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Survival of deterministic dynamics in the presence of noise and the exponential decay of power spectra at high frequency

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Power spectra from continuous-time dynamical systems exhibiting deterministic chaos decay exponentially at high frequency. Power spectra from noisy systems decay via a power law. Since noise is always present in real systems, one can, in practice, observe only a finite region of exponential decay before the spectrum flattens into the power-law decay characteristic of noise. Numerical results are presented that show that, in the Lorenz and Rossler models [J. Atmos. Sci. **20**, 130 (1963); Phys. Lett. **57A**, 397 (1976)], the preservation of a portion of the region of exponential decay in the presence of noise is equivalent to the preservation of a portion of the scaling region of the attractor giving the correct correlation dimension. This suggests that the observation of a finite region of exponential decay is a sufficient condition for the dynamics of the system to be essentially deterministic. In addition, theoretical arguments are presented that suggest that preservation of the exponential decay is a sufficient condition for the existence of finite-time shadowing orbits. These results are applied to the numerical simulation of ordinary differential equations leading to the conclusion that the survival of the region of exponential decay in the power spectrum should guarantee that round-off error and truncation error arising from the discretization of time are not changing the dynamics of the simulation from the dynamics of the original ordinary differential equation. It is conjectured that analogous results should hold for the wave number spectrum in spatiotemporally chaotic systems, that is, that the survival of a region of exponential decay in the wave number spectrum should guarantee that truncation error arising from the discretization of space is not fundamentally changing the dynamics of the system. This is shown to be true for the special case of simulations of fully turbulent flows.

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I. INTRODUCTION

It has been generally accepted for some time that the power spectra of time series taken from continuous-time dynamical systems[1] exhibiting deterministic chaos will decay exponentially at high frequency [1]. On the other hand, Sigeti and Horsthemke have proven that the spectrum decays as an inverse even power of the frequency for systems subject to a very broad class of noises [2]. This has led to the suggestion that an examination of the fall-off of the power spectrum at high frequency may provide a way of distinguishing systems exhibiting deterministic chaos from systems subject to noise [3,2].

In practice, there are several problems with doing this. It is impossible, of course, to measure to infinite frequency. More importantly (because the amplitude is falling off exponentially), it is impossible to measure to infinitessimal amplitude. In experimental systems, the size of the region of exponential decay tends to be restricted in amplitude by instrumental noise which sets a floor to the spectrum. This raises the question of what it means to observe a finite region of exponential decay.

A more fundamental problem arises from the fact that noise is ubiquitous, at least on the scale of Brownian motion. Thus, one expects to see a power-law decay in *any* system if one goes to high enough frequency. In practice, one would expect a system with very low noise to exhibit a long region of exponential decay followed, at very high frequencies, by a flattening to the power-law decay characteristic of noise. This again raises the question of the significance of a finite region of exponential decay. It also suggests that the naive dichotomy of deterministic versus stochastic systems must be replaced by more sophisticated distinctions if one is to make meaningful statements about real systems.

In this paper, results are presented from numerical simulations of the Lorenz and Rossler models subject to additive white noise that suggest that the preservation of a part of the deterministic asymptotic (the exponential decay) in the presence of noise is equivalent to the preservation of the "correct" (that is, deterministic) dynamics at longer time scales and larger amplitude scales. In other words, it appears that the exponential decay effectively insulates the deterministic dynamics at larger scales from the noise that must dominate at smaller scales in real systems. This approach to the subject deals directly with the question of the significance of a finite region of ex-

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[1]The use of high-frequency power spectra is restricted to continuous-time dynamical systems because a discrete-time system with a sample time of Δt has a power spectrum that is defined only within the Nyquist interval, $-1/2\Delta t \leq f \leq 1/2\Delta t$. Thus, for discrete-time systems, the notion of a high-frequency asymptotic behavior for the spectrum is not even defined.

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ponential decay and avoids the problem of a naive dichotomy between deterministic and stochastic systems.

In addition to the results of numerical simulations theoretical arguments are presented that suggest that the preservation of a region of exponential decay is equivalent to the existence of a so-called "shadowing" orbit, an orbit of the corresponding deterministic (zero-noise) system that stays within a small distance of a typical orbit of the noisy system for a long period of time. The existence of shadowing orbits would provide an explanation for the preservation of the correct deterministic dynamics at long time and large amplitude scales that has been observed in numerical simulations. Conversely, an argument connecting the existence of shadowing orbits to the preservation of a region of exponential decay provides additional evidence for the thesis that the preservation of a region of exponential decay corresponds to the preservation of the correct deterministic dynamics.

The approach taken in this paper has the advantage of making the results relevant to the question of whether or not simulations of ordinary differential equations are valid. Because the errors introduced in simulating ordinary differential equations on digital computers—round-off and truncation error—can be considered to be white or nearly white noise, the result of Sigeti and Horsthemke [2] applies and we may expect these kinds of error to produce a flattening of the spectrum at high frequencies. Thus, the results presented in this paper suggest that the resolution of a region of exponential decay in a power spectrum from a simulation of a set of differential equations can be taken as an indication that round-off and truncation error are not fundamentally changing the dynamics from those of the "true" (continuous time, infinite precision) system. This may, in fact, be the area in which the results reported here will be the most useful, since numerical simulations can show a dynamic range in the power spectrum of as much as 30 orders of magnitude before the fundamental noise floor set by double precision arithmetic is reached. This provides far more room for detection of a region of exponential decay than one is ever likely to have in experimental systems where one is lucky to achieve seven orders of magnitude of dynamic range in the spectrum.

In Sec. VIII, I show that the resolution of the region of exponential decay in *the wave-number spectrum* is equivalent to a well-known practical criterion for the validity of simulations of fully turbulent flows. This follows from certain well-known constraints on numerical simulation of turbulent flows together with some recent results on the wave-number spectrum of turbulent flows in the far dissipation regime. The fact that results analogous to those presented here for power spectra can be shown to hold for the wave-number spectrum for a particular spatiotemporally chaotic system suggests further support for our results. Conversely, the fact that the results presented for power spectra are quite general suggests that the analogous results for wave-number spectra may apply much more generally than is suggested by the energy cascade arguments used for the case of turbulent flow. The results may, in fact, apply to any system exhibiting spatiotemporal chaos.

The remainder of this paper will proceed as follows. Section II reviews the origins of exponential and power-law decays in deterministic and noisy systems, respectively. This section also discusses the relationship between the smoothness of the time series and the high-frequency fall-off of the power spectrum. Section III briefly reviews the theory of shadowing. Section IV gives an example of the preservation of a region of exponential decay in the presence of noise. Section V presents the argument for a connection between the preservation of a region of exponential decay and the existence of shadowing orbits. Section VI defines the criteria used to determine if deterministic dynamics are being preserved at larger scales and presents the results from numerical simulations of the Lorenz and Rossler models. This section also includes a general discussion of the problem of calculating power spectra accurately in the region of exponential decay. Section VII discusses the question of whether or not the survival of a region of exponential decay can be equivalent to the survival of deterministic dynamics in systems that are extremely sensitive to noise. Section VIII discusses the connection of our results to the question of determining the validity of numerical simulations of differential equations. This section also presents the case of turbulent flow and discusses the possible extension of our results to the wave-number spectrum in spatiotemporally chaotic systems. Conclusions are summarized in Sec. IX.

II. ORIGINS OF EXPONENTIAL AND POWER-LAW DECAYS

A. Exponential decay of power spectra in deterministic systems

The original argument for the exponential decay of power spectra from deterministic systems was given by Frisch and Morf [1]. This section is a review of their argument.

We consider a dynamical system given by a set of ordinary differential equations,

$$\dot{x} = f(x) \ , \tag{1}$$

where x is a vector with n components. We restrict our attention to cases where the functions $f(x)$ are analytic functions of their arguments for finite x. If we further restrict our attention to motion on a bounded attractor, then the components of $x(t)$, $x_i(t)$, are bounded functions of t for all real t, $-\infty \leq t \leq \infty$, and are analytic functions of t for all finite, real t. It is then possible to extend the $x_i(t)$ to analytic functions of complex t. Although the $x_i(t)$ have no singularities on the real t axis, they can—and in general will—have singularities at complex times. It is the locations of these singularities that determine the rate of exponential decay of power spectra at high frequencies.

