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Saddles, singularities, and extrema in random phase fields

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Four simple topological rules are derived that constrain the arrangement of critical points (saddles, singularities, and extrema) in a random phase field. These rules relate the signs of the singularities to the nature of the extrema (maxima or minima) and the topology of the saddles. Once the latter is fixed, only a *single* degree of freedom remains and if, for example, some extremum is chosen to be a maximum, this choice automatically determines the nature of *all other* extrema and the signs of *all* singularities. Thus, even in a random wave field there are extensive, topologically mandated correlations between *all* critical points. Higher-order gradient fields derived from the phase are considered and the rules and their induced correlations are shown to apply also to these fields. Other aspects of the phase field are discussed and it is shown, for example, that the number of saddles very nearly (but not necessarily exactly) equals the number of singularities plus the number of extrema and that during the evolution of the wave field, large numbers of different, specific features must appear simultaneously.

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I. INTRODUCTION

Fluctuations [1–13], correlations [1–4,14–25], and weak [26–38] and strong [39–42] localization of light in random optical wave fields [43–45] (speckle patterns) have been the subject of numerous recent investigations. These studies are almost all concerned with the wave field *intensity* (amplitude). Previous studies on the *phase* of random wave fields have concentrated on the unusual phase singularities (vortices) that are found in such fields [46–65], but there is relatively little known about other aspects of the phase of such a wave field. Here, we consider the whole phase field, and we explore the relationship between phase saddles, phase extrema (maxima and minima), and phase singularities. We begin in Sec. II with a discussion of an important cardinal rule for random wave fields that simplifies the treatment of these fields. We also formulate two propositions that have far reaching consequences for the phase field structure. In Sec. III we consider the relationship between phase saddles and phase extrema, and we develop a simple deterministic rule that we call the first loop rule that connects these two features. In Sec. IV we review basic properties of phase singularities, including a previously developed deterministic rule we called the sign principle that relates the signs of neighboring singularities [61,62]. We also discuss a vector field description of these singularities that proves especially useful in later sections. In Sec. V we review the topological index theorem which is then used throughout the remainder of this study. In Sec. VI we discuss the relationship between phase saddles and phase singularities, and we promulgate an enlarged version of the sign principle that includes these two very different wave field features. In Sec. VII we discuss interrelationships between phase saddles, singularities, and extrema, and we develop two additional deterministic rules—the second and third loop rules—which connect all three wave field features. In Sec. VIII we consider higher-order gradient fields obtained from the phase, and we show that the number of phase saddles approximately (but not necessarily exactly) equals the sum of the numbers of phase extrema and phase singularities. In Sec. IX we consider the scaling properties of the wave function and the boundary conditions to the wave equation, and we show that there cannot exist an *exact* rule relating the numbers of saddles, singularities, and extrema in a random phase field. This is in contrast to the intensity, for which an exact rule does exist that relates the numbers of saddles and extrema. In Sec. X we discuss the free space evolution of the wave field and show that the creation of a new phase singularity is accompanied by the simultaneous creation of myriad additional new wave field features. We summarize our findings in Sec. XI. Throughout, we use simple, physically intuitive arguments backed by large scale computer simulations and illustrated by means of numerous figures to provide a more complete picture of the structure of random phase fields than may have hitherto been available.

II. RANDOM WAVE FIELDS

An important simplifying principle for random wave fields is that all accidental coincidences (degeneracies) have vanishingly small probability and can be neglected. Accordingly, we need only concern ourselves with *generic* features that remain stable under small perturbation. For the two-dimensional random wave fields that are of interest here, all features for which the wave function must take on some *particular* value at some *particular* point in the (x,y) plane are nongeneric, and are therefore excluded. This cardinal rule excludes, for example, second- (and higher-) order phase singularities that require a saddle point of the real part of the wave function to exactly coincide with a saddle point of the imaginary part [62]. Also excluded are edge dislocations [46], which are lines (continuous sets of connected vortices) along which the phase changes discontinuously. The generic phase singularities in a random wave field are thus isolated points corresponding to simple first-order zeros of the wave function, a result first shown by Berry [47] using Gaussian statistics. The converse of the foregoing is also true, and all zeros of a random wave function are both isolated and of first order. Other important accidental coincidences that are excluded are two or more phase extrema or phase saddles with exactly the same value for the wave function, phase ridges or phase canyons that have a perfectly constant height or depth over some finite length, and features that require the intersection of more than two lines, such as higher-order stationary points. At ordinary (second-order) stationary points of the phase φ, for example, both $\varphi_x = \partial\varphi/\partial x$ and $\varphi_y = \partial\varphi/\partial y$ vanish. Stationary points thus lie at intersections of the zero crossings of φ_x and φ_y (the set of points or continuous curves on which the function vanishes). At higher-order (degenerate) stationary points also higher-order derivatives vanish, so such points require additional zero crossings to pass through an ordinary stationary point and may be excluded as nongeneric. Accordingly, all stationary points in a two-dimensional random wave field are both isolated and nondegenerate, and therefore can be described locally by a second-order polynomial homogeneous in x and y (Morse's lemma [66,67]).

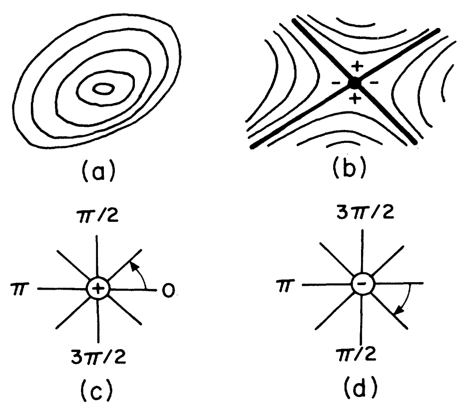

FIG. 1. Equiphases of generic critical points in a random phase field. (a) Extremum (maximum or minimum) surrounded by closed contour lines. (b) Saddle point. The simplest symmetric saddle is described by $f(x,y) = y^2 - x^2 = (y-x)(y+x)$. This function is zero at the saddle center (origin) $x = y = 0$, symbolized throughout by a black dot, is positive in regions marked $+$, negative in regions marked $-$, and vanishes on the lines $y = \pm x$, here called bifurcation lines. Throughout, bifurcation lines of saddles are symbolized by heavy lines and other equiphases by light lines. For the example shown, the sign of the wave function changes each time a bifurcation line is crossed, while traversing the saddle point along the vertical y axis (horizontal x axis) one passes through a phase minimum (maximum). (c) Positive and (d) negative first-order phase singularities. The phase circulates once from 0 to 2π counterclockwise (clockwise) around positive (negative) singularities and changes discontinuously by π along every line that passes through the center of the singularity. Throughout, singularities are generally symbolized by circles which contain the sign of the singularity.

We will call lines of constant phase "equiphases." Of special interest are the equiphases that surround phase extrema [Fig. 1(a)], phase saddles [Fig. 1(b)], and phase singularities [Figs. 1(c) and 1(d)]. We note that, in order to maintain the wave function single valued everywhere, the phase is "folded back" into the primary interval 0 to 2π (or, when convenient, $-\pi$ to $+\pi$), so that after passing 2π the phase starts over again at 0. This *apparent* discontinuity, which can always be moved "elsewhere" by a uniform phase shift, is not physically meaningful, however, and so does not show up as discontinuities in the topology of contour maps of the phase field. Accordingly, in what follows we ignore the $2\pi \to 0$ phase "discontinuity."

We assume that our wave function (and hence both its real and imaginary parts) is everywhere regular (single valued, continuous, and differentiable), which implies that the phase field itself must also be regular everywhere except at zeros of the wave function (the phase singularities). From this it follows that two different equiphases can never intersect (contact one another) except at a singularity, so equiphases that do not form closed circuits either terminate at phase singularities or continue on to the boundaries of the wave field. Based on the study of large scale computer simulations [59], we postulate that only saddles and singularities that are close to the wave field boundary can have equiphases that reach the boundary. In what follows we therefore assume that in the wave field interior the following two propositions hold.

