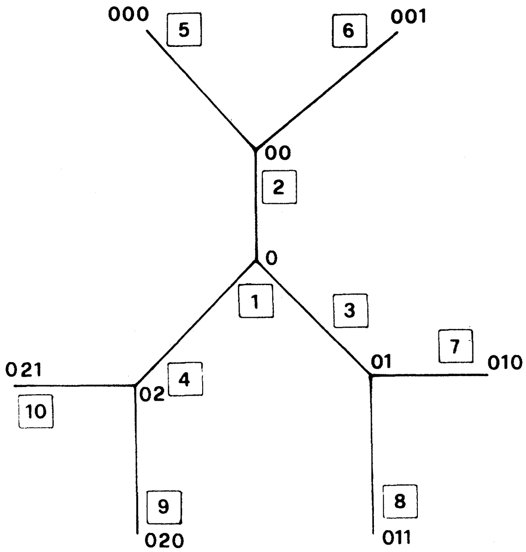

FIG. 1. Cayley tree with three generations and the labeling scheme.

evolution rules and remain synchronized. With parameters

$$o_0 + 3o' = b_0 + b' = h_0 + h_p + 2h_d = 1 \ , \qquad (10)$$

another simple pattern is possible. This is also node-homogeneous, with $z_i = z$ for all i; i.e., all the points on the lattice are synchronized since the evolution is essentially that of a single map f.

These "allowed" patterns can be observed in practice if and only if they are linearly and convectively stable against small perturbations. In the present work, we have mainly dealt with linear stability analysis of this system in the stationary frame, and while we have not analyzed convective stability, numerical experiments suggest that no extra instabilities other than the ones in an equivalent one-dimensional model creep in. This directly evolves from the fact that there are no loops on the lattice; there is only one direction in which instabilities can be enhanced in a moving frame of reference, and these are the same as in the equivalent one-dimensional model.

For the linear stability analysis the eigenvalues and eigenvectors of the matrix $J = \lim_{\tau\to\infty} J(\tau)$, where $J(\tau) = J_\tau \cdots J_2 J_1$, are (asymptotically) relevant. The Jacobian matrix at time t, i.e., J_t, is given by $J_t(i,j) = I(i,j) f'(x_j(t))$ and $x_j(t) = x(t)$ for all j. Thus the Jacobian matrix is $J = \lim_{\tau\to\infty} [I]^\tau f'(x_\tau) f'(x_{\tau-1}) \cdots f'(x_1)$. The eigenvalues of J are $\lim_{\tau\to\infty} \lambda_i^\tau$, where $\lambda_i = v_i \lambda$, where v_i, $i = 1, 2, \ldots, g(k)$ are the eigenvalues of the interaction matrix I and $\lambda = \lim_{t\to\infty} |f'(x(t)) f'(x(t-1)) \cdots f'(x(1))|^{1/t}$. The relevant eigenvectors are those of I, and the problem reduces to a study of the eigenvalues and eigenvectors of the interaction matrix.

The fact that coherent patterns are allowed [by the condition in Eq. (10)] implies that a right eigenvector of the interaction matrix is $e_1 = (1, 1, \ldots, 1)$. This is a characteristic of row stochastic matrices and corresponds to the eigenvalue λ for the product of the J's. From Greshgorin's theorem [22] this is the largest eigenvalue. Consider a small deviation, $\Delta_0 = (\delta_1, \delta_2, \ldots, \delta_{g(k)})$ from the homogeneous pattern $(x, x, \ldots)$. We can reexpress Δ_0 on the basis of eigenvectors $e_1, e_2, \ldots, e_{g(k)}$ as

$$\Delta_0 = a_1 e_1 + a_2 e_2 + \cdots + a_{g(k)} e_{g(k)} . \qquad (11)$$

After t iterations the deviation from the homogeneous condition will be

$$\Delta_t = a_1 \lambda_1^t e_1 + a_2 \lambda_2^t e_2 + \cdots + a_{g(k)} \lambda_{g(k)}^t e_{g(k)} . \qquad (12)$$

If the only eigenvalue with modulus greater than unity is $\lambda_1 = \lambda$ and $|\lambda v_j| < 1$ for $j > 1$, i.e., the rest are less than unity in magnitude, then for large enough t we can write

$$\Delta_t \simeq a_1 \lambda_1^t e_1 \ . \qquad (13)$$

The perturbation grows along the direction $e_1 = (1, 1, \ldots, 1)$, and any random deviation will eventually be homogenized.