To show this, we consider the Fourier integral

$$\int_{-\infty}^{\infty} dt\, e^{i\omega t} x_i(t). \tag{2}$$

Under quite general conditions, this integral may be extended to a contour integral by closing with an infinite semicircle in the upper half-plane. It is then possible to deform the part of the contour lying on the real t axis upward until one encounters a singularity at $t = t_*$ (Fig. 1). If the singularity is a pole or essential singularity, one is left with a contour integral around the singularity, giving a contribution to the Fourier integral proportional to the residue of the integrand at t_*. If the singularity is an algebraic branch point, one must also consider the parts of the integral along the branch cut, which may be taken along the direction of the positive imaginary axis. In either case, the final result for the contribution of the singularity to the Fourier integral has the form

$$Ce^{i\sigma\omega}\omega^{\rho}e^{-\tau\omega}, \tag{3}$$

where the singularity is located at $t_* = \sigma + i\tau$ and C and ρ are constants that are independent of ω but dependent on the properties of the singularity. Thus, the contribution of each singularity to the Fourier integral has an amplitude that decreases exponentially with ω at a rate, τ, given by the distance of the singularity from the real t axis. In the limit of large ω, then, the contribution of the singularity closest to the real axis will come to dominate the integral, giving the Fourier integral the form of expression (3) in the high-frequency regime where τ is now the minimum distance from the real time axis to a singularity in the complex time plane. Since the power spectrum is essentially the absolute square of this integral, it will have the form

$$|C|^2\,|\omega|^{2\rho}\,e^{-2\tau\omega}. \tag{4}$$

This does not constitute a proof, in part because we have ignored the fact that we cannot take Fourier transforms of time series from chaotic attractors because the functions $x_i(t)$ do not go to zero at $\pm\infty$. Presumably, in such cases, the result can be converted to a result about power spectra by multiplying the integrand of expression (2) by some analytic windowing function or by putting finite bounds on the integral and taking the limit as the bounds go to infinity. Such a proof would also need

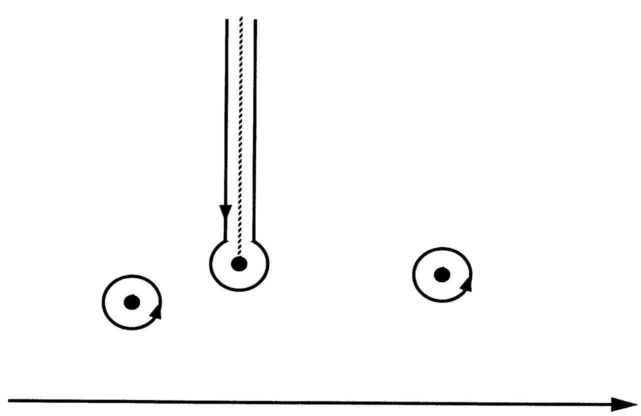

FIG. 1. Moving the contour to surround singularities in the upper half-plane.

to involve conditions on the statistics of the complex-time singularities of $x(t)$ which would probably be impossible to verify for any real system. It is probably for this reason that (to the best of the author's knowledge) no such proof has yet appeared. In practice, exponential decays of power spectra from time series from chaotic systems are commonly observed both in model systems [3,4] and in experiments [5–7] [8, section 3].

B. Power-law decay of power spectra in noisy systems

Sigeti and Horsthemke [2], building on earlier partial results [9,10], have proven that the power spectrum decays as an inverse even power of the frequency for a broad class of noisy systems. Here, we will give an intuitive explanation for the power-law decay and describe the class of systems for which the result holds.

We consider systems satisfying the stochastic differential equation:

$$\dot{x} = f(x) + \sigma(x)\xi\ , \tag{5}$$

where x is an n-dimensional vector, σ is a matrix (the diffusion matrix), and ξ is a vector of independent, unit variance Gaussian white noises.

Because x is a vector, this actually covers a very wide range of systems. It is possible, for example, to include higher order stochastic differential equations such as

$$\ddot{x} = g(x,\dot{x}) + \rho(x,\dot{x})\xi \tag{6}$$

by the well-known trick of turning $\dot{x}$ into an independent variable as in

$$\begin{aligned}\dot{x} &= v\ ,\\ \dot{v} &= g(x,v) + \rho(x,v)\xi.\end{aligned} \tag{7}$$

Note that Eqs. (7), when regarded as a special case of Eq. (5), have a degenerate diffusion matrix, since the original variable x is not driven directly by white noise. Thus, we may expect Eqs. (7) to exhibit behavior which is nongeneric for the general system given by Eq. (5).

It is also possible to include colored noises as in the system

$$\begin{aligned}\dot{x} &= g(x,z)\ ,\\ \dot{z} &= h(z) + \rho(z)\xi.\end{aligned} \tag{8}$$

Once again, these equations have a degenerate diffusion matrix and we may expect nongeneric behavior.

The full generality of the class of systems being considered may be seen from the fact that Eq. (5) is valid for a general n-dimensional diffusion process. Thus, the model includes all stochastic processes that are part of some finite-dimensional Markov process and which have, with probability 1, sample paths that are continuous functions of time.

We may understand, intuitively, what the high-frequency behavior of the power spectrum of one of the components, $x_j(t)$, of $x(t)$ should be via the following argument. We write the Fourier transform of $x_j(t)$ as

$\tilde{x}_j(\omega)$. Then the Fourier transform of the derivative of $x_j(t)$ is just $i\omega\tilde{x}_j(\omega)$. From Eq. (5), it is clear that the Fourier transform of the derivative is composed of two parts, one of which is essentially the Fourier transform of white noise—which is flat. It is clear that this term will dominate for high enough frequency because the other term depends only on $x_j(t)$, which is obtained from ξ_t via a smoothing process (that is, integration). Thus, we should have, as $\omega \to \infty$,

$$i\omega\tilde{x}_j(\omega) \sim C \tag{9}$$

or

$$\tilde{x}_j(\omega) \sim \frac{C}{i\omega}, \tag{10}$$

where C is a constant.

Now, the power spectrum, $S_{x_j}(\omega)$, of x_j is just the average of the absolute square of the Fourier transform, $\tilde{x}_j$, so we should have, as $\omega \to \infty$,

$$S_{x_j}(\omega) \sim \frac{C}{\omega^2} . \tag{11}$$

The only way that this can fail to be the case is if, for a particular x_j, the noise term on the right-hand side of the stochastic differential equation turns out to be identically zero. This will be true, for example, in the two special cases that we considered earlier (colored noise and a second-order SDE). Then, one must differentiate at least once more before one has a derivative with a white-noise part. If one reaches white noise after differentiating n times, we say that x_j is n times removed from white noise. The same arguments that we made above lead to the conclusion that the spectrum will go as

$$S_{x_j}(\omega) \sim \frac{C}{\omega^{2n}} \tag{12}$$

when x_j is n times removed from white noise.

All this can be rigorously defined and proven. See Ref. [2] for details. In fact, it is possible to obtain the coefficients for the full asymptotic series for $S_{x_j}(\omega)$ in the limit $\omega \to \infty$. In particular, if the diffusion matrix, $\sigma(x)$, is diagonal with elements $\sigma_i(x)$ then the power spectrum of $x_i(t)$ goes as $\langle\sigma_i^2\rangle/\omega^2$ (times a numerical factor that depends on the conventions one uses for defining the power spectrum).

C. Smoothness of time series and high-frequency fall-off of power spectra

For the class of systems discussed in the preceding subsection, a stochastic process $x_j(t)$ being n times removed from white noise is equivalent to the sample paths of $x_j(t)$ being differentiable $n-1$ times and no more. This will be obvious to anyone familiar with the solutions of stochastic differential equations such as Eq. (5). It may be seen intuitively from the observation that any stochastic process that has a spectrum that is flat to infinitely high frequency cannot have sample paths that are ordinary functions of time. This is why white noise is ordinarily described as physically unrealizable. Thus, the nth derivative of a process that is n times removed from white noise does not technically exist (as an ordinary stochastic process) and an attempt to measure the nth derivative of the sample paths of such a variable will not find a sensible limit as Δt approaches 0.

Thus, we see that the power-law decays characteristic of systems subject to noise are a consequence of the fact that the sample paths of such systems are not C^∞ functions of time. Conversely, the arguments presented in Sec. II A imply that the faster-than-power-law decay of the power spectra of deterministic systems is a consequence of the analytic character of the solutions which in turn implies that the solutions are C^∞ functions of time. In fact, one can make even more general arguments that any variable that is a C^n function of time should have a spectrum that decays faster than ω^{-2n} and that, consequently, any variable that is a C^∞ function of time should have a spectrum that decays faster than any power law [2, section II].

III. SHADOWING

In this section, we will briefly review the theory of shadowing. The development of the theory has been motivated principally by the need to determine how well a dynamical system can be simulated with finite-precision arithmetic. The question is particularly important for chaotic systems since these systems show an exponential magnification of small differences between orbits, guaranteeing that computations done at different levels of precision will produce orbits that are very different after a time on the order of the inverse of the largest positive Lyapunov exponent. The theory provides assurance that simulations are valid by showing that, under certain conditions, a typical computed orbit is "shadowed" by a true orbit. The theory makes contact with the question of the effects of noise because the effects of finite-precision arithmetic are modeled by the addition of noise to the equations of motion.