(i) All equiphases that are not closed terminate on singularities.

(ii) All equiphases that do not terminate on singularities are closed.

These complementary (nontrivial) propositions have the most far reaching consequences, and will be seen to be major determinants of the phase field structure.

III. SADDLES AND EXTREMA

The classical picture of a saddle is a minimum along one principal axis and a maximum along the other. An important topological property of a saddle point is that it is a self-intersection of an equiphase, and is thus a bifurcation point for the field of lines defined by all the different equiphases. This is illustrated in Fig. 1(b). The converse of the foregoing is also true, and wherever a bifurcation of equiphases occurs a saddle point must be present. The lines passing through the center of the saddle, which are uniquely labeled by the value of φ at the saddle point, have directions that cannot be uniquely determined by a limiting process based upon the directions of neighboring contours, as the limit depends upon whether one approaches from the right or from the left [Fig. 1(b)]. We call these special lines "bifurcation" lines, and restrict use of the word "contours" to equiphases whose direction $(dy/dx = -\varphi_x/\varphi_y)$ is well defined everywhere. Worth noting is that, although the position of a saddle point and/or the value of the wave function at this point may be changed by an arbitrarily small, local perturbation, such a perturbation cannot eliminate the saddle point altogether, since saddles, extrema, and singularities are *individually* topologically stable features of the wave function [66,67].

We will need a name for the segments of a bifurcation line that radiate outward from a saddle point, and although the term "strands" is sometimes used, here we will refer to these segments as the "arms" of the saddle. In analyzing the properties of complicated phase fields it is sometimes convenient to move a particular saddle point into the (x,y) plane by adding a suitable (positive or negative) constant to the wave function. This turns the bifurcation lines into zero crossings of φ, which now changes sign across every arm of the saddle [Fig. 1(b)].

As indicated at the end of the previous section, all equiphases in the interior of a generic random wave field either terminate on singularities or close on themselves. This is also true for the equiphases that form the arms of a saddle. We characterize saddles with no joined arms as "open" [Fig. 2(a)], those with one pair of arms joined as "half closed" or "one loop" [Fig. 2(b)], and those with both pairs of arms joined as (fully) "closed" or "two loop" [Figs. 2(c) and 2(d)]. A topologically important property of closed saddles which makes them equivalent in many ways to extrema is that there are no equiphases emanating from these structures. Closed saddles are further subdivided into "figure eights" [Fig. 2(c)] and "reentrant" [Fig. 2(d)]. We will refer to the two loops of a reentrant saddle as the "interior" and "exterior" loops, in obvious notation. A reentrant saddle may have the arms of its interior loop joined directly [Fig. 2(d)], or these arms may be connected via a pair of positive and negative phase singularities (the interested reader may enjoy sketching this configuration). Although apparently possible in principle, we note that we have never seen singularities *inside* saddles, and so we will not dwell further on this possibility.

Extrema, each of which is encircled by a set of closed contours [Fig. 1(a)], and saddle points, each of which radiates a pair of bifurcation lines [Fig. 1(b)], are always found in close association, since, as indicated in Fig. 3(a), neighboring extrema must be separated by a bifurcation line. The question then quite naturally arises as to how large numbers of these two very different types of structures with their topologically very different field lines coexist in close association. The interesting, possibly surprising answer, illustrated in Fig. 3(b), is that every extremum lies in the *intimate* embrace of the arms of a saddle. Here, "intimate" implies the absence of any other feature within the embrace, and by definition the exterior loop of a reentrant saddle [Fig. 2(d)] does not embrace the interior loop. We summarize the relationship between saddles and extrema in the following rule.

First loop rule. All extrema and closed saddles lie in the intimate embrace of the arms of a saddle (bifurcation loop).

This rule is based on exhaustive analysis of numerous examples, and follows from the fact that the closed contours of extrema and closed saddles must be cordoned off by bifurcation lines in order to preserve continuity of the field lines between singularities. The converse of this rule is shown in Sec. V to also be true, and whenever the arms of a saddle are joined they necessarily embrace either an

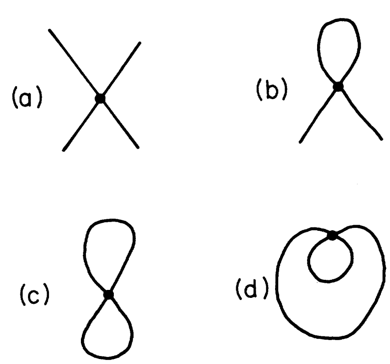

FIG. 2. Generic four-armed saddles in a random phase field. Only saddle points and the bifurcation lines that pass through these points are shown. (a) Open saddle, no arms joined. (b) One loop saddle, one pair of adjacent arms joined. (c),(d) Closed saddles, both pairs of adjacent arms joined. The closed saddle in (c) is dubbed a "figure eight," and the one in (d) is called "reentrant." The two loops of a reentrant saddle are labeled interior and exterior in obvious notation.

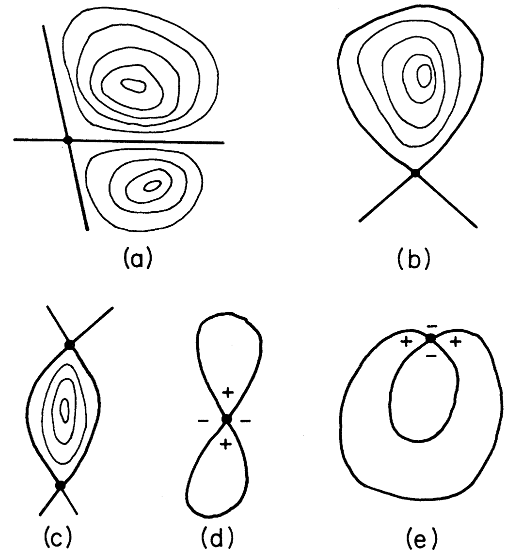

FIG. 3. Saddles and extrema. (a) Neighboring extrema *must* be separated by a bifurcation line. (b) Extremum embraced by the joined arms of a single saddle. This is the generic arrangement in a random phase field. (c) Two saddles join arms to embrace one extremum yielding a topologically possible but nongeneric arrangement. (d) Both loops of a figure eight embrace the same type of extremum (maximum or minimum). In the example shown the saddle point has been moved into the (x,y) plane by addition of a suitable constant to the phase function so that the value of the phase is zero at the saddle point and along the bifurcation lines. Inside the two closed loops of the saddle the phase is positive [Fig. 1(b)] and continuously increases as one moves away from the bifurcation lines towards the loop centers. Accordingly, both loops embrace maxima. (e) The loops of a reentrant saddle embrace extrema of opposite type. In the example shown the exterior loop embraces a maximum and the interior loop a minimum.

extremum or a closed saddle. Worth noting is that, although in principle two or more saddles can join arms to embrace one extremum [Fig. 3(c)], such a structure requires two different saddle points to have the same value for the wave function and is nongeneric. The first loop rule and its converse are applicable not only to phase fields but also to the wave field intensity (amplitude), and are one of the major determinants of the structure of these fields. Extrema enclosed within deeply nested closed saddles may also occur, and are the generic structures in random amplitude fields (speckle patterns). We note, however, that we have never seen nested saddles in a *phase* field, random or otherwise.

Both of the extrema enclosed by the arms of a figure eight must be of the same type, i.e., two maxima or two minima [Fig. 3(d)], while for a reentrant saddle, if one extremum is a maximum the other must be a minimum, and vice versa [Fig. 3(e)]. These conclusions follow from the fact that the direction of increase of the phase (or intensity) as one crosses any one of the arms of a saddle suffices to determine what happens at all other arms, and this in turn determines whether the embraced extremum is a maximum or a minimum [Figs. 3(d) and 3(e)].

Although the first loop rule specifics where extrema will be found, it does not specify the extremum type. We fill this gap in Sec. VII where we derive two simple rules that determine whether a given extremum will be a maximum or a minimum.