Thus the necessary (though not sufficient) condition for the synchronized pattern to exist (and evolve chaotically in time) is that λ_1 be the only eigenvalue greater than unity and all others be less than unity in magnitude; a linearly stable coherent pattern—in the infinite lattice limit—therefore requires a finite *gap* in the eigenvalue spectrum of the interaction matrix.

The interaction matrix is analogous to the tight-binding Hamiltonian on the Bethe lattice [20], although the eigenvectors are different (since the matrix is not necessarily symmetrical or Hermitian). However, using similar arguments [20], one can see that all the sites at a given generation are equivalent in the sense that, if sites at every generation are synchronized, this pattern will continue to exist in the absence of small perturbations or noise since the evolution rule is the same for all of them. Furthermore, one can see that if any two sites that have the same parent are interchanged along with their subtrees, the system is left unchanged. Using the equivalence of all points at a generation and the permutation symmetries of the lattice, the similarity transformation that will block-diagonalize the interaction matrix can be deduced to be

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} . \qquad (14)$$

(The first three vectors follow from the fact that lattice points at each generation are equivalent. The fourth and sixth vectors simply represent the two linearly independent and mutually orthogonal interchanges possible between points at the second generation $[(1, 0, -1) \pm$

$(0,1,-1)]$, while the fifth and seventh vectors are similar interchanges within siblings with the phase derived from the parent site. The last three vectors arise from the interchange among the siblings of the same parent.)

Thus the block-diagonalizing matrix is written using permutation symmetries of the underlying lattice; the blocks are as follows:

$$\begin{pmatrix} o_0 & 3o' & 0 \\ h_p & h_0 & 2h_d \\ 0 & b' & b_0 \end{pmatrix} . \tag{15}$$

The two doubly degenerate eigenvalues are the eigenvalues of the matrix given below. They correspond to the fact that one can have two independent permutations in the three branches at the first node:

$$\begin{pmatrix} h_0 & 2h_d \\ 3b' & b_0 \end{pmatrix} . \tag{16}$$

Finally we have the triply degenerate eigenvalue b_0 (reflecting the fact that one can have permutations among the daughters of any of the three branches at the second node without affecting the matrix). One can see that the consecutive blocks giving eigenvalues are just like earlier blocks except that the first row and column of the matrix are removed. This construction can be trivially extended to a matrix of higher order. The matrix $S^{-1}IS$ is block diagonal.

This scheme can be generalized to higher dimensions and the diagonalizing matrix for the kth stage can be deduced as follows. Specifying the nonzero components of the column vectors [in the notation of Eq. (4) to denote the components] the first k vectors are as follows:

$$\begin{aligned} &v^1_{n(a_1)} = 1, \\ &v^2_{n(a_1,a_2)} = 1, \\ &v^3_{n(a_1,a_2,a_3)} = 1, \end{aligned} \tag{17}$$

..., and

$$v^k_{n(a_1,a_2,a_3,\ldots,a_k)} = 1 \tag{18}$$

[e.g., the first three columns of the matrix defined in Eq. (14)]. Then we have two sets of $k-1$ vectors. One is

$$\begin{aligned} &\{v^{k+1}_{n(0,1)} = v^{k+1}_{n(0,2)} = 1, v^{k+1}_{n(0,3)} = -2\}, \\ &\{v^{k+2}_{n(0,1,a_3)} = v^{k+2}_{n(0,2,a_3)} = 1, v^{k+2}_{n(0,3,a_3)} = -2\}, \end{aligned} \tag{19}$$

..., and

$$\begin{aligned} \{v^{k+(k-1)}_{n(0,1,a_3,\ldots,a_k)} &= v^{k+(k-1)}_{n(0,2,a_3,\ldots,a_k)} \\ &= 1, v^{k+(k-1)}_{n(0,3,a_3,\ldots,a_k)} = -2\}. \end{aligned} \tag{20}$$