The theory generally assumes discrete time. Thus, the dynamics are given by a map, $f : \mathbb{R}^n \to \mathbb{R}^n$, that maps the state of the system at one time, x_m, to the state at the next time, x_{m+1}, via the relation $x_{m+1} = f(x_m)$. In the case of differential equations, this implies that a time step, Δt, has been chosen and that f maps the state of the system to the state at a time Δt later. This is not really any restriction since the truncation error introduced by discretizing time can be included in the "noise" used to model finite-precision arithmetic.

We begin with a pair of definitions.

(i) One calls a sequence, $\{x_i\}$, $a \leq i \leq b$, a δ *pseudo-orbit* of a map f, if $d(f(x_i), x_{i+1}) < \delta$ for all i, $a \leq i < b$. $a = -\infty$ and $b = \infty$ are permitted.

(ii) One says that a point, x, ϵ *shadows* a sequence $\{x_i\}$ if $d(f^i(x), x_i) < \epsilon$ for all i, $a \leq i \leq b$.

The original "shadowing lemma" [11,12] applies to maps with "uniformly hyperbolic" attractors. An attractor is uniformly hyperbolic if the angle between the stable and unstable subspaces at each point is bounded below by an angle greater than zero. For such systems, the lemma states that for any desired shadowing distance, ϵ, there is a noise level, δ, such that any δ pseudoorbit of f is ϵ shadowed by some point, x. This includes infinite sequences.

Unfortunately, it appears that most dynamical systems with which researchers actually work are not uniformly hyperbolic. However, it has been observed that, typically, the regions of the attractor that break the uniform hyperbolicity are very small and are visited only very occasionally. On this basis, Yorke and collaborators [13–16] have developed a theory of finite-time shadowing. What happens in typical systems is that the noisy orbit is shadowed very closely by a true orbit for a long period of time. Then the noisy orbit enters a region that violates uniform hyperbolicity and the noisy orbit loses all its shadowing orbits. It picks up new shadowing orbits, however, as soon as it leaves the nonhyperbolic region which it typically does very quickly. The shadowing time grows, apparently without bound, as the amplitude of the noise is decreased.

IV. THE TRANSITION FROM EXPONENTIAL TO POWER-LAW DECAY

For a system subject to sufficiently weak noise, we would expect to see a substantial region of exponential decay in the spectrum until we reach a high enough frequency for the power-law decay characteristic of noise to take over. This is, in fact what is seen in model systems. Figure 2 shows several spectra taken from time series of the variable x from the Lorenz model. (See Sec. VI B for a description of the model and the way in which noise has been introduced.) One series comes from a simulation of the model with no noise. The other series come from simulations with noise amplitudes of $\mu = 2.5\times10^{-1}$, 2.5×10^{-3}, and 2.5×10^{-5}. The spectra of the noisy systems follow the spectrum of the deterministic system for a portion of the exponential decay and then flatten out into a power-law decay. As the noise amplitude is increased, the amplitude of the power-law decay increases and the frequency at which the two spectra diverge decreases. The author has observed the same behavior in the Rossler model. Since the amplitude of the power-law decay can be made arbitrarily small, it should, in principle, be possible to make the frequency range over which the noisy system shows an exponential decay arbitrarily large. (The flattening of the spectrum of the deterministic system at extremely low amplitude is an effect of finite precision computations. See Sigeti [4, appendix] for a discussion of this effect.)

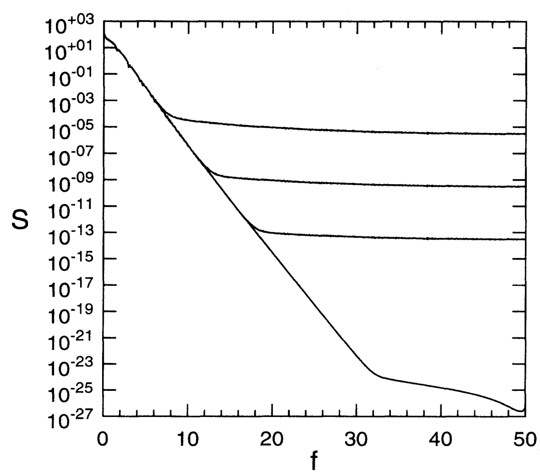

FIG. 2. Power spectra, S, versus frequency, f, from the time series of the variable x from the Lorenz model for four noise values ($\mu = 0$, 2.5×10^{-1}, 2.5×10^{-3}, and 2.5×10^{-5}). See Sec. VI B for details of the model and computations.

V. SHADOWING AND THE PRESERVATION OF EXPONENTIAL DECAY IN THE PRESENCE OF NOISE

This raises the question of how a noisy system can show an exponential decay. The decay and its characteristic parameters (the rate of decay, the coefficient, and the power law prefactor) are all determined by the complex time singularities of the solution of the deterministic equations of motion which are, in turn, consequences of the fact that the solution of the deterministic equations of motion is an analytic function of time for real t. But the introduction of the slightest amount of noise of the kind considered here completely destroys the analytic character of the function $x(t)$ because $x(t)$ is no longer a C^∞ function of time. Thus, for a system subject to arbitrarily weak noise, there is no longer an extension of $x(t)$ to complex time and therefore no complex-time singularities to give the parameters of the exponential decay.

It appears that, when the noise is weak enough to reveal part of the region of exponential decay, the orbits of the noisy system must somehow be associated with orbits of the equivalent deterministic system, which alone have the complex-time singularities that determine the parameters of the decay. But, we already know of an association between orbits of the noisy system and orbits of the corresponding deterministic system that occurs for sufficiently weak noise—it is the shadowing association. Moreover, the existence of a shadowing orbit will give exactly the behavior we expect and which we have just observed in the Lorenz model. The fact that the deviations of the noisy orbit from the deterministic shadowing orbit are small in amplitude guarantees that the power spectrum of the noisy orbit will be the same as that of the deterministic orbit over the range of frequencies where the amplitude is large. If the noise is white or approximately white, the deviations will have a short correlation time and the difference between the noisy and deterministic spectra will appear at high frequencies where the spectrum of the deterministic dynamics is very weak.

Therefore, it seems likely that the preservation of the region of exponential decay in the presence of noise is a sufficient condition for the shadowing of orbits of the noisy system by the equivalent deterministic system, at least for the long but finite times described by Yorke *et al.*

VI. NUMERICAL SIMULATIONS

In this section we present results from simulations of two well-known dynamical systems, the Lorenz and Rossler models. We want to verify that the survival of the region of exponential decay in the presence of noise is equivalent to the survival of the "correct" deterministic dynamics at larger amplitudes and longer time scales. In order to determine whether or not the correct deterministic dynamics are being preserved, we will examine the scaling region for the attractor. We will demonstrate that the survival of the region of exponential decay is equivalent to the survival of a scaling region that gives the correct correlation dimension for the attractor. In addition, we will see that the survival of the region of exponential decay is equivalent to the existence of a well-defined separation of scales in the power spectrum and, for the Lorenz model, in the function that is used to identify the scaling region and the associated value of the correlation dimension.

The following subsection describes the method used to identify the scaling region and the value of the correlation dimension. The next two subsections present our results from the Lorenz and Rossler models. A final subsection presents a general discussion of the problem of calculating power spectra accurately in the region of exponential decay.

A. Finding the scaling region and the correlation dimension

The correlation dimension is defined using the correlation integral, $C(r)$. For a random sampling of the attractor, $C(r)$ is the number of pairs of points on the attractor which lie within a distance r of each other, normalized by the total number of pairs of points. For systems with a well-defined correlation dimension, $C(r)$ scales as r^d where d is the correlation dimension.

The most simple-minded method of computing the correlation dimension is to plot $\log C(r)$ versus $\log r$ and to look for a region where the plot is a straight line. The slope of this line is then d. Where the data permit, it is usually better to plot the derivative of the log-log plot—that is, $d[\log C(r)]/d(\log r)$—versus $\log r$. $d[\log C(r)]/d(\log r)$ is an effective scale-dependent correlation dimension and a region where the plot is flat is a scaling region for the attractor. It is this method that is used in the following subsections. (See Sec. VI C 1 for a discussion of cases where this method does not work even with large amounts of very precise data.)

The correlation integral itself was calculated using computer software provided by James Theiler which implements his "box-assisted" algorithm for calculating the integral [17].

B. Lorenz model

The Lorenz model [18] is defined by the equations

$$\begin{aligned} \dot{x} &= \sigma y - \sigma x \ , \\ \dot{y} &= x(R-z) - y \ , \\ \dot{z} &= xy - bz. \end{aligned} \tag{13}$$

We have done our calculations at the parameter values

$$\sigma = 10.0, \quad R = 28.0, \quad b = 8/3. \tag{14}$$

At these parameter values, the model has an attractor with a Lyapunov dimension of approximately 2.07 [4, section 6.1].