IV. SINGULARITIES

Although in optics the wave function is conventionally described in terms of the amplitude $A=(G^2+H^2)^{1/2}$ and phase $\varphi=\arctan(H/G)$ of a complex field $F(x,y)=G(x,y)+iH(x,y)$, it is sometimes convenient to use a vector field description, as is often done in solid state physics [68,69]. In this representation one defines a vector field $\mathbf{\Psi}(x,y)=\Psi_X\mathbf{x}+\Psi_Y\mathbf{y}$, together with an associated order parameter $\Theta(x,y)=\arctan(\Psi_Y/\Psi_X)$ that measures the angle the vectors make with respect to the x axis. Here $\mathbf{x}$ and $\mathbf{y}$ are unit vectors along x and y, and to avoid confusion with partial derivatives we use upper case subscripts X and Y to denote the components of a vector. The equivalence of the two descriptions is established by writing $\Psi_X(x,y)=G(x,y)$ and $\Psi_Y(x,y)=H(x,y)$, which yields $\Theta(x,y)=\varphi(x,y)$. In the complex field representation, topological singularities of the phase φ are located at the intersections of the zero crossings of G and H, while in the vector field representation the equivalent topological singularities of the order parameter Θ are located at the intersections of the zero crossings of Ψ_X and Ψ_Y. The utility of the vector field is that it is easily generalized to permit the definition of topological singularities for functions other than the phase (Sec. VIII and Ref. [64]).

For the complex field representation, the sign ($+$ or $-$) of a singularity [Figs. 1(c) and 1(d)] is given by the sign of $d\varphi/d\theta$, where the local polar angle θ is conventionally measured counterclockwise from the x axis of a local x,y coordinate system centered on the singularity. Equivalently, the sign of a singularity in the vector field representation is given by the sign of $d\Theta/d\theta$. Single valuedness of the wave function guarantees that in both cases this sign is the same everywhere along a convex path that encircles only the given singularity. Using the fact that the singularity is a first-order zero, the sign of $d\varphi/d\theta$ is found by direct calculation to equal the sign of the Jacobian $J=\partial(G,H)/\partial(x,y)$ [58,59], while the sign of $d\Theta/d\theta$ equals the sign of $\partial(\Psi_X,\Psi_Y)/\partial(x,y)$.

Even in a random wave field there exists a *deterministic* principle that relates the signs of neighboring singularities that we called the "sign principle" [61,62]. This principle reads: Adjacent singularities on any zero crossing of G or H (Ψ_X or Ψ_Y) must be of opposite sign. Since this rule implies that we can unambiguously determine which singularities are adjacent on a zero crossing, self-intersections of zero crossings must be eliminated by a small, local perturbation of the wave function, which can always be done without penalty [46]. Numerous far reaching implications of the sign principle are discussed in Refs. [61–64].

All topological singularities in our random wave field are first-order zeros, so that both φ and Θ change discontinuously by π on any line passing through the center of the singularity. This leads us to define lines of reduced phase $\varphi^*=\varphi(\mathrm{mod}\pi)$ and lines of reduced order parameter $\Theta^*=\Theta(\mathrm{mod}\pi)$ that thread their way continuously through the wave field from one singularity to another. We note that the definition of Θ^* is equivalent to removing the heads of the arrows comprising the vector field $\mathbf{\Psi}$. The zero crossings of G (Ψ_X) correspond to equiphases with φ (Θ)$=0$ or π, and those of H (Ψ_Y) correspond to equiphases with φ (Θ)$=\pi/2$ or $3\pi/2$. But the phase of a wave field may be uniformly shifted by an arbitrary amount without changing its internal structure, so that any equiphase can be turned into a zero crossing of G or H. Similarly, Θ may be uniformly shifted by a coordinate rotation, so that any line of constant Θ can be turned into a zero crossing of Ψ_X or Ψ_Y. Accordingly, the sign principle is extended to read [62] as follows.

Sign principle. Adjacent singularities on any contour of constant reduced phase (order parameter) must have opposite signs.

The restriction to *contours* again arises from the requirement that we must be able to unambiguously determine which are the adjacent singularities, and this, in turn, requires that the direction of the equiphase be everywhere well defined. Thus the sign principle cannot be applied to the bifurcation lines of a saddle. We recall that, unlike the self-intersections of zero crossings, isolated saddle points are only shifted by small local perturbation but not eliminated altogether. Accordingly, a new principle is required for singularities that terminate bifurcation lines. This new principle is derived in Sec. VI.

V. THE TOPOLOGICAL INDEX

Associated with each type of stationary point and singularity (collectively critical points) is a quantity called the topological index. The value of this index is obtained by observing how lines of equal value of the function rotate over one complete circuit around the critical point

[66,67]. If the lines rotate through $2n\pi$ in the same (opposite) direction as the circuit, the index is $+n(-n)$. The indices for an extremum (maximum or minimum) and for an isolated first-order point singularity (positive or negative) equal $+1$, while for a generic four-armed (second-order) saddle the index is -1. (Illustrations for these and other critical points may be found in Sec. 36 of Ref. [66].) For nongeneric saddles containing $2j$ arms with $j>2$, the index may be seen to be $1-j$. (Odd armed saddles cannot occur in a regular wave field.) The index associated with some region of the wave field is the sum of the individual indices of the critical points in this region. The index associated with a *closed* saddle, for example, is the same as that of a single extremum, $+1$. This follows from the fact that the saddle, whose index is -1, must contain either two extrema or a pair of positive and negative phase singularities, whose indices sum to $+2$. By straightforward extension, deeply nested closed saddles with arbitrary combinations of enclosed extrema and/or singularities also always yield a net index of $+1$. Accordingly, extrema and closed saddles are for the most part topological equivalents, as is suggested by the fact that both are surrounded by closed contours.

We can now give a formal justification for the converse of the first loop rule. As the index of a closed contour located just inside the joined arms of a saddle that form a bifurcation loop is $+1$, we conclude that the indices of all critical points contained within the loop sum to unity. But the bifurcation loop is also an equiphase, so that the net change in phase during traversal of the loop vanishes. This implies that the topological charges of all critical points contained within the loop sum to zero. These twin requirements of a net index of unity and a net topological charge of zero imply that the loop can embrace only a *single* extremum or closed saddle, which is the converse of the first loop rule. The first loop rule itself, although correctly describing the interior of random wave fields, cannot be proven in this way, as the rule is not universally true—a simple counterexample to the rule is provided by the function $f(x,y)=\exp[-(x^2+y^2)]$. We note, however, that a closed circuit that lies just outside the arms of a bifurcation loop and that includes the saddle point has an index of zero, so that both the index and topological charge of a bifurcation loop with its saddle point and enclosed critical points are zero. Accordingly, bifurcation loops are topologically "neutral" and can be inserted anywhere within the wave field using only local perturbations.

An important property of the index of some bounded region is that it is conserved under continuous perturbation of the wave field, provided only that no critical point crosses the boundary *(index theorem)* [66,67]. This latter restriction is a weak point of the index theorem, since there is no general rule that permits one to decide in advance when and where critical points will move in or out of the region of interest. This is illustrated in Fig. 4 using a simple model phase field [60]. From this figure it may be noted that the mobile critical points are saddle points, which often leave the wave field altogether, being expelled to infinity. On the other hand, in a closed space from which no feature can be expelled such as the surface of a sphere, the index theorem can be used to good effect, as is done in an interesting, instructive paper by Nye, Hajnal, and Hannay, who studied two-dimensional ocean waves on the surface of a spherical planet [70].