[See, e.g., the fourth and fifth columns in Eq. (14).] The other set is

$$\begin{aligned} &\{v^{2k}_{n(0,1)} = 1, v^{2k}_{n(0,2)} = -1\}, \\ &\{v^{2k+1}_{n(0,1,a_3)} = 1, v^{2k+1}_{n(0,2,a_3)} = -1\} , \end{aligned} \tag{21}$$

..., and

$$\{v^{2k-1+(k-1)}_{n(0,1,a_3,\ldots,a_k)} = 1, v^{2k-1+(k-1)}_{n(0,2,a_3,\ldots,a_k)} = -1\}. \tag{22}$$

[See, e.g., the sixth and seventh columns in Eq. (14).] The next three sets of $k-2$ vectors each are given as follows. The first is of the type

$$\begin{aligned} &\{v^{3k-1}_{n(0,1,1)} = 1, v^{3k-1}_{n(0,1,2)} = -1\}, \\ &\{v^{3k}_{n(0,1,1,a_4)} = 1, v^{3k}_{n(0,1,2,a_4)} = -1\}, \end{aligned} \tag{23}$$

..., and

$$\{v^{3k-2+(k-2)}_{n(0,1,1,a_4,\ldots,a_k)} = 1, v^{3k-2+(k-2)}_{n(0,1,2,a_4,\ldots,a_k)} = -1\}. \tag{24}$$

The second is

$$\begin{aligned} &\{v^{4k-3}_{n(0,2,1)} = 1, v^{4k-3}_{n(0,2,2)} = -1\}, \\ &\{v^{4k-2}_{n(0,2,1,a_4)} = 1, v^{4k-2}_{n(0,2,2,a_4)} = -1\}, \end{aligned} \tag{25}$$

..., and

$$\{v^{4k-4+(k-2)}_{n(0,1,1,a_4,\ldots,a_k)} = 1, v^{4k-4+(k-2)}_{n(0,1,2,a_4,\ldots,a_k)} = -1\}. \tag{26}$$

The last set is

$$\{v^{5k-5}_{n(0,3,1)} = 1, v^{5k-5}_{n(0,3,2)} = -1\} ,$$

and so on. [The last three columns in Eq. (14) are v^{3k-1}, v^{4k-3}, and v^{5k-5}. For $k > 3$ newer sets will appear.] Now we will have sets of vectors that will give blocks of size $k-3$. The next six blocks of $k-3$ vectors arise from permutations between the points on the fourth generation and their descendants and are of the same type as the three sets of $k-2$ vectors mentioned above, which result from the three independent permutations possible between the six points on the third generation. One can continue this scheme until reaching the boundary. The number of points on the boundary is $g(k) - g(k-1) = 3 \times 2^{k-2}$, and $[g(k) - g(k-1)]/2 = 3 \times 2^{k-3}$ permutation vectors are possible [see Eq. (5)], which will give a block of size 1 with the same degeneracy as the number of permutations possible on the boundary.

For boundary conditions in which $o_0 = h_0 + h_p$, $o' = 2h_d/3$, $b_0 = h_0 + 2h_d$, $b' = h_p$, at stage k, the first block of the block-diagonal form is

$$\begin{pmatrix} h_0+h_p & 2h_d & 0 & \cdots & 0 & 0 \\ h_p & h_0 & 2h_d & & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & & h_0 & 2h_d \\ 0 & 0 & 0 & \cdots & h_p & h_0+2h_d \end{pmatrix} , \tag{27}$$

which exploits the equivalence symmetry of all the sites at a given generation. The second block, which exploits the permutation symmetry of the points on the first generation, is

$$\begin{pmatrix} h_0 & 2h_d & 0 & \cdots & 0 & 0 \\ h_p & h_0 & 2h_d & & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & & h_0 & 2h_d \\ 0 & 0 & 0 & \cdots & h_p & h_0+2h_d \end{pmatrix} , \tag{28}$$

and so on. The last two blocks are

$$\begin{pmatrix} h_0 & 2h_d \\ h_p & h_0+2h_d \end{pmatrix}, \quad \begin{pmatrix} h_0+2h_d \end{pmatrix} . \tag{29}$$

The first block of order k appears once in the block-diagonal form, the second of order $k-1$ appears twice, and next blocks of order $k-n$ $(k-1 \geq n \geq 2)$ appears $3 \times 2^{n-2}$ times. The first k eigenvalues are therefore nondegenerate; then $k-1$ eigenvalues are doubly degenerate, $k-n$ eigenvalues have degeneracy $3 \times 2^{n-2}$ for $k-1 \geq n \geq 2$, etc.