We have added noise to this model by adding independent, Gaussian white noises with equal variances to the right-hand sides of each of the equations (13). Thus, our stochastic differential equations are

$$\begin{aligned} \dot{x} &= \sigma y - \sigma x + \mu\xi_t \ , \\ \dot{y} &= x(R-z) - y + \mu\eta_t \ , \\ \dot{z} &= xy - bz + \mu\zeta_t \ , \end{aligned} \tag{15}$$

where ξ_t, η_t, and ζ_t are independent, unit-variance Gaussian white noises and μ is the common strength of the added noises.

In the following, we will consider three values of the noise strength, μ: 0, 0.25 (which we call the low-noise case), and 1.25 (which we call the high-noise case). Details of the simulation of these systems and of the computation of spectra and dimensions are given in Sec. VI B 1 below.

Figure 3 shows the convergence of the spectra from time-series of the variable x from the zero-noise and low-noise systems. The horizontal axis is the frequency, f (not the angular frequency, ω) in inverse model time units. The rough, solid curve is the zero-noise spectrum. The dashed curve is the low-noise spectrum. The smooth solid curve is a function of the form $Cf^n e^{-kf}$ where the parameters C, n, and k have been fit to the zero-noise spectrum in the region where the spectrum has clearly become asymptotic. The important point to note is that the low-noise spectrum joins the zero-noise spectrum before the latter has separated from the asymptotic. Thus, we can say that this value of the noise is low enough to preserve a part of the region of exponential decay.

Figure 4 shows the low-noise and zero-noise spectra at the lowest frequencies and largest amplitudes. Here, the two spectra are virtually indistiguishable. This means that this value of the noise is low enough for there to be a genuine separation of scales in the power spectrum.

Figure 5 shows the convergence of the spectra from the zero-noise and high-noise systems. As in the previous two figures, the horizontal axis is the frequency and the vertical axis is a logarithmic plot of the power spectrum. Again, the rough, solid curve is the zero-noise spectrum, the dashed curve is the high-noise spectrum, and the smooth solid curve is the zero-noise asymptotic. The important point to note is that the high-noise spec-

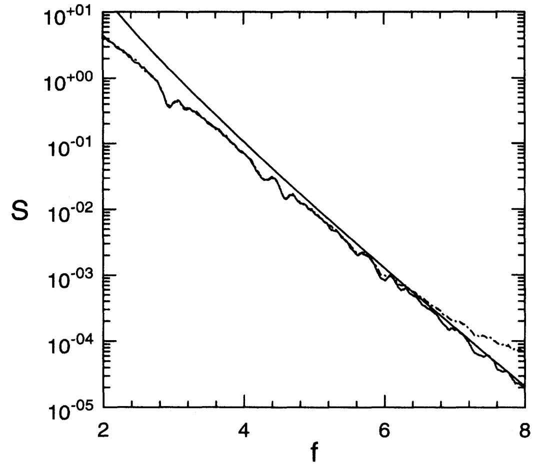

FIG. 3. Low-noise, zero-noise, and asymptotic spectra, S, versus frequency, f, from time series of the variable x from the Lorenz model. The rough, solid curve is the zero-noise spectrum. The dashed curve—which is distinguishable from the rough, solid curve only in the lower right corner of the figure—is the low-noise spectrum. The smooth solid curve is a function of the form $Cf^n e^{-kf}$ where the parameters C, n, and k have been fit to the zero-noise spectrum in the region where the spectrum has clearly become asymptotic. The frequency is in inverse model time units. See Sec. VI B 1 for details of the computation of the spectra including details of the normalization.

trum only joins the zero-noise spectrum after the latter has separated from the asymptotic. Thus, we can say that this value of the noise is high enough to destroy the region of exponential decay.

Figure 6 shows the high-noise and zero-noise spectra at the lowest frequencies and largest amplitudes. The important point to note is that the two spectra remain

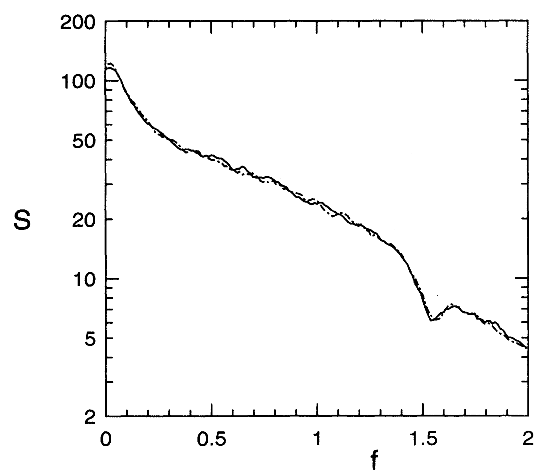

FIG. 4. Low-noise and zero-noise spectra, S, versus frequency, f, from time series of the variable x from the Lorenz model at low frequencies. The solid curve is the zero-noise spectrum. The dashed curve is the low-noise spectrum. The frequency is in inverse model time units.

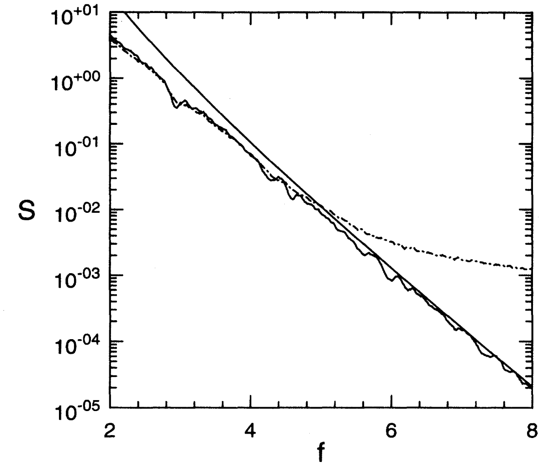

FIG. 5. High-noise, zero-noise, and asymptotic spectra, S, versus frequency, f, from time series of the variable x from the Lorenz model. The rough, solid curve is the zero-noise spectrum. The dashed curve is the high-noise spectrum. The smooth solid curve is a function of the form $Cf^n e^{-kf}$ where the parameters C, n, and k have been fit to the zero-noise spectrum in the region where the spectrum has clearly become asymptotic. The frequency is in inverse model time units.

distiguishable even down to zero frequency. This means that this value of the noise is high enough to destroy any separation of scales in the power spectrum.

Thus, we see from the preceding four plots that the preservation of a portion of the region of exponential decay is equivalent to the preservation of separation of scales in the power spectrum.

Figure 7 shows a plot $d[\log C(r)]/d(\log r)$ versus $\log r$ for the three values of the noise. In other words, the plot shows the effective correlation dimension as a function of scale. The lower solid curve is the plot for zero noise. The

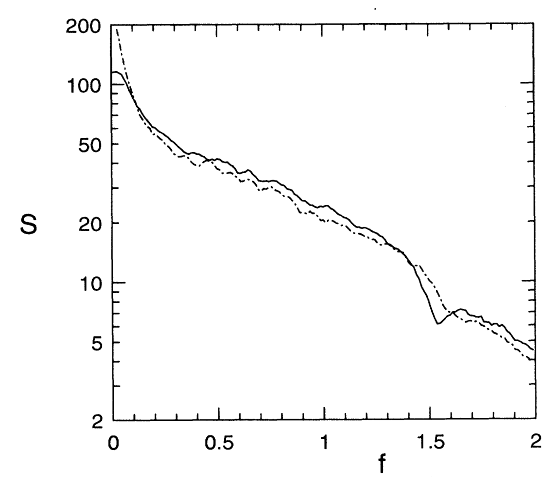

FIG. 6. High-noise and zero-noise spectra, S, versus frequency, f, from time series of the variable x from the Lorenz model at low frequencies. The solid curve is the zero-noise spectrum. The dashed curve is the high-noise spectrum. The frequency is in inverse model time units.