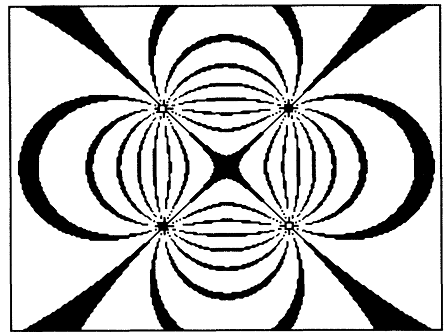

FIG. 4. Three saddles needed to satisfy the index theorem are missing from this model phase field. The field was created using the product wave function of Eq. (1a) of Ref. [60] with alternating positive (closed squares) and negative (open squares) isotropic ($\alpha=\pm1$, $\sigma=\rho=0$) singularities arranged at the corners of a square of side s to yield an arrangement whose net topological *charge* is zero. When $s=0$, the four singularities annihilate one another, the phase is everywhere uniform, and the topological *index* of the (empty) phase field is also zero. When $|s|>0$ (the wave function is invariant to the sign of s), four singularities (index $+4$) and one saddle (X shaped black region at figure center, index -1) are created and the net index of the phase field is $+3$. Since s is made to vary *continuously* through zero, the index ought to be continuous throughout the act of creation, implying that additional saddles with a total index of -3 are created together with the singularities. These missing saddles are presumed to be expelled to infinity at the moment of creation.

In the deep interior of an infinitely extended planar random wave field, no feature can be expelled from the region of interest, as every finite region is bounded by numerous contour lines that connect the surrounding (fixed) phase singularities. On its way out of the wave field the feature cannot cross these contours as this would make the wave function multivalued. The feature also cannot open a clear channel for its escape, as this would require a major reconstruction of a large segment of the wave field. Accordingly, we conclude that all features are "trapped" by the surrounding contour lines. An example of this trapping is shown in Fig. 5. In later sections we assume this trapping and use the index theorem to discuss local configurations of critical points in the interior of a random wave field.

VI. SADDLES AND SINGULARITIES

We now enlarge the sign principle to include bifurcation lines. We first define the *order* of an equiphase in terms of the number of self-intersections (saddle points) that it contains. An equiphase that is a bifurcation line of an isolated saddle is of order 1, a nongeneric equiphase that is a bifurcation line which runs through two saddle points is of order 2, etc., while a *contour* is an equiphase of order zero. Within this definition the sign principle is enlarged to read as follows.

Enlarged sign principle. Adjacent singularities on equiphases of even (odd) order must be of opposite (same) sign.

The proof of this assertion is obtained by shifting the phase (order parameter) of the wave field so as to make the equiphase of interest into a zero crossing of either G (Ψ_X) or H (Ψ_Y), and then using the (unenlarged) sign principle together with the method of contour extensions described previously [61,62]. This is illustrated in Fig. 6(a), while Fig. 6(b) shows that the bifurcation loop formed by joining together the arms of a single saddle is an equiphase of order 2, since in traversing the loop one passes twice through the saddle point. In what follows we reserve the term bifurcation *line* for the "straight" lines that pass only once through the saddle point.

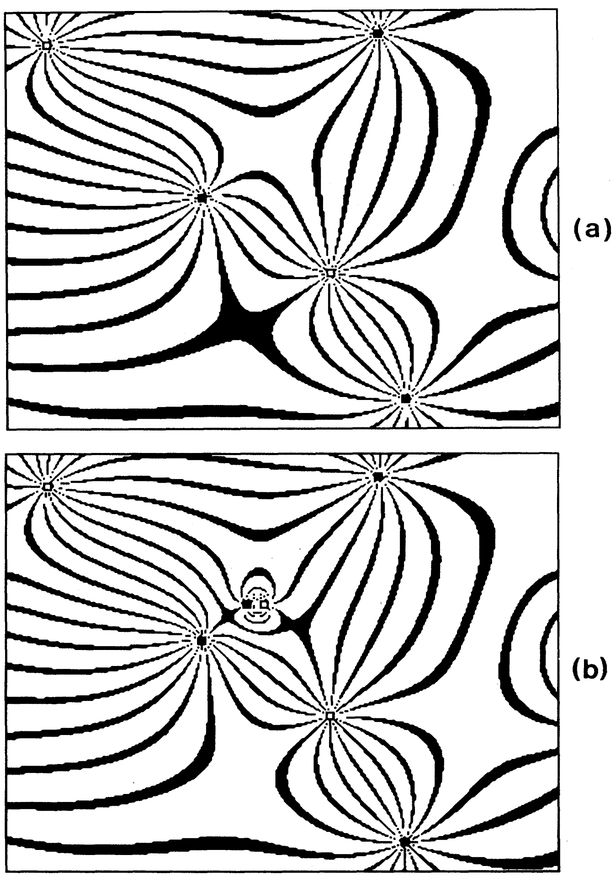

FIG. 5. Trapping of saddles in the interior of a random phase field. The field was created using the product wave function of Ref. [60] with a large number of randomly distributed isotropic singularities. (a) Interior of initial configuration. Positive (negative) singularities are symbolized by closed (open) squares. Saddles show up as distorted X shaped regions which are either black or white depending on the value of the phase, with the saddle point itself located at the center of the X. (b) A positive and negative singularity twin is created near the center of the (white) saddle just above the center line of (a). Both topological charge and topological index are conserved here since the saddles created together with the singularities are trapped by the surrounding contour lines and cannot escape to infinity.

The enlarged sign principle implies that in a generic random phase field bifurcation lines always terminate on singularities with the same sign, while contours and bifurcation loops always terminate on singularities with opposite signs. In a random phase field the topological

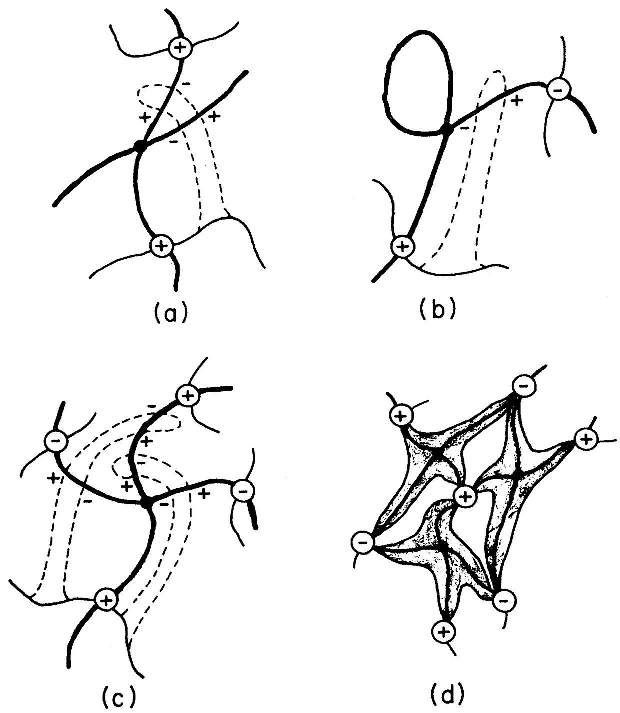

FIG. 6. The enlarged sign principle. (a)–(c) Zero crossings of the real (imaginary) part of the wave function are shown by heavy (light) lines. Contour extensions of the imaginary part of the wave function are shown by dashed light lines. Virtual vortices created when these contour extensions cross a zero crossing of the real part of the wave function are labeled by their signs + or −. The phase of the wave is shifted to turn a given saddle point into a self-intersection of the real part of the wave function. (a) Application of the (unenlarged) sign principle demonstrates that the singularities that terminate a bifurcation line of an open saddle must be of the same sign. The interested reader may enjoy extending the argument to show that singularities with the same (opposite) sign terminate bifurcation lines containing an odd (even) number of saddle points. (b) The singularities that terminate the bifurcation line of a one-loop saddle are shown to have opposite sign. As the saddle point is traversed twice in passing between the singularities, the bifurcation line contains two saddle points. (c) The singularities that terminate the arms of a generic four-armed open saddle are shown to alternate in sign from one arm to the next. The interested reader may enjoy showing that this sign alternation is true also for n-order (degenerate) saddles containing n bifurcation lines [e.g., $f(x,y)=y(y^2-x^2)$, $n=3$, $y=0,\pm x$]. By removing the degeneracy [e.g., $f(x,y)=(y-\varepsilon)(y^2-x^2)$] the order of a bifurcation line of an n-order saddle may be seen to be $n-1$, so that the bifurcation lines of a third-order saddle, for example, are terminated by singularities of opposite sign. (d) The generic arrangement of saddles and singularities in a random phase field. The X shaped regions of nearly constant phase surrounding each saddle point are shown shaded.

charges (± 1) of the singularities that terminate the arms of an open saddle are easily seen [Fig. 6(c)] to always sum to zero. As discussed below, open saddles predominate in random phase fields. Accordingly, in a random phase field the generic arrangement of saddles and singularities is that shown in Fig. 6(d).