From the structure of the matrices and their degeneracies, one sees that for the Cayley tree with one more generation, the $k-1$ degenerate blocks are retained (with, however, the doubly degenerate block becoming triply degenerate and other blocks doubling their degeneracy) with an additional block that has degeneracy 2. The block corresponding to nondegenerate eigenvalues is, however, completely changed. The density of states has to be singular continuous since the new eigenvalues that are created have a lower degeneracy; the eigenvalue spectrum is a sum of δ peaks and is nowhere differentiable, as is common in hierarchical systems [11,21].

In the situation where the synchronized state is linearly stable in the stationary frame, the typical degeneracy structure of eigenvalues is as shown in Fig. 2 for the parameters and boundary conditions as discussed below. The structure is generic if the system is linearly stable, but the width of the gap varies with the parameters. For piecewise linear maps; e.g., $f(x) = rx \bmod y$, the Jacobian is constant in time and the spectrum of eigenvalues of the interaction matrix determines the Lyapunov spectrum of the CML's, and thus (via the Lyapunov dimension) the fractal dimension.

We can see from the degeneracy structure that about a quarter of the eigenstates have their support fully from the boundary. The next layer is approximately half of this number, and so on, with the number of states that have their support up to a length l from the boundary reducing exponentially. This is in keeping with the expectation that the rate must be faster than that in finite dimensional spaces where the number of modes with wave number $|\kappa| < \kappa$ is proportional to κ^d.

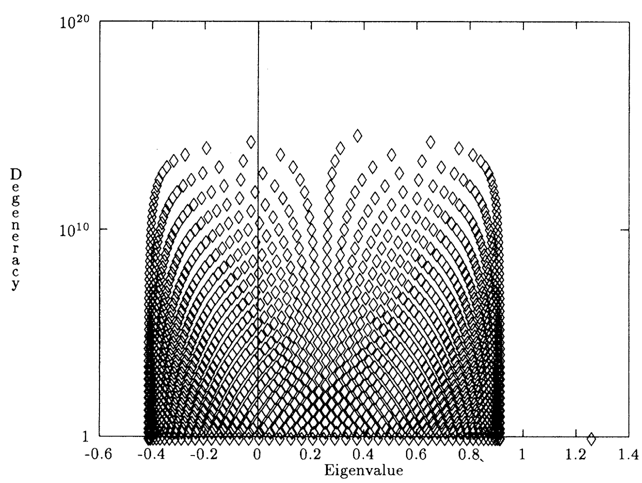

FIG. 2. Eigenvalues and their degeneracies for the synchronized state for $k = 50$ generations. The parameters are $h_p = 0.7$, $h_0 = 0.2$, $h_s = 0.05$, and $\lambda = 1.26$. One can clearly see the gap that separates a nondegenerate eigenvalue greater than unity and all the others below unity. Degeneracies are on a logarithmic scale for clarity.

Note that in the block-diagonal matrix, the blocks are tridiagonal and (for positive couplings) all elements are positive. Such a matrix can be transformed to symmetric form [22] and thus all its eigenvalues are real; there can be no Hopf bifurcation leading to the instability of a synchronized state.

Since the consecutive blocks are the principal (tridiagonal) submatrices of the earlier block, the eigenvalues are interlaced [23]. In other words, the bounds for the eigenvalues of the lower block are contained in the bounds for the eigenvalues for the higher block, and it is enough to consider the first two blocks in order to study the stability of a spatially synchronized state.

For the first block, the nondegenerate eigenvalues are given by $h_0 + h_p + 2h_d$, which is set to 1 by definition, and the other $k-1$ eigenvalues are $h_0 + 2\sqrt{2h_d h_p}\cos(\theta)$, where $\theta = 2\pi i/k$, $i = 1, 2, \ldots, k-1$. The eigenvalues will have a gap if $2h_d \neq h_p$.