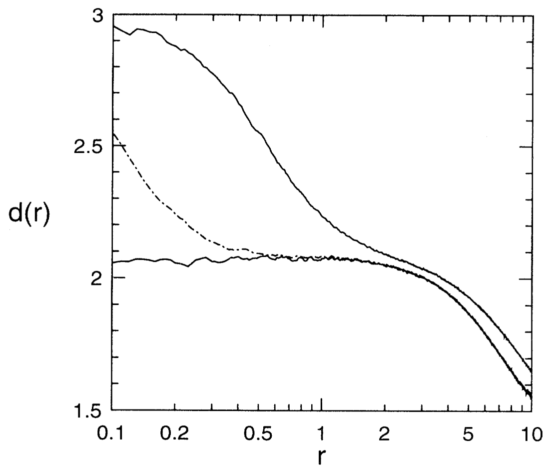

FIG. 7. Dimension curves from the Lorenz model. The vertical axis is the effective (scale-dependent) correlation dimension, $d(r)$, which is just the derivative of the log-log plot of the correlation integral versus length scale $\{d[\log C(r)]/d(\log r)\}$. The horizontal axis is the logarithm of length scale, r. The lower solid curve is the dimension curve for the zero-noise case, the dashed curve gives the low-noise case, and the upper solid curve gives the high-noise case.

dashed curve is the low-noise case and the upper solid curve is the high-noise case. The low-noise curve joins the zero-noise curve while the latter is still in its scaling region and is completely indistinguishable from the zero-noise curve at larger scales. Thus, we can say that the low-noise case (which we have earlier shown preserves a portion of the region of exponential decay in the power spectrum) preserves a part of the scaling region of the attractor and shows separation of scales in the dimension curves.

The high-noise curve, on the other hand, never joins the zero-noise curve and does not have a scaling region of any kind. Thus, we can say that the high-noise case (which we have earlier shown does not preserve a portion of the region of exponential decay in the power spectrum) does not preserve the scaling region of the attractor and does not show separation of scales in the dimension curves.

The up-shot of all this is that, for the Lorenz model, the preservation of a portion of the region of exponential decay in the presence of noise is equivalent to the preservation of a portion of the scaling region of the attractor giving the correct correlation dimension and to separation of scales in the power spectrum and the dimension curves.

1. Numerical simulations of the Lorenz model with noise

Equations (15) were simulated using a fixed time-step, second-order integrator for stochastic differential equations due to Rümelin [19]. The model was integrated with a time step of 0.001 model time units. For each of the three values of the noise strength, μ, the simulation was run for 32 000 time units and was sampled once every 0.01 time units.

Power spectra from the resulting time series for x were calculated by fast-Fourier transforming overlapped segments of the series of length 2^{17} samples. All spectra are normalized so that the integral of the spectrum over frequency (not angular frequency) from $f = 0$ to the Nyquist frequency equals the variance in the time series. The Hann window was applied to the segments of the time series before Fourier transforming to minimize end effects. Spectra were smoothed in the first place by averaging the squared amplitude of the discrete Fourier transform over the values from the different overlapped segments and in the second place by performing local smoothing in the frequency domain. This was done with a second-order Savitsky-Golay smoothing filter applied to the logarithm of the spectrum over a window of 65 frequency bins (corresponding to a smoothing range of 0.05 inverse time units). (See Sec. VI D for a general discussion of methods for calculating power spectra accurately in the region of exponential decay.)

The asymptotic curve that appears in some of the spectra was obtained by fitting a function of the form $Cf^n e^{-kf}$ to the zero-noise spectrum over the range $f = 12$ to $f = 28$, where the spectrum is clearly asymptotic. The resulting parameter values for C, n, and k were checked by fitting over narrower ranges. Although the values of C and n varied significantly with the different fitting ranges—as is typical with fits to this functional form—the resulting plots of the asymptotic were almost indistiguishable in the critical region where the spectra from the noisy systems diverge from the zero-noise spectrum. Thus, the variation in the fitted parameters does not effect the conclusions drawn from the spectra.

For calculation of the correlation integral, the three-variable time series for x, y, and z was used but the series was thinned by using only a fraction of the points. For $\mu = 0$, every 60th point was used for a total of 40 000 points. For the other values of μ, every 50th point was used for a total of 60 000 points each.

C. Rossler model

The Rossler model [20] is defined by the equations

$$\begin{aligned} \dot{x} &= -z - y \ , \\ \dot{y} &= x + ay \ , \\ \dot{z} &= b + z(x - c) \ . \end{aligned} \tag{16}$$

We have done our calculations at the parameter values

$$a = 0.15, \quad b = 0.20, \quad c = 10.0 \ . \tag{17}$$

At these parameter values, the model has an attractor with a Lyapunov dimension of approximately 2.01 [4, section 6.2].

As with the Lorenz model, we have added noise to this model by adding independent, Gaussian white noises with equal variances to the right-hand sides of each of the equations (16). Thus, our stochastic differential equations are

$$\begin{aligned} \dot{x} &= -z - y + \mu\xi_t \ , \\ \dot{y} &= x + ay + \mu\eta_t \ , \\ \dot{z} &= b + z(x - c) + \mu\zeta_t \ , \end{aligned} \tag{18}$$

where ξ_t, η_t, and ζ_t are independent, unit-variance Gaussian white noises and μ is the common strength of the added noises.

As with the Lorenz model, we present results from cases with a relatively low noise, a relatively high noise, and no noise. For the low-noise case, the noise strength, μ, was 0.01. For the high-noise case, μ was 0.05. Details of the simulation of these systems and of the computation of spectra and dimensions are given in Sec. VI C 2 below.

Figure 8 shows the full spectrum for the zero-noise Rossler system. Note that the structure that is visible in the asymptotic region—which amounts to a variation of about one-quarter of a decade on either side of the asymptotic—is not statistical. Unlike the Lorenz model, the Rossler model shows a great deal of structure in the exponential regime that does not appear to fade out as one goes to higher frequencies. The author suspects that the structure is related to the well-known "phase coherence" of the Rossler model.

Figure 9 shows the convergence of the spectra from the zero-noise and low-noise systems. As with the Lorenz model, the horizontal axis is the frequency, the vertical axis gives the power spectrum on a logarithmic scale, the rough, solid curve is the zero-noise spectrum, the dashed curve is the low-noise spectrum, and the smooth solid curve is the deterministic asymptotic. The low-noise spectrum may be seen diverging from the zero-noise spectrum in the lower right-hand corner of the figure. Unlike the Lorenz model, the zero-noise spectrum for the Rossler model does not completely separate from the asymptotic until it is clearly well out of the asymptotic regime. Rather, the spectrum develops "dips" that destroy the exponential character of the spectrum and which make the average value of the spectrum much less than the value of the asymptotic. A careful examination of the spectrum shows that the dips begin to be visible and the average value of the spectrum begins to diverge from the asymptotic at a frequency between 1.5 and 2.0 inverse time units. Since the low-noise spectrum joins the zero-noise spectrum at a frequency of about 3.0 inverse time units we can say that this value of the noise is low enough to preserve a part of the region of exponential decay.

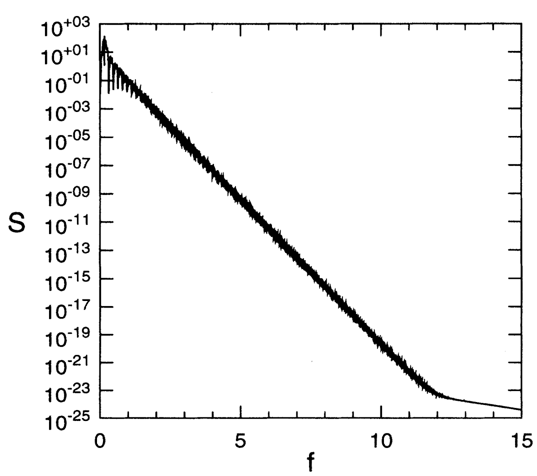

FIG. 8. Power spectrum, S, versus frequency, f, from time series of the variable x from the Rossler model with no noise.

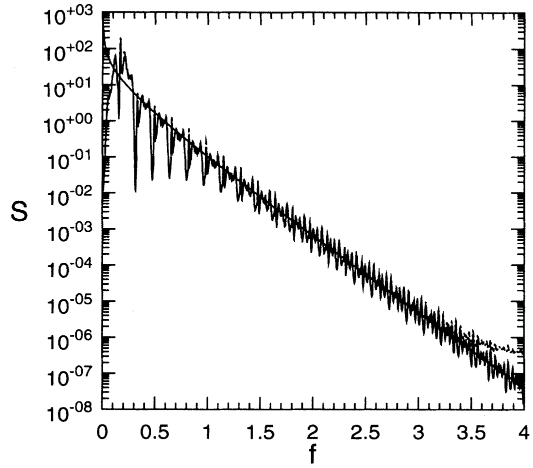

FIG. 9. Low-noise, zero-noise, and asymptotic spectra, S, versus frequency, f, from time series of the variable x from the Rossler model. The rough, solid curve is the zero-noise spectrum. The dashed curve—which is distinguishable from the rough, solid curve only in the lower right corner of the figure—is the low-noise spectrum. The smooth solid curve is a function of the form $Cf^n e^{-kf}$ where the parameters C, n, and k have been fit to the zero-noise spectrum in the region where the spectrum has clearly become asymptotic. The frequency is in inverse model time units. See Sec. VI C 2 for details of the computation of the spectra including details of the normalization.