Since all singularities in a generic random phase field are of first order, the phase circulates once from 0 to 2π along a convex closed path that encircles a single singularity. Accordingly, a given equiphase can contact a singularity only once. This implies that only one arm of a saddle can terminate at a singularity, so that open saddles grip four different singularities, while saddles with a bifurcation loop grip two. But how many *different* saddles can hold one singularity? The answer may be developed in the following way. We start with two singularities A (+) and B ($-$) connected to each other by contours. Examining the phase map of this structure we observe that A, for example, must be contacted by one bifurcation line [Fig. 7(a)]. Adding another singularity C ($-$) and connecting A to C requires that a second bifurcation line contact A [Fig. 7(b)], while connecting still another singularity D ($-$) to A requires still another bifurcation line to contact A [Fig. 7(c)], etc. We thus conclude that the number of bifurcation lines that contact A equals the number of singularities to which A is connected by contours. But each bifurcation line *must* belong to a different saddle point, so if A is connected to m different singularities it must be held by m different saddles. The converse of this is also true, and if a given singularity is held by m saddles it must be connected to m other singularities.

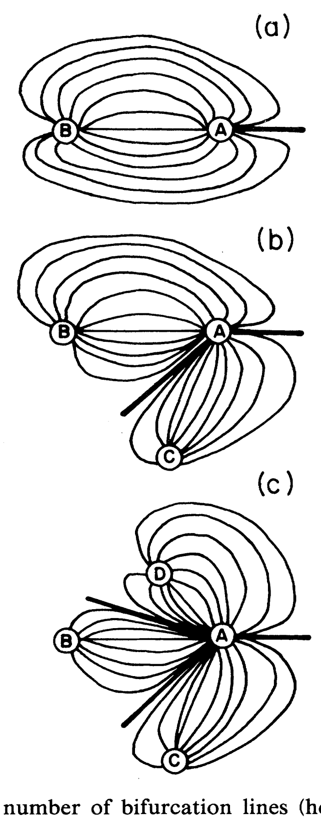

FIG. 7. The number of bifurcation lines (heavy lines) contacting positive singularity A equals the number of negative singularities to which it is connected by contours (light lines). A connects to (a) one singularity B, (b) two singularities B and C, and (c) three singularities B, C, and D, and is contacted in turn by one, two, and three bifurcation lines, respectively.

What is the average value of m? A first-principle calculation based on, say, Gaussian statistics appears to be quite impossible at the present time, so we supplement theory with empirical information to estimate $\langle m \rangle$. We recall that nested saddles are never seen in random phase fields, so they are either completely absent or of negligible statistical weight. Accordingly, only one-loop saddles [Fig. 2(b)] are ever found. Neglecting edge effects (wave field boundary), this permits us to divide the total number of saddle points (N_{sad}) into two classes, those that contain four *free* arms [n_4 in number, Fig. 2(a)], and those that contain two [n_2 in number, Fig. 2(b)]. Since each bifurcation loop embraces a single extremum (N_E in number), we have $N_E = n_2$. In a later section we show that the number of singularities (N_{sing}) plus the number of extrema substantially equals the total number of saddles. From this follows $N_{\text{sing}} = n_4$. The total number of free arms is $4n_4 + 2n_2$, and these arms must be divided among n_4 singularities, so $\langle m \rangle = 4 + 2n_2/n_4$. From large scale computer simulations [59] we have found that within a statistical uncertainty of $\sim 2\%$ $N_{\text{sad}} \approx 14N_E$, so that $n_4 \approx 13n_2$. Adopting this value, we have $\langle m \rangle \approx 4 + \frac{2}{13} = 4.15$. Thus, on average each singularity is held by slightly more than four different saddles and is connected to slightly more than four different singularities. In a similar fashion we easily conclude that the average number m' of singularities held by each saddle is $\langle m' \rangle = (4n_4 + 2n_2)/(n_4 + n_2) \approx 3.86$.

In light of the fact that there are so few phase extrema, it is natural to inquire why these features which are so abundant in random *intensity* fields are so rare in random *phase* fields. As discussed previously [59], the average number density of phase singularities is 1 for every two coherence areas. Since it is a topological necessity that there be one phase saddle for each singularity, saddles and singularities together have a joint number density of 1 per coherence area. But the wave field is not expected to vary appreciably on length scales less than the coherence length, so that one feature per coherence area essentially exhausts the possible spatial variation of the wave field. Accordingly, we may conclude that phase extrema are so rare because there is little room left in the phase field to accommodate them.

VII. SADDLES, SINGULARITIES, AND EXTREMA

The rule that the two ends of a bifurcation loop terminate on singularities with opposite signs [Fig. 6(b)] leads to a rule for phase extrema. We consider traversing a bifurcation loop starting always at the negative singularity and ending at the positive singularity (Fig. 8). If we traverse the loop counterclockwise (the positive θ direction by our sign convention) we assign a positive sign to the loop, and if we traverse it clockwise (negative θ direction) we assign to the loop a negative sign. As discussed

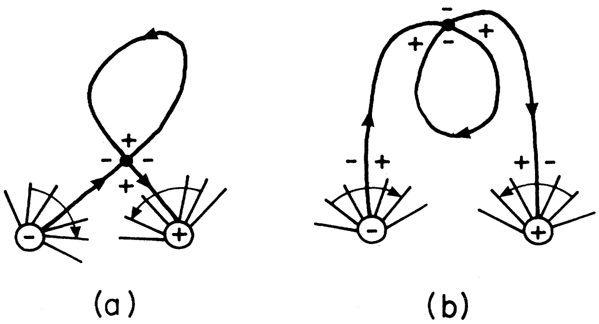

FIG. 8. The sign of a bifurcation loop is determined by its terminating singularities. The saddle point is moved into the (x,y) plane and its bifurcation lines become zero crossings. The bifurcation loop is traversed by always starting at the negative singularity and ending at the positive singularity. The direction of increase of the phase about each terminating singularity is shown by arrows. These directions determine which side of the bifurcation line is positive and which side is negative, and thus fix the sign of the bifurcation loop. (a) The loop is traversed counterclockwise (positive direction) and its sign is seen to be positive. (b) The loop is traversed clockwise (negative direction) and its sign is seen to be negative. The interested reader may enjoy working through the six additional cases in which the signs of the singularities are interchanged and/or the loops shown are reflected about the line joining the singularities. In all cases the sign of a loop is positive (negative) if it is traversed in the positive (negative) direction.

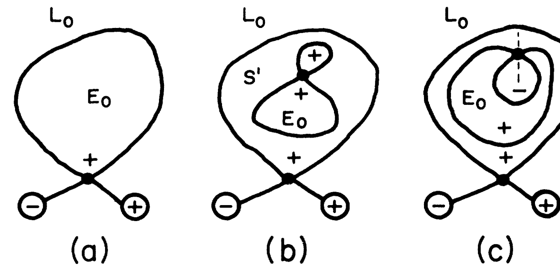

FIG. 9. The sign of an internal loop is the same as the sign of the external loop that embraces it. (a) Starting point for the analysis in the text. The sign of the outermost loop L_0 is assumed here to be positive and the extremum E_0 that it embraces is a maximum. (b) A new saddle S' which is a figure eight is created inside L_0. Since E_0 must be inside one loop of this figure eight and is a maximum, and since both loops of a figure eight have the same sign [Fig. 3(d)], both loops of S' are positive. Both loops of S' are embraced by L_0 and both are seen to have the same positive sign as L_0. (c) The new saddle S' is reentrant. As discussed in the text, upon passing through the saddle point of S' along the dashed line one passes over a maximum, so the interior loop of S' is negative. Since the interior and exterior loops of a reentrant saddle have opposite signs [Fig. 3(e)] the exterior loop is positive. This exterior loop, which is the only loop embraced by L_0, is seen to have the same positive sign as L_0.

in Sec. III, every loop must contain either a single extremum or else a closed saddle, and for the case of an extremum we have the following.