Consider the second block of order $m = k-1$, which is tridiagonal and can be symmetrized by using a similarity transformation involving a diagonal matrix with elements $D_{i,i} = [\sqrt{2h_d/h_p}]^{i-1}$. This yields a tridiagonal matrix O such that the diagonal elements remain unchanged and all the elements on upper and lower diagonal are $\sqrt{2h_d h_p}$. Using Greshgorin's theorem [22] again, one can see that the largest eigenvalue cannot exceed $h_0 + 2\sqrt{2h_p h_d}$ if $h_p > 2h_d$. As explained above, the analysis of the first two blocks suffices to explore the stability of the synchronized state and thus the other blocks do not modify the gap in the eigenvalue spectrum of the first block if $h_p > 2h_d$.

Aranson, Golomb, and Sompolinsky [14] consider asymmetrically coupled one-dimensional (1D) chains with open boundary conditions where there is a convective instability of synchronized patterns; perturbations *grow* in the moving frame of reference, destroying macroscopic coherence. As we have shown above, under these conditions macroscopic chaos is linearly stable in a stationary frame also on the Cayley tree. However, the difficulty in synchronizing large systems is less pronounced in this case. Because of the ultrametric topology, much larger systems can be synchronized under the conditions above. With open boundary conditions and asymmetrical coupling, coherence is more easily established in the present case. For example, for $h_1 = 0.7$ in one direction and $h_2 = 0.1$ in the other direction, Aranson, Golumb, and Sompolinsky [14] have a coherence length of around 55 for the choice of map $f(x) = a - x^2$, with a value of a such that the eigenvalue for a single map is 1.26. With $h_p = 0.7$, $2h_d = 0.1$, Fig. 2 shows a plot of eigenvalues as a function of degeneracies at these parameter values for 50 generations. One can clearly see the gap between a single nondegenerate eigenvalue above unity and the others below unity. For the above parameters we can easily obtain a coherent pattern for $k = 20$ with random initial conditions; a CML with $\approx 10^6$ sites is easily synchronized [24] to within 10^{-4}, even under single precision (16 binary digits) evolution. This is in stark contrast to one-dimensional 1D coupled CML, which has a coherence length of about 55 sites. This example is a good illustration of the dramatic increase in stability with hierarchical connectivity.

To check that no other instabilities than the ones expected from an equivalent one-dimensional model come into the picture, we looked at the function $f(x) = 1.39\, x \bmod 1$ with the same choice of coupling constants as above. Here the coherence is within 10^{-5} for the first six sites on a one-dimensional lattice, and even on the Cayley tree it is maintained for six generations. This is expected since there are no closed loops and the only direction in which the instabilities can *flow and grow* is the one from the center to the boundary. However, we can see that since the number of sites synchronized is exponentially higher on a tree with equivalent generations than on a one-dimensional lattice, an exponentially larger number of sites are synchronized on trees at equivalent parameters. The base of the exponent is related to the number of branches.

Auerbach [15] has shown that one can circumvent the difficulty arising from convective instabilities on a 1D lattice by using system-size dependent feedback control. In essence, we achieve the same ends through a change in geometry, without extra controls. The boundary conditions $o_0 = b_0 = h_0$, $o_p = 3b_d = h_p + 2h_s$ also give the same result, which indicates that some more variants are possible for open boundary conditions and asymmetric coupling.

Now consider the node-homogeneous pattern. The stability matrix is given by

$$J = \prod_{t=1}^{\infty} J_t \ . \tag{30}$$

Thus analysis of the eigenvalues of the product of matrices is reduced to the analysis of the eigenvalues of the product of blocks. This is a great simplification since instead of considering matrices of order 2^k, where k is the number of generations, we only need to consider k matrices of order k and below. The analysis of the Jacobian matrix reduces to analysis of the matrices

$$\prod_{t=1}^{\infty} \begin{pmatrix} o & 3o' & 0 & \cdots & 0 & 0 \\ h_p & h_0 & 2h_d & & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \ldots & h_0 & 2h_d \\ 0 & 0 & 0 & \cdots & b' & b_0 \end{pmatrix} \begin{pmatrix} f'[z_1(t)] \\ f'[z_2(t)] \\ \vdots \\ f'[z_{k-1}(t)] \\ f'[z_k(t)] \end{pmatrix},$$

$$\prod_{t=1}^{\infty} \begin{pmatrix} h_0 & 2h_d & \cdots & 0 & 0 & \\ h_p & h_0 & 2h_d & & 0 & 0 \\ \vdots & & & & & \vdots \\ & 0 & 0 & & h_0 & 2h_d \\ & 0 & 0 & \cdots & b' & b_0 \end{pmatrix} \begin{pmatrix} f'[z_2(t)] \\ f'[z_3(t)] \\ \vdots \\ f'[z_{k-1}(t)] \\ f'[z_k(t)] \end{pmatrix}, \tag{31}$$

and so on.