Figure 10 shows the low-noise and zero-noise spectra at the lowest frequencies and largest amplitudes. Although the two spectra are not quite as indistiguishable as for the Lorenz model, they are quite close. Thus, we can say that this value of the noise is low enough for there to be a separation of scales in the power spectrum.

Figure 11 shows in detail the convergence of the spectra from the zero-noise and high-noise systems. The high-noise spectrum joins the zero-noise spectrum at a frequency just below $f = 2.0$, right in the region where we have identified the end of the region of exponential decay. Thus, we can say that this value of the noise is high enough to destroy the region of exponential decay.

Figure 12 shows the high-noise and zero-noise spectra at the lowest frequencies and largest amplitudes. Here, the two spectra are clearly distinct even at the peaks that contain the bulk of the power. This means that this value of the noise is high enough to destroy any separation of scales in the power spectrum.

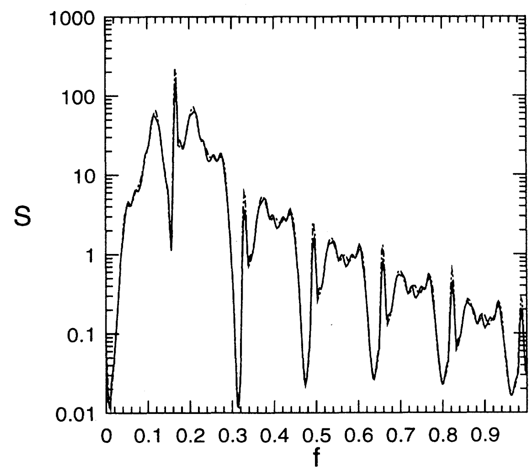

FIG. 10. Low-noise and zero-noise spectra, S, versus frequency, f, from time series of the variable x from the Rossler model at low frequencies. The solid curve is the zero-noise spectrum. The dashed curve is the low-noise spectrum. The frequency is in inverse model time units.

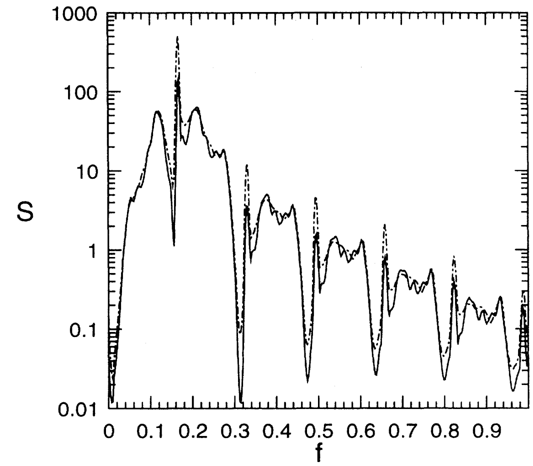

FIG. 12. High-noise and zero-noise spectra, S, versus frequency, f, from time series of the variable x from the Rossler model at low frequencies. The solid curve is the zero-noise spectrum. The dashed curve is the high-noise spectrum. The frequency is in inverse model time units.

Figure 13 shows a plot of effective correlation dimension, $d(r)$, as a function of scale, r (Sec. VI A). The lower solid curve is the plot for zero noise. The dashed curve is the low-noise case and the upper solid curve is the high-noise case.

The dimension curves from the Rossler model differ from the curves from the Lorenz model in a number of ways. The most obvious difference is that, even in the zero-noise case, the Rossler model shows scaling behavior much less clearly than does the Lorenz model. The zero-noise dimension curve shows slow variations with scale down to the smallest scale measured ($r = 0.1$). Between the smallest scale and approximately $r = 4.0$—where $d(r)$ begins to drop off precipitously due to the finite size of the attractor—$d(r)$ wanders between approximately 1.86 and 1.93.

The lack of a region in the graph of $d(r)$ versus r that is as flat as in the Lorenz model makes it difficult to determine the upper limit (in r) of the scaling region. We need to be able to do this in order to determine if the low-noise curve is preserving a part of the scaling region. The simplest assumption is that the scaling region extends up to the point where $d(r)$ begins to drop precipitously—in

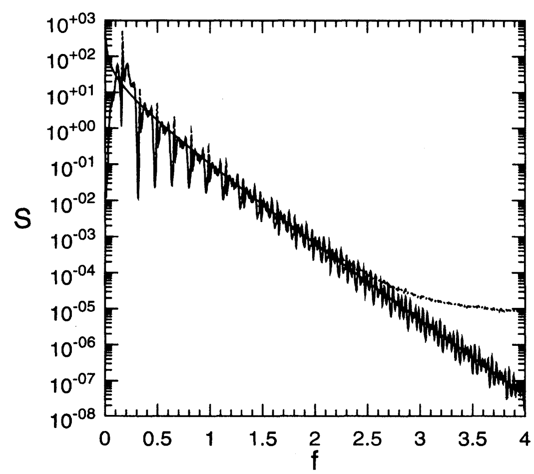

FIG. 11. High-noise, zero-noise, and asymptotic spectra, S, versus frequency, f, from time series of the variable x from the Rossler model. The rough, solid curve is the zero-noise spectrum. The dashed curve is the high-noise spectrum. The smooth solid curve is a function of the form $Cf^n e^{-kf}$ where the parameters C, n, and k have been fit to the zero-noise spectrum in the region where the spectrum has clearly become asymptotic. The frequency is in inverse model time units.

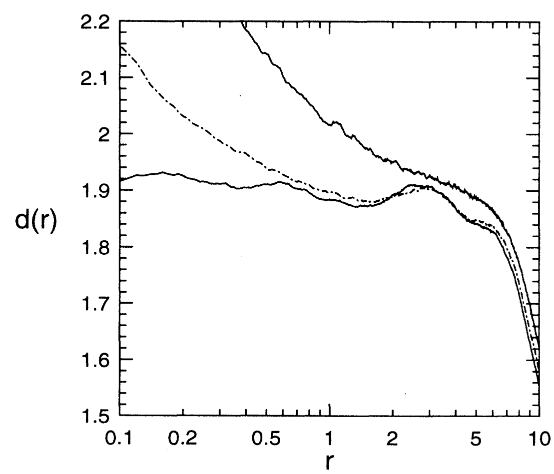

FIG. 13. Dimension curves from the Rossler model. The vertical axis is the effective (scale-dependent) correlation dimension, $d(r)$, which is just the derivative of the log-log plot of the correlation integral versus length scale $\{d[\log C(r)]/d(\log r)\}$. The horizontal axis is the logarithm of length scale, r. The lower solid curve is the dimension curve for the zero-noise case, the dashed curve gives the low-noise case, and the upper solid curve gives the high-noise case.

other words, up to about $r = 4.0$. Further reasons for locating the upper limit of the scaling region at about $r = 4.0$ are discussed in Sec. VI C 1.

Given our identification of the upper limit of the scaling region at approximately $r = 4.0$, then it is clear that—in the Rossler model as in the Lorenz model—the low-noise curve joins the zero-noise curve while the latter is still in its scaling region. As with Lorenz, then, we have shown that a noise value low enough to preserve a part of the region of exponential decay in the power spectrum preserves a part of the scaling region for the correlation dimension.

Another important difference between the dimension curves for the two models is that, in the case of the Rossler model, the dimension curve for the low-noise case is distinguishable from the zero-noise curve at larger scales (although it is still quite close to the zero-noise curve). Thus, we can say that the low-noise case (which we have earlier shown preserves a portion of the region of exponential decay in the power spectrum) preserves a part of the scaling region of the attractor but that it does not show a complete separation of scales in the dimension curves.

The high-noise curve, on the other hand, does not have a scaling region of any kind, never joins the zero-noise curve, and remains much further from the deterministic curve at large scales. Thus, we can say that the high-noise case (which we have earlier shown does not preserve a portion of the region of exponential decay in the power spectrum) does not preserve the scaling region of the attractor and clearly does not show separation of scales in the dimension curves.

For the Rossler model, then, the preservation of a portion of the region of exponential decay in the presence of noise is equivalent to the preservation of a portion of the scaling region of the attractor giving the correct correlation dimension and to separation of scales in the power spectrum. Preservation of the region of exponential decay does not guarantee complete separation of scales in the dimension curves but it clearly results in a dimension curve that fits the zero-noise curve at large scales much better than does a curve from a noise value that is high enough to destroy the region of exponential decay.

One peculiarity of the Rossler model is that it has basin boundaries quite close to the attractor. In fact, in simulating the high-noise case for purposes of calculating the correlation integral, the simulation passed over a basin boundary at about 150 000 model time units and soon was producing NaN's. The low-noise case showed no signs of escaping from the basin of attraction of the attractor when run for 1 050 000 model time units. It is interesting that the value of the noise strength that produces a breakdown of the region of exponential decay is the same value that produces escape from the basin of attraction with a substantial probability in a reasonable amount of model time.