Second loop rule. Maxima (minima) are contained only in positive (negative) bifurcation loops.

This simple but important rule is an immediate consequence of the fact that the wave function increases (decreases) for a positive (negative) loop as one moves towards the loop center from any point on the boundary (Fig. 8).

We now develop a rule for handling extrema within a closed saddle that *nests* within a loop of some other saddle. Although there are various routes to such a rule, the route we use employs the index theorem together with the second loop rule to find the sign of an internal loop based on the sign of the external loop that embraces it (Fig. 9). Specifically, we imagine recording the structure of the nested loops on a "loop map," and then having all extrema and saddle points internal to some outermost loop L_0 annihilate one another in pairs as required by the index theorem, leaving a single extremum E_0 embraced by L_0 [Fig. 9(a)]. We assume the sign of L_0 is fixed by its terminating phase singularities as described previously, so that E_0 is uniquely determined by the second loop rule. Consulting our loop map, we introduce into L_0 the next outermost saddle, which we label S'. One of its loops must necessarily embrace E_0. If S' is a figure eight [Fig. 9(b)], the extrema contained within its two loops are both of the same type as E_0, so the signs of both loops are the same as the sign of L_0. We have thus determined that when dealing with figure eights the sign of an inner loop is the same as the sign of the outer loop that embraces it.

If S' is reentrant [Fig. 9(b)], however, we are faced with the problem of deciding into which of its two loops E_0 is to be placed. Suppose, for example, the sign of L_0 is positive, so that E_0 is a maximum (second loop rule) and the phase increases as we move away from the boundary of L_0 towards the loop center. Inside L_0, as we cross the saddle point of S' along one of its principal axes [Fig. 9(c)] we pass either over a maximum or through a minimum. But the phase is increasing along all directions that lead directly (i.e., not across a bifurcation line) from the boundary of L_0 to the saddle point of S', so that entry into the interior loop of S' from L_0 must be over a maximum. Coming down from this maximum into S' we conclude that inside the interior loop of S' the phase decreases as we move away from the arms that form this loop, so this interior loop contains a minimum. By the second loop rule its loop sign is negative. Since the two loops of a reentrant saddle are always of opposite sign, the exterior loop of S' must then be positive and contain a maximum, which is E_0. Repeating the above argument for the case in which the sign of L_0 is negative and E_0 is a minimum leads us to conclude that in this case the sign of the interior loop of S' is positive and the sign of the exterior loop of S' is negative. Accordingly, the *exterior* loop of the reentrant saddle S' always has the same sign as L_0, the loop that embraces it. As this was also the case for the loops of a figure eight, we obtain the following.

Third loop rule. The sign of an internal loop is the same as the sign of the external loop that embraces it.

We recall that since a saddle does not embrace itself the exterior loop of S' does not embrace its interior loop, so the third loop rule is not contradicted by the fact that the interior and exterior loops of a reentrant saddle always have opposite signs. With the assignment of loop signs via the third loop rule, the second loop rule for phase extrema holds also for nested loops. We again note that in actual fact we have never seen an example of a nested loop in a random (or any other) *phase* field. The third rule, however, is of the greatest importance for amplitude (intensity) fields, where deep nesting of saddles is the generic arrangement.

The sign principle implies that, for a fixed topology of saddles (saddle points together with their bifurcation lines), changing the sign of *one* singularity in a phase field forces the signs of all *other* singularities to change. The loop rules show that maxima must then change into minima and vice versa. A similar result holds for extrema, and if a single maximum within a phase field is converted into a minimum, then all maxima must be converted into minima and vice versa, and all positive phase singularities must become negative singularities and vice versa. Clearly, the sign principle and the loop rules together eliminate all but one degree of freedom for the critical points, establish stringent constraints on the possible structure of random (and other) phase fields, and induce extensive, unexpected, topologically mandated correlations between *all* the different critical points of the field.

As mentioned, deeply nested loops are generic to random fields without singularities such as the intensity, since saddle arms have nowhere to go and must close on themselves. This is verified by our computer simulations. The loop rules hold also for these fields, so if an extremum anywhere within some set of deeply nested loops is identified (or arbitrarily chosen) as, say, a maximum, then the loop rules (working both outward and inward) suffice to fix the nature of all remaining extrema within this set. This implies that neighboring intensity maxima, minima, and saddles in a speckle pattern are not independent, in agreement with our previous conclusions based upon the sampling theorem [64].

VIII. SADDLES AND EXTREMA AS SINGULARITIES

We now turn phase saddles and extrema into *singularities* by defining the vector field $\Xi(x,y)=\nabla\varphi$ together with its associated order parameter $\Theta'=\arctan(\varphi_y/\varphi_x)$. As before (Sec. IV), the signs of the singularities of Θ' are given by $d\Theta'/d\theta$, which equals the sign of the Jacobian (Hessian) $J'=\partial(\varphi_x,\varphi_y)/\partial(x,y)$, when this exists. Stationary points and singularities of φ are the *only* singularities of Θ', since φ is regular everywhere between its singularities. Now, J' exists at the stationary points of φ (recall Morse's lemma [66,67]), so that extrema of φ are *positive,* and saddle points of φ are *negative,* isolated first-order singularities of Θ'. J', on the other hand, is singular at the singularities of φ, but $d\Theta'/d\theta$ is not. Moving the origin to the center of some arbitrarily chosen φ singularity, noting that leading terms in an expansion of the wave function (G and H) around the (first-order) singularity are linear in x and y, and using $\varphi=\arctan(H/G)$ yields after straightforward calculation $\Theta'=\theta+\pi/2$ $(\theta+3\pi/2)$ for positive (negative) φ singularities. From this it follows that both positive and negative φ singularities are *positive,* isolated, (isotropic [60]) first-order Θ' singularities. Thus *all* singularities of Θ' are isolated and of first order, and since Θ' is regular everywhere between its singularities, the phase field Θ' is generically the same as the phase field φ, and contains the by now familiar mix of positive and negative singularities, four armed saddles, and extrema enclosed in bifurcation loops. The two phase fields differ mainly in the number of their singularities, since the number of Θ' singularities is $N_{\Theta'}=2n_2+2n_4\approx 2.15N_{\rm sing}$. All of the foregoing is confirmed in detail by our computer simulations, and in Fig. 10(a) we present an example of a random phase (φ) field generated by these simulations, while in Fig. 10(b) we display the associated Θ' field.