Again the degeneracy structure is the same as for the interaction matrix; the Lyapunov spectrum is the sum of δ peaks and is an everywhere discontinuous function, as for the fully synchronized state (which is a special case of the node-homogeneous structure). We can similarly argue that the condition for stability of the node-homogeneous state (evolving chaotically in time) is that the first block corresponding to the nondegenerate eigenvectors is the only one with eigenvalues of modulus greater than unity, all other blocks having eigenvalues with modulus less than unity. (This is because the first block corresponds to eigenvectors that have a contribution from all the generations, and the contribution from all the points of the same generation is the same.)

A simple example of such stable patterns can be constructed for $f(x) = rx \bmod 1$, with boundary conditions $o_0 = 0, b_0 = 0, o' = h_d, b' = h_p$ and parameters $r = \sqrt{(3)/2}, h_0 = 0,\ h_d = h_p = \frac{1}{3}$. For the Cayley tree with five generations, i.e., 46 sites, it can be shown that the eigenvalues are higher than unity for the first block alone. Numerically, one can easily get node-homogeneous patterns, starting from random initial conditions. Thus the possible coherent patterns are *characteristically* different from those on regular lattices, and the stability analysis is also distinct [26].

IV. INFINITE DIMENSIONAL CHARACTER

We now study the properties of this model, which should reflect the fact that it is embedded in infinite dimensions, where a mean-field-like treatment can be expected to be valid. A collective variable [2, 20] $h(t)$ is defined as

$$h(t) = \frac{1}{g(k)} \sum_{i=1}^{g(k)} x(i,t). \tag{32}$$

where $g(k)$ is the total number of sites on the Cayley tree with k generations, as noted above; $f(x) = \mu x(1-x)$ with $h_0 = 1-\epsilon$, $h_p = h_d = \epsilon/3$, $b_0 = o_0 = h_0$, $b' = \epsilon$, $o' = \epsilon/3$; while the parameter values are $\epsilon = 0.1$ and $\mu = 4$. The return map of this variable, i.e., $h(t+1)$ vs $h(t)$, is a filled ellipse, whose size decreases rapidly with the number of generations. We conjecture that in the macroscopic limit it tends to a fixed point; i.e., though the evolution is chaotic for the system, the collective variable is invariant in time. The mean square deviation of $h(t)$ decays like $1/N$, where N is the number of sites (see Fig. 3), quite unlike the case of globally coupled maps [27], where some reorganization occurs in such a way that the total number of independent degrees of freedom is not linearly proportional to the number of sites. This is not totally unexpected [28] since the values being summed are not independent random variables. This also means that the mean field is not valid in these systems. However, this expectation is fulfilled for the Bethe lattice, although the variables that are being summed are not only not independent but are also not identically distributed; the boundary evolves differently from the bulk, and boundary effects are not negligible in any limit since half the points reside at the boundary. Figure 4 shows the probability distribution of the central sites and the boundary for the above case, and they are clearly different. However, the sum behaves in a way that is expected from the

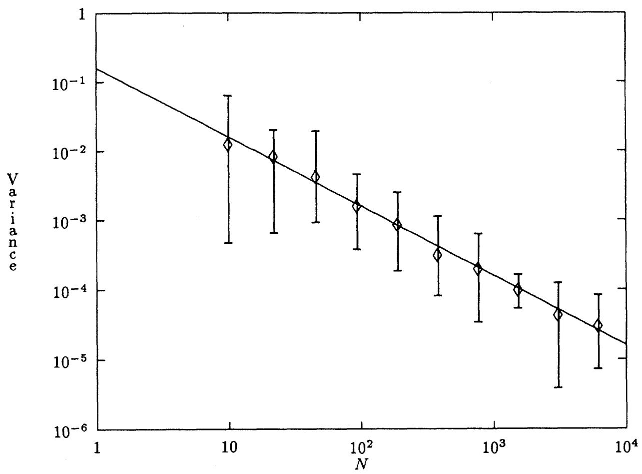

FIG. 3. Standard deviation of the mean field as a function of total number of sites for the CML with function $f(x) = \mu x(1-x)$, with $\mu = 4.0$ and $\epsilon = 0.1$. Similar behavior is also obtained for other values of μ, ϵ.