1. The upper limit of the scaling region for the Rossler attractor

In this subsection, we discuss our identification of an upper limit of the scaling region for the correlation dimension of the reconstructed Rossler attractor at approximately $r = 4.0$, where r is the scale for the variable x used to reconstruct the attractor. The first point to be made is that the actual requirement for scaling is that the correlation *integral*, $C(r)$, scale as r^d where d is the correlation dimension—in other words that

$$C(r) \sim r^d. \tag{19}$$

It is possible to satisfy Eq. (19) without having a flat region for $d(r) = d[\log C(r)]/d(\log r)$ if, for example, $C(r)$ grows discontinuously in the manner of a set of stairs. This is normally the case with, for example, deterministic fractals. Thus, the sort of slow variation of $d(r)$ with r that we see in the Rossler model is not inconsistent with a scaling region. This suggests that we should take as the scaling region the entire range of r over which $d(r)$ remains close to its average value at the smallest scales. This region ends at approximately $r = 4.0$, where $d(r)$ begins to drop off precipitously due to the finite size of the attractor.

We can see that this is a sensible choice for the scaling region by examining the reconstructed attractor that was used to compute the dimension curves. Figure 14 shows one view of the attractor—a plot of $x(t)$ versus $x(t+1.5)$, where 1.5 is approximately one-quarter of the fundamental period of the system. (See Sec. VI C 2 for more information on the reconstruction.) The attractor consists of an almost two-dimensional band which folds back on itself, as one would expect for an almost two-dimensional attractor. The band has a well defined width of approximately 10.0 units. If we think of scaling behavior around a "typical" point, then it is clear that no points will show the proper number of neighbors for distances greater than about 5.0 but that, below that distance, scaling behavior should appear fairly quickly, especially since the proper number of neighbors will be present for distances considerably greater than 5.0 in the two directions that lie along the band, in the one direction that points toward the interior of the band, and in the two directions orthogonal to the band. This suggests

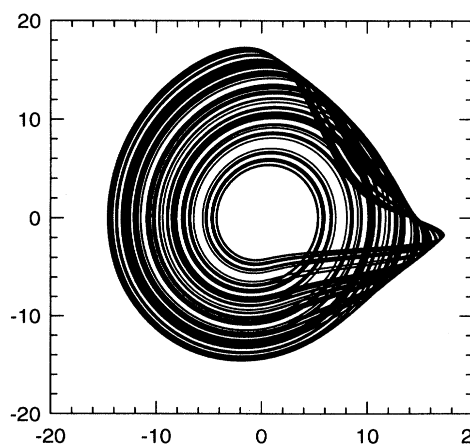

FIG. 14. The Rossler attractor reconstructed from a time series of the variable x as described in Sec. VI C 2.

that an upper limit of about 4.0 for the scaling region is quite reasonable.

Examination of Fig. 14 also provides clues to the origin of the slow variation of $d(r)$ with r (along with the surprisingly low value of approximately 1.9 for the correlation dimension). These are probably effects of the extreme "banding" of the Rossler attractor in this parameter regime. (See Sec. VI C 2 for a discussion of the fact that the curves give a value less than 2.0 for the correlation dimension.) This banding is clearly visible in the reconstructed attractor shown in Fig. 14.

2. Numerical simulations of the Rossler model with noise

As with the Lorenz model, Eqs. (18) were simulated using a fixed time-step, second-order integrator for stochastic differential equations due to Rümelin [19].

For calculation of power spectra, the simulation for each of the three values of the noise was run for 80 000 model time units with a time step of 0.001 model time units and was sampled once every 0.025 time units. Power spectra from the resulting time series for x were calculated by fast-Fourier transforming overlapped segments of the series of length 2^{17} samples. As with Lorenz, all spectra are normalized so that the integral of the spectrum over frequency from $f = 0$ to the Nyquist frequency equals the variance in the time series and the Hann window was applied to the segments of the time series before Fourier transforming to minimize end effects. Spectra were smoothed in the first place by averaging the squared amplitude of the discrete Fourier transform over the values from the different overlapped segments and in the second place by performing local smoothing in the frequency domain. This was done with a second-order Savitsky-Golay smoothing filter applied to the logarithm of the spectrum over a window of 65 frequency bins (corresponding to a smoothing range of 0.02 inverse time units). (See Sec. VI D for a general discussion of methods for calculating power spectra accurately in the region of exponential decay.)

The curve that gives the zero-noise asymptotic was obtained by fitting a function of the form $Cf^n e^{-kf}$ to the zero-noise spectrum over the range $f = 4$ to $f = 10$, where the spectrum is clearly asymptotic. As with Lorenz, the curve was checked by fitting over smaller ranges and the different curves are indistinguishable in the region in which the zero-noise asymptotic breaks down.

For calculation of the correlation integral, the simulations were run with a time step of 0.005 time units. The resulting time series for x was sampled once every 1.5 time units, which is approximately one-quarter of the fundamental period of the system. A three-dimensional phase space reconstruction was produced from this series using a time delay equal to the sample time (that is, 1.5 time units). For the zero-noise and low-noise cases, the simulation was run for 1 050 000 time units. Only every 15th reconstructed point was used for computation of the correlation integral, for a total of 45 000 points for each case.

For the high-noise case, the simulation passed over a basin boundary at approximately 150 000 time units so only the first part of the series for this case could be used. In this case, every third reconstructed point was used for a total of 20 000 points.

Some readers may be disturbed by the fact that Fig. 13 gives a value of 1.9 for the correlation dimension of the Rossler attractor. This may arise from the mistaken impression that attractors from continuous-time dynamical systems must have dimensions that are greater than or equal to 2.0. Actually, the only dimension of such attractors that must always be greater than or equal to 2.0 is the Lyapunov dimension. Since this is generally believed to be equal to the information dimension, D_1, and since the Hausdorf dimension, D_0, is constrained to be greater than or equal to the information dimension, it makes sense to be very suspicious if an algorithm is giving values less than 2.0 for any of these three dimensions. The correlation dimension, on the other hand, is always less than or equal to the information dimension and there does not appear to be any reason that it cannot be less than 2.0. Smith has obtained the same value for the correlation dimension of the Rossler model using a different algorithm [21].

D. Computation of power spectra in the region of exponential decay

Computation of power spectra in the region of exponential decay presents difficult numerical problems because the amplitude is declining very rapidly with frequency. The most serious result of the rapid decline in amplitude is that the exponential decay can be completely obscured by end effects. The end effects take the form of a plateau in the spectrum at high frequency that looks qualitatively the same as the kind of plateau that is produced by noise. End effects may be minimized by using long time segments in the Fourier transform and by "windowing" with smooth windows like the Hann window. Thus, whether a given plateau at high frequency is produced by end effects or by noise may be determined by recomputing the spectrum using longer time segments or a smoother window. If the plateau is due to end effects, longer time segments or a smoother window will lower the plateau while a plateau due to noise will be unaffected. The author has found that the use of segments that are 2^{16} samples long together with the Hann window is sufficient to push the plateau that is due to end effects down to approximately the level of the fundamental plateau due to double precision arithmetic (at about 25 to 30 decades below the largest values in the spectrum) in a variety of chaotic systems [4, appendix].

Because spectra are usually smoothed by averaging over many segments, the use of long segments can, for a fixed amount of data, result in very rough spectra. Fortunately, long segments also provide a very fine resolution in frequency so spectra may be smoothed locally in the frequency domain. This approach—long time segments, smooth windows, and local smoothing in the frequency domain—has been used for the computation and plotting of spectra in this paper. See Sigeti [4, appendix] for a more extensive discussion of methods for calculating power spectra accurately in the region of exponential decay.

VII. STRUCTURALLY SENSITIVE SYSTEMS

There exist flows which are extremely sensitive to noise in the sense that an arbitrarily small amount of noise will produce a qualitative change in the dynamics of the system. In this section, we will discuss the possibility that, in such systems, the presence of a region of exponential decay in the power spectrum may not correlate with the preservation of essentially deterministic dynamics.

We shall call systems in which an arbitrarily small amount of noise produces a qualitative change in the dynamics *structurally sensitive systems.* The term has been chosen to emphasize the similarity between structural *sensitivity* and structural *instability.* A structurally *unstable* system will exhibit a qualitative change in its dynamics when one makes an arbitrarily small *static change* in the system parameters—a structurally *sensitive* system will show a qualitative change in its dynamics when one places an arbitrarily small *noise* on the system parameters. Although the two concepts are closely related they are not necessarily identical. In particular, it is possible for a system to be structurally stable to a limited class of changes in the system parameters—those that perserve certain symmetries, for example—while being structurally sensitive to noises that preserve the same symmetries—see the article by Stone discussed below.

