Natural symmetries and regularization by means of weak parametric modulations in the forced pendulum

Ricardo Chacón
Departamento de Física Aplicada, Escuela Universitaria Politécnica de Almadén, Universidad de Castilla-La Mancha, 13400 Almadén, Ciudad Real, Spain
*and Departamento de Física, Facultad de Ciencias, Universidad de Extremadura, 06071 Badajoz, Spain**
(Received 10 January 1995)

The suppression of chaos in the driven pendulum due to a small parametric modulating term is studied theoretically with Melnikov's method, and the results are compared with those from classical perturbation theory. I obtained coherent results for the initial phase differences between the two modulations for which the chaotic dynamics is regularized. It is also shown that these values are compatible with the surviving natural symmetry under the parametric modulation.

PACS number(s): 05.45.+b, 05.40.+j

I. INTRODUCTION

Control of chaos by applying only small resonant parametric perturbations [1–4] seems to be an effective procedure due both its robustness *vis-à-vis* noise and its applicability to experimental situations. The choice of a parametric modulation (PM) instead of, e.g., a forcing term was previously considered from the observation that PM's can change the stability properties of hyperbolic or elliptic fixed points in the phase space of linear systems. It was conjectured that this result is also valid for nonlinear systems. (See Ref. [2] for a more thorough discussion.) The validity of this method in the general case has not been rigorously proved, although its effectiveness was demonstrated for the case of a Duffing-Holmes (DH) oscillator subjected to a PM of the cubic term [2,4]. In fact, it has been stated that the effect of a PM of a sine term—in the example of a driven pendulum—is to favor chaos [1]. On the contrary, in this paper I will show how such a PM stabilizes the chaotic motion provided there is a suitable phase difference between it and the original forcing, and for certain values of its amplitude. These phase differences are distinct from those found for the DH oscillator [4]. Generally speaking, one can expect them to depend on both the underlying conservative system (potential shape and homoclinic orbits) and the PM. In particular, as will be shown, they are closely related to the natural symmetries of the system.

Taking this into account, all the evidence indicates that the stabilization by a resonant PM at subharmonic frequencies of the main driving term is a generic method for perturbed Hamiltonian systems.

The rest of the paper is organized as follows. Section II gives the results concerning the suppression of chaos in the driven pendulum with a PM of the sine term by means of the analysis of the previous results arising from Melnikov's method (MM). In Sec. III, it is shown how an analysis of the natural symmetries of the complete system explains some results arising from MM. The inclusion of some numerical examples illustrates the scope and accuracy of the theoretical derivations. Section IV discusses the