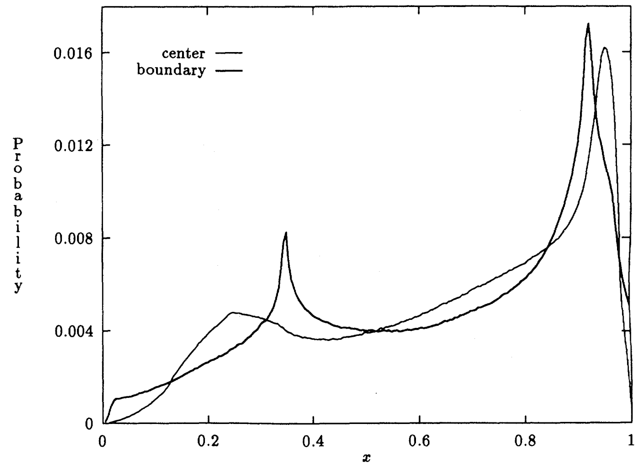

FIG. 4. Probability distributions of the CML's at a central point and at the boundary for the same set of parameters as in Fig. 3. The lattice has $k = 6$, and the first 10^4 iterates are treated as transients. The distribution is clearly very different for these two sites.

sum of *iid* variables. (We have verified that this behavior holds at some other values of the parameters.)

The recovery of the mean field in infinite dimensions is interesting. Chaté and Manneville [29] have found that the mean-field-like approach works better in higher dimensions in spatially extended systems and that connectivity plays a relatively marginal role. Further studies will be necessary to determine the upper critical dimension for this problem. We are exploring this question.

V. CONCLUSIONS

In summary, in this paper we consider a coupled map lattice in an ultrametric space. We show that synchronized but temporally chaotic systems can be stabilized more easily in this space. We present a simple method of obtaining the eigenvalue spectrum for the node-homogeneous and spatially synchronized structures using symmetries. We emphasize different properties that owe their existence to the hierarchical connections and to the infinite embedding dimension.

In an ecological model, Hogg, Huberman, and McGlade [25] have suggested that a hierarchically ordered random system should be more stable compared to unstructured systems, and they argue that the stability should scale like log (*system size*) instead of (*system size*): thus exponentially larger systems should be stabilized in hierarchical organization. Our results are in conformity with this expectation, although the system studied in [25] has a random branching. They have also found that asymmetric interactions give a higher stability. This is also expected from our analysis. It is difficult to find such a naturally occurring system with a clearly demarcated tree structure. However, the properties we are trying to emphasize are qualitatively unchanged as long as the connectivity remains free of loops (and the resulting feedback).

Our results have immediate relevance to the problem of synchronization in chaotic systems, which is currently evoking considerable interest [30]. In one dimension, this problem has been extensively discussed [16, 17]. Our results demonstrate a method of stabilizing large systems and are of practical utility in this context.

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[1] J. P. Crutchfield and K. Kaneko, in *Directions in Chaos*, edited by Hao Bai-Lin (World Scientific, Singapore, 1987), Vol. 2, and references therein.
[2] K. Kaneko, Physica D **34**, 1 (1989); Prog. Theor. Phys. **72**, 480 (1984).
[3] Y. Oono and S. Puri, Phys. Rev. Lett. **58**, 836 (1987); Phys. Rev. A **38**, 434 (1988); **38**, 1542 (1988).
[4] D. A. Kessler, H. Levine, and W. N. Reynolds, Phys. Rev. A **42**, 6125 (1990).
[5] Wei Wang, G. Perez, and H. A Cerdeira, Phys. Rev. E **47**, 2893 (1993).
[6] G. Perez *et al.*, Phys. Rev. A **45**, 5469 (1992).
[7] D. Barkley, in *Nonlinear Structures in Dynamical Systems*, edited by L. Lam and H. C. Moriss (Springer-