One example of a structurally sensitive system is discussed qualitatively by Eckmann and Ruelle [22, pp. 640–641]. (The example is originally due to Bowen [23] and to Katok [24].) This is a two-dimensional flow in which the behavior is dominated by a saddle point. Both branches of the unstable manifold of the saddle point wrap around to join branches of the stable manifold. The manifolds thus form a "figure 8" meeting at the saddle point. This figure is globally attracting. (See Eckmann and Ruelle for an explanatory figure.) In the absence of noise, time series from the system take the form of finite-amplitude bursts of activity that occur less and less frequently as time goes on but which never cease. The system never settles down to a stationary behavior. In the presence of any amount of noise, the bursts will occur with some average frequency and the behavior will become stationary.

Sigeti [25] has shown that the well-known case of a saddle-node bifurcation leading to relaxation oscillations is also a structurally sensitive system in the sense that we are discussing. In this system, the center manifold of the saddle-node point forms a closed curve, insuring that the behavior of the system is bounded. When such a system is exactly at the saddle-node bifurcation, the system with zero noise settles down to a fixed point (the saddle node) while an arbitrarily small noise will induce finite-amplitude oscillations by inducing passage through the saddle node.

Stone, in working on a specific model that has the qualitative dynamics described by Eckmann and Ruelle, has suggested that it should be possible for such a system subject to weak noise to show behavior that is qualitatively different from what would be seen with no noise while showing a substantial region of exponential decay in the power spectrum [26]. This seems plausible because even an arbitrarily small noise (presumably weak enough to allow a substantial region of exponential decay in the power spectrum) will result in profound changes in the dynamics. She has presented results from numerical simulations that show an exponential decay in the power spectrum with noise values that are qualitatively affecting the dynamics.

It should be pointed out that, in both the systems discussed above, even the presence of finite-time shadowing orbits is not a sufficient criterion for the dynamics to be essentially deterministic. Both these systems have dynamics that is dominated by an unstable fixed point—a simple saddle in the case discussed by Eckmann and Ruelle and Stone and a saddle node in the case discussed by Sigeti. Finite-time shadowing is not a sufficient condition for essentially deterministic dynamics for either system because the system will have a shadowing orbit except for the relatively brief period of noise-induced passage through the region around the fixed point. Since the period of quiescence goes to infinity as the noise goes to zero, the shadowing time will also go to infinity, just as is typical of finite-time shadowing. Thus, the considerations discussed in this section do not affect the conjecture presented in Sec. V of this paper connecting the preservation of the region of exponential decay to the presence of shadowing orbits.

Despite the probable violation of the thesis presented in this paper in structurally sensitive systems, the examination of high-frequency power spectra can still provide a useful test for deterministic chaos if structurally sensitive systems can be reliably identified. In practice, this should not prove very difficult. If an actual experimental apparatus is accessible, the extreme sensitivity to noise should be obvious. Even if the only information available is a time series, time series from structurally sensitive systems have characteristic features that should be fairly easy to detect. In particular, these series show relatively long periods of quiescence punctuated by relatively brief periods of very rapid activity. Sigeti [25] shows a time series that exhibits this behavior. This behavior and the accompanying wide separation of time scales should not be difficult to observe in the time series and should serve as a warning that the system may be structurally sensitive.

One may wonder whether the presence of a region of exponential decay will serve as a reliable indicator of essentially deterministic dynamics in systems that may not be structurally sensitive to noise in the manner of the examples discussed above but which are nonetheless, in some rough sense, quite sensitive to noise. The Rossler model is actually such a system. It is quite sensitive to noise in at least two ways. The activation of z that occurs when x exceeds c involves an exponential growth that is quite sensitive to noise. Moreover, as pointed out in Sec. VI C above, there are basin boundaries quite close to the attractor so that even a fairly small noise has a reasonable probability of inducing escape from the attractor in a finite amount of time. In light of this sensitivity to noise it is very encouraging for the thesis presented here that the preservation of a region of exponential decay in the Rossler model corresponds to the preservation of the scaling region in the attractor.

VIII. NUMERICAL SIMULATIONS AND THE CASE OF TURBULENT FLOW

As discussed in the Introduction, the results presented so far are relevant to the question of whether or not simulations of ordinary differential equations are valid. The connection between our results and numerical simulations arises because the errors introduced in simulating ordinary differential equations on digital computers—round-off and truncation error—can be considered to be white or nearly white noise. Round-off error is certainly white on the scale of a single time step while truncation error, especially for chaotic systems, should not be correlated over more than a few time steps. Since these errors behave essentially like white noise, our results indicate that the preservation of a region of exponential decay in the power spectrum should guarantee that these errors are not fundamentally effecting the dynamics at longer time and larger amplitude scales. In other words, the preservation of the region of exponential decay should correspond to a simulation with essentially the same dynamics as the original differential equation.

In simulations of partial differential equations, truncation error arises not only from the discretization of time but also from the discretization of space. Truncation error produced by a finite spatial grid is essentially a *spatially* white noise. Thus, in determining the effect of this kind of truncation error, one would expect the wave-number spectrum to play the role played by the power spectrum in determining the effect of truncation error associated with a finite time step. In other words, one would expect a spatiotemporally chaotic system to show an exponential decay in the wave-number spectrum for sufficiently high wave number and one would expect the preservation of a part of the region of exponential decay to indicate that the discretization of space was not fundamentally altering the dynamics of the system. We will now show that this is in fact the case for a particular spatiotemporally chaotic system—a flow exhibiting homogeneous, isotropic turbulence.

For homogeneous, isotropic turbulence, there exists a length scale, λ_0 (the "dissipation" or "Kolmogorov" scale), above which viscous dissipation is negligible and below which viscosity dominates the dynamics [27]. It is well known that direct numerical simulations of turbulent flow which do not employ some kind of phenomenological modeling of subgrid turbulence must resolve the dissipation scale in order to correctly simulate the flow dynamics. The reason is obvious—if one does not resolve the dissipation scale, one cannot dispose of the energy being pumped into the system at larger scales and energy builds up without limit. Conversely, if one does resolve the region of exponential decay, one may be reasonably assured that one's simulation is producing correct results.

It has recently been shown by Chen *et al.* [28] that the wave-number spectrum of turbulent flow decays exponentially in the dissipation regime, $k > 1/\lambda_0$. Thus, in the case of turbulence, it is necessary to resolve the region of exponential decay in the wave-number spectrum (corresponding to the dissipation range) in order to obtain correct simulations.

The fact that results analogous to those presented earlier for power spectra can be shown to hold for the wave-number spectrum for a particular spatiotemporally chaotic system suggests further support for the results for power spectra. Conversely, the fact that the arguments presented for power spectra are quite general suggests that the analogous results for wave-number spectra may apply much more generally than is suggested by the energy cascade arguments used for the case of turbulent flow. The results may, in fact, apply to any system exhibiting spatiotemporal chaos.

IX. CONCLUSIONS

We conclude with the following points

(1) For the Lorenz and Rossler models, the preservation of a portion of the region of exponential decay in the presence of noise is equivalent to the preservation of a scaling region that gives the correct correlation dimension for the attractor. In addition, preservation of the region of exponential decay is equivalent to separation of scales in the power spectrum and, for the Lorenz model, in the dimension curve. This suggests that the detection of a region of exponential decay in power spectra from time series from a continuous-time (apparently chaotic) dynamical system can serve as a good test for determining when the dynamics of the system are essentially deterministic.

(2) There is good theoretical reason to believe that the preservation of a portion of the region of exponential decay in the power spectrum is equivalent to the existence of finite-time shadowing orbits.

(3) In systems that are structurally sensitive to noise, it seems likely that the preservation of a part of the region of exponential decay in the power spectrum will *not* be equivalent to the preservation of the correct deterministic dynamics at larger amplitude and longer time scales. However, it should not be difficult to identify structurally sensitive systems even if the only information available is a time series. The fact that the criterion of preservation of the region of exponential decay works fairly well for the Rossler model—which is quite sensitive to noise—suggests that the existence of structurally sensitive systems will not present insurmountable difficulties in applying the criterion.

(4) The results and arguments presented in this paper suggest that, in simulations of ordinary differential equations, the preservation of a region of exponential decay in the power spectrum should guarantee that round-off error and truncation error associated with the discretization of time are not fundamentally effecting the dynamics at longer time and larger amplitude scales. In other

words, the preservation of the region of exponential decay should correspond to a simulation with essentially the same dynamics as the original differential equation.

(5) The necessity to resolve the region of exponential decay in the power spectrum in order to obtain valid simulations of ordinary differential equations is analogous to the necessity to resolve the dissipation scale in the wave-number spectrum in simulations of turbulent flows. This suggests that the detection of a region of exponential decay in the wave-number spectrum may be a good criterion for determining when truncation error due to the discretization of space is not fundamentally altering the dynamics of spatiotemporally chaotic systems.

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