The sign principle for the singularities of $\varphi=\arctan(H/G)$ was originally derived for fields G and H that are everywhere regular and do not themselves contain singularities [61,62]. We now rederive this rule

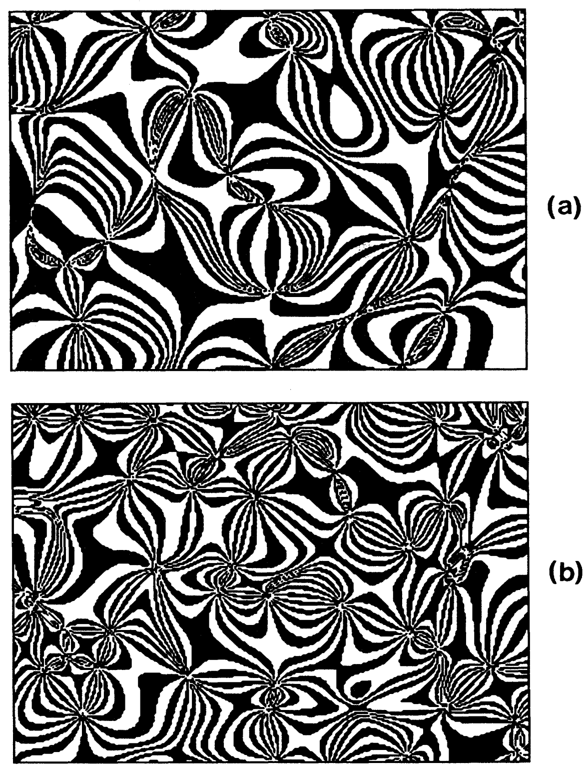

FIG. 10. (a) Generic random phase field φ created by the computer simulation described in Ref. [59]. (b) Order parameter Θ' of the gradient field $\nabla\varphi$. In both (a) and (b) singularities are located at the centers of the black and white striped "pinwheels," and saddles appear as distorted black or white X shaped regions. In (a) a single extremum appears to the upper right of center as a white teardrop shape on a black background, while the single extremum in (b) appears to the lower right of center as a small black teardrop shape on a white background. The interested reader may enjoy matching the singularities of (b) with the critical points of (a).

for the phase field $\Theta'=\arctan(\varphi_y/\varphi_x)$, since here both φ_x and φ_y do contain singularities. Consider (Fig. 11) two singularities 1 and 2 that are adjacent on a *contour* of constant Θ' labeled by C. C leaves singularity 1 with local polar angle θ_1, arrives at singularity 2 with local polar angle θ_2, and corresponds to some value Θ'_{12} for the order parameter. As the order parameter is continuous and differentiable between singularities, there exists at least one neighboring contour C' connecting the two singularities that leaves singularity 1 with local polar angle $\theta_1+d\theta_1$, arrives at singularity 2 with local polar angle $\theta_2+d\theta_2$, and corresponds to the value $\Theta'_{12}+d\Theta'$ for the order parameter. Since C and C' never intersect between singularities, $d\theta_1$ and $d\theta_2$, and therefore $d\Theta'/d\theta_1$ and $d\Theta'/d\theta_2$, *must* be of opposite sign (Fig. 11). We have thus established that adjacent singularities on contours of constant Θ' are of opposite sign. Now, contours on which $\Theta'=0$ or π are zero crossings of φ_y, and contours on which $\Theta'=\pi/2$ or $3\pi/2$ are zero crossings of φ_x. Accordingly, we have also established that Θ' singularities that are adjacent on zero crossings of φ_x or φ_y *must* alternate in sign, which completes the proof of the sign principle. But we can go further. Since every singularity of Θ' contacts (by a limiting process) contours on which Θ' equals 0, $\pi/2$, π, and $3\pi/2$, all singularities of Θ' lie at the intersections of the zero crossings of φ_x and φ_y. Now the singularities of Θ' include the saddle points, extrema, and singularities of φ. Accordingly, we conclude that (i) not only the stationary points of φ but also its *singularities* lie on the zero crossings of φ_x and φ_y, and (ii) on these zero crossings saddle points alternate with either singularities or extrema. As both φ_x and φ_y *diverge* at the singularities of φ, these conclusions do not appear to be self-evident. They are, of course, fully confirmed by our computer simulations.

As the sign principle holds for Θ' (as do the loop rules), on closed contours of reduced Θ' (modπ) the number of positive singularities must equal the number of negative singularities if the sign principle is to be satisfied everywhere along the contour. On average the same is very nearly true for open contours that reach the wave field boundaries. Accordingly, we may conclude that there are approximately (but not necessarily exactly) equal numbers of positive and negative singularities in the phase field Θ'. But positive Θ' singularities correspond to

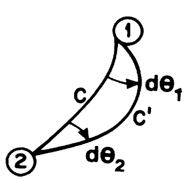

FIG. 11. The signs $(d\Theta'/d\theta)$ of adjacent singularities on a contour of constant order parameter Θ' are opposite. Singularities 1 and 2 are connected by contour C with order parameter Θ'_{12}. In going from C to neighboring contour C' with order parameter $\Theta'_{12}+d\Theta'$, the local polar angles $d\theta_1$ and $d\theta_2$ are of opposite sign, so that singularities 1 and 2 also have opposite signs.

φ singularities and φ extrema, while negative Θ' singularities correspond to φ saddles, so we conclude that for the phase field φ the number of phase saddles approximately (but not necessarily exactly) equals the sum of the numbers of phase singularities and phase extrema. This approximate equality is consistent with the index theorem, which may be interpreted as suggesting that the total index of the phase field ought to be the Euler characteristic $\chi=+1$ of a plane [66,67]. We note, however, that the initial phase structure of the wave is created when the wave peels away from the scattering surface or source distribution and lifts off to begin its journey through free space. But this is a *discontinuous* process, while the structure of the scattering surface or the arrangement of sources may be completely arbitrary. Under these circumstances the index theorem may not be invoked, and *physical* wave fields exist that violate topological conservation laws which are based on continuous evolution. (The sign principle and the loop rules, however, are never violated.) One well known example is given by the work of Arecchi and co-workers [56] who produced laboratory wave fields with a 6:1 preponderance of positive to negative phase singularities, showing thereby that one may not invoke conservation of topological charge during the process of wave field creation. Similarly, using both theory and experiment, Brambilla and co-workers [71] demonstrated the existence of laser beams containing very different numbers of phase singularities and phase saddles (but no phase extrema) showing that one may also not invoke the index theorem during wave field creation. Thus, the index theorem by itself is insufficient to obtain the approximate equality relating the numbers of saddle, singularities, and extrema developed above. In the next section we show that, in spite of the fact that this approximate equality does provide a good description of a random phase field, no *exact* rule can ever exist for the numbers of phase saddles, singularities, and extrema.

Previously, we found that the sign principle applied to the phase field φ induced strong correlations between the internal structures of neighboring φ singularities [61,63]. We therefore expect that application of the sign principle to Θ' will similarly induce strong correlations between neighboring Θ' singularities. As neighboring Θ' singularities correspond to neighboring φ extrema, φ singularities, and φ saddles, we are now led to predict that all these different features ought to be strongly correlated. The many new, previously unsuspected phase correlations that have emerged in this way will be reported on elsewhere.

We conclude this section by noting that the enlarged spin principle and its induced correlations, together with the loop rules and their induced correlations, remain true also for the field Θ'' defined as the order parameter of $\nabla\Theta'$, as well as for the field Θ''' defined as the order parameter of $\nabla\Theta''$, etc. The same is also true for the vector fields and associated order parameters that are based upon the wave field *intensity* (amplitude) [64]. This infinite regress, which establishes a dense network of correlations between all points in the wave field, will undoubtedly prove to have the most profound consequences for the structure of random (and other) wave fields.