Verlag, New York, 1990).
[8] H. Chaté and P. Manneville, Physica D **32**, 409 (1988).
[9] R. E. Amritkar and P. M. Gade, Phys. Rev. Lett. **70**, 3408 (1993); P. M. Gade and R. E. Amritkar, Phys. Rev. E **49**, 2617 (1994).
[10] The number of sites in a d-dimensional lattice with lateral length l scales as l^d in any finite dimensional lattice. On the Cayley tree, this number scales as $\exp(l)$. See for example, R. J. Baxter, *Exactly Solvable Models in Statistical Mechanics* (Academic, New York, 1982).
[11] M. G. Cosenza and R. Kapral, Phys. Rev. A **46**, 1850 (1992).
[12] R. Kopelman, Science **241**, 1620 (1988); and for additional applications, see *The Fractal Approach to Heterogeneous Chemistry*, edited by D. Avnir (Wiley, New York, 1989).
[13] See, e.g., D. Amit, *Modelling Brain Function: The World of Attractor Neural Nets* (Cambridge University Press, Cambridge, 1989).
[14] I. Aranson, D. Golomb, and H. Sompolinsky, Phys. Rev. Lett. **68**, 3495 (1992); see also I. Aranson *et al.*, Nonlinearity **3**, 639 (1990).
[15] D. Auerbach, Phys. Rev. Lett. **72**, 1184 (1994).
[16] J. F. Heagy, T. L. Caroll, and L. M. Pecora, Phys. Rev. E **50**, 1874 (1994).
[17] T. L. Caroll, Phys. Rev. E **50**, 2580 (1994).
[18] In Ref. [16], where the control of cascaded dynamical systems is considered, the case of vector coupling in free-end, symmetrical one-dimensional chains of oscillators is worked out, and the condition for synchronization is a special case of the condition for synchronization in [14]. We would also like to point to G. A. Johnson *et al.*, Phys. Rev. E **51**, 1625 (1995), where the case of convectively unstable open flow is considered in coupled diode resonators.
[19] See, e.g., N. F. Rulkov *et al.*, Phys. Rev. E **51**, 980 (1995).
[20] M. S. Chen *et al.*, J. Chem. Phys. **46**, 1850 (1972).
[21] M. G. Cosenza and R. Kapral, Chaos **1**, 99 (1994).
[22] See, e.g., S. Barnett, *Matrices, Methods and Applications* (Oxford University Press, Oxford, 1990)
[23] A. Ostaszewski, *Advanced Mathematical Methods* (Cambridge University Press, Cambridge, 1990).
[24] The choice of a in [14] gives a larger eigenvalue than that quoted, so we chose $a = 1.5$, corresponding to the eigenvalue ≈ 1.22. The evolution with map $1.26x$ mod 1 also gives similar results. Note also that the extent of synchronization depends on the noise level in the evolution. Our linear stability analysis merely guarantees the stability of the macroscopic coherent pattern under infinitesimal perturbations in stationary frame.
[25] T. Hogg, B. A. Huberman, and J. M. McGlade, Proc. R. Soc. London, Ser. B **237**, 43 (1989).
[26] Similar coherent structures on regular lattices are discussed in P. M. Gade, Ph.D. thesis, Poona University, 1993 (unpublished).
[27] K. Kaneko, Phys. Rev. Lett. **65**, 1391 (1990).
[28] S. Sinha, Phys. Rev. Lett. **69**, 3306 (1992); M. Grinasty and V. Hakim, Phys. Rev. E **49**, 2661 (1994); A. S. Pikovsky and J. Kurths, Phys. Rev. Lett. **72**, 1644 (1994); G. Perez *et al.*, Physica D **63**, 341 (1993); D. Dominguez and H. Cerdeira, Phys. Rev. Lett. **71**, 3359 (1993).
[29] H. Chaté and P. Manneville, Prog. Theor. Phys. **87**, 1 (1992).
[30] Some of the references are L. Pecora and T. M. Caroll, Phys. Rev. Lett. **64**, 821, (1990); Phys. Rev. A **44**, 2374, (1991); N. Gupte and R. E. Amritkar, Phys. Rev. E **48**, 1620 (1993); K. Pyragas, Phys. Lett. A **181**, 203 (1993). The list is prohibitively long.