IX. WAVE FUNCTION SCALING

We now show by means of an explicit counterexample that in a random phase field there cannot exist an *exact* rule for the numbers of phase extrema, saddles, and singularities. This is in contrast to the wave field intensity, for which there does exist an exact rule due to Longuet-Higgins [72] that requires the number of saddles to be one less than the number of extrema. Consider a speckle field projected onto the surface of a large hemisphere of radius R centered on a sample that occupies a square aperture of side L in a planar, infinitely extended opaque mask. Measuring position on the sample surface by coordinates ξ,η, we take the random amplitude and phase of the wave on this surface to be a function of the scaled coordinates ξ/L, η/L. A physically acceptable sample realization [59] that exhibits this scaling is a set of $n=1-N$ point sources with random amplitudes, phases, and coordinates $\xi(n)=r_\xi(n)L/2$, $\eta(n)=r_\eta(n)L/2$, where r_ξ and r_η are two different fixed sets of random numbers lying between ± 1. In order to eliminate superfluous phase factors, we choose R such that $kR=(2j+\frac{1}{2})\pi$, where $k=2\pi/\lambda$ with λ the wavelength, and we make the integer j sufficiently large such that $kL^2/R \ll 1$. Measuring position on the sphere surface by means of the usual polar angle θ and azimuthal angle ϕ, and using the Rayleigh-Sommerfeld formulation of Huygens' principle [73,74], the real (G) and imaginary (H) parts of the wave function may be written $G=\gamma g(u,v)\cos\theta$ and $H=\gamma h(u,v)\cos\theta$, where $u=kL\sin\theta\cos\phi$, $v=kL\sin\theta\sin\phi$, $\gamma=L^2/(\lambda R)$, and g and h are random Fourier transforms determined by the specifics of the random sample. Due to the $\cos\theta$ factor, G and H both go to zero at the equator where $\theta=\pi/2$, so the wave field terminates at the equator. Because of the form of the variables u and v and the scaling properties of the source distribution, the functions g and h and therefore $\varphi=\arctan(h/g)$ scale with sample size L such that $\varphi(\theta',\phi')=\varphi(\theta,\phi)$ when $\sin\theta'=(L/L')\sin\theta$ and $\phi'=\phi$. Thus decreasing L causes the phase map to expand in such a way that every feature on this map is shifted towards the equator along a line of constant longitude. But features that are initially close to the equator cannot be shifted past the equator, and so these features fall off the edge of the map and disappear. Taking account of the $\cos\theta$ foreshortening of the apparent sample size which leads to a $1/\cos\theta$ elongation of the wave field coherence area [43], the fraction of all features initially present that fall away when L decreases to L' is easily seen to be $1-(L'/L)^2$. Halving L, for example, causes $\frac{3}{4}$ of all features to cross the equator and disappear. Our computer simulations [59] confirm that in a random wave field different regions bounded by contours of constant latitude contain randomly different selections of phase saddles, minima, maxima, and singularities, so that decreasing L causes undetermined numbers of each of these features to vanish from the wave field. (If this were not the case, definite relative numbers of each kind of feature would have to be present on lines of constant latitude, implying a regularity of the phase field structure that is not seen.) The foregoing demonstrates that no exact rule can exist for the numbers of maxima, minima, saddles, and singularities in a random phase field, in contrast to the wave field intensity for which Longuet-Higgins's rule does hold [73].

X. MULTIPLE EVENTS

A new phase singularity is created when an ordinary intensity minimum dips down to become a zero of intensity. But the birth of a new topological singularity abruptly alters the wave field everywhere. As this is forbidden by continuity, in *free* space the new intensity zero must immediately split into two, yielding twin positive and negative phase singularities that conserve topological *charge* [22,46,48]. Conservation of the topological *index* requires, however, that in free space each phase singularity (index $+1$) must be created together with a phase saddle (index -1) [70]. We call this sequence of events the "primary route" for the creation of singularities. In principle, the index theorem also permits a secondary route. Here, creation of twin positive and negative phase singularities takes place from the paired phase maximum and minimum contained within the loops of a reentrant saddle *without* creation of additional phase saddles. (The interested reader may enjoy visualizing this process and determining the relationship between the signs of the singularities and the nature of the extrema.) We note, however, that the resultant structure—singularities *inside* saddles—is never observed, so this secondary route is evidently never taken in practice, and the primary route described above is the generic route for the creation of new phase singularities. Since phase singularities are intensity minima (zeros), conservation of index also requires that the second singularity of a new phase singularity twin must be created together with a new *intensity* saddle. The free space creation of a new phase singularity is thus seen to be a multiple event that heralds the creation of *sextuplets* (twin positive and negative phase singularities, two new phase saddles, a new intensity minimum, and a new intensity saddle).

Conservation of index holds also for the order parameter Θ' of the gradient field $\nabla\varphi$, and this enlarges the multiple event. When four new features are added to the phase field φ during a multiple event (twin positive and negative phase singularities and two phase saddles), four new saddles must be added to the gradient field Θ', which brings the number of new wave field features that are created to ten *(dectuplets)*. But index conservation holds also for the gradient field ∇I derived from the intensity I, so that the creation of two new intensity features during a multiple event (the second new zero minimum and its accompanying new saddle) must be accompanied by the creation of two new ∇I saddles, bringing the number of new features to twelve *(duodectuplets)*. Clearly these arguments may be endlessly extended to higher-order gradient fields, and to other starting points such as the creation of a new intensity extremum. Since all these new features are created at the same time and in the same place, they initially are highly correlated one with another. As the wave field continues to evolve, these features separate along continuous trajectories constrained by the sign principle and loop rules. But, structurally correlated at creation, these new features are likely to remain correlated even when widely separated, suggesting the existence of extended, topologically mandated but previously unsuspected correlations between many different aspects of the phase and amplitude of a random wave field.

XI. SUMMARY

Saddles, singularities, and extrema in a generic random phase field are arranged as follows: Approximately $\frac{13}{14}$ of all saddles are strung between four different phase singularities, of which two are positive and two are negative in sign, with singularities of the same sign terminating each of the two equiphases (bifurcation lines) that pass straight through the saddle point. The remaining saddles contain a single loop and are terminated by a pair of positive and negative phase singularities. This loop houses a phase extremum, and, conversely, extrema are found *only* within such loops. The relationship between saddles and singularities is governed by a deterministic rule, the enlarged sign principle, and three additional deterministic rules, the loop rules, govern the relationship between maxima and minima and the saddles that house them. These loop rules apply also to the wave field intensity, and govern the relationship between saddles and extrema in this field. We note that the sign principle and loop rules are not only topologically necessary constraints, but are also sufficient in the sense that configurations for which the rules appear to be mute turn out under closer scrutiny to be forbidden.

On average, each phase singularity is connected to slightly more than four different saddles, while each saddle is connected to slightly less than four different singularities. These connections are made via bifurcation lines, which form a highly connected network that runs throughout the phase field. There are two independent networks of bifurcation lines; one connects the positive, and the other connects the negative phase singularities. These two networks are weakly coupled through the relatively small number of looped saddles that house phase extrema. Based on our computer simulations we conjecture that, starting at any given singularity, any other singularity can be reached by moving along these two weakly coupled networks of bifurcation lines. Saddles and their bifurcation lines therefore form a scaffold that underlies the phase field structure.

The sign principle and the loop rules strongly couple together all the singularities, saddles, and extrema in the phase field, so that for a given topology of saddles changing the sign of a single singularity changes the signs of all other singularities, turns all maxima into minima, and all minima into maxima. Similarly, changing a single maximum into a minimum causes all maxima and minima to be interchanged and reverses the signs of all phase singularities.

Phase saddles, singularities, and extrema are all singularities of an infinite regress of higher-order gradient fields. The sign principle and loop rules apply also to these higher-order fields. This leads to an infinite regress of couplings of the spatial variation of the phase throughout the phase field.

The number of phase saddles approximately (but not necessarily exactly) equals the sum of the numbers of phase singularities and phase extrema. Unlike the intensity, for which an exact rule exists relating the number of intensity saddles and extrema, no exact rule can exist for the phase. This difference between the two fields reflects the fact that, unlike the phase, the intensity goes to zero at the wave field boundary with zero slope (gradient).

The field scattered by a multiple scattering medium has a complex spatial variation of polarization that is determined by the phase and amplitude structure of its two orthogonal components. The spatial variation of the wave field polarization is therefore expected to be at least indirectly constrained via the topological rules developed here that constrain the phase and amplitude fields of the individual polarization components. Direct topological constraints on the polarization are also likely and are worth searching for.

During the free space evolution of the wave field, phase singularities and saddles are necessarily created together with intensity minima and intensity saddles in a multiple event. Here too there exists an infinite regress of higher-order gradient fields of the intensity and the phase that couple together the spatial variations of these two different aspects of the wave field. Clearly, the sign principle, loop rules, interactions present during multiple events, and other constraints that are likely to be found, must lead to a broad range of previously unsuspected, topologically mandated correlations between the phase, amplitude, and polarization of a random wave field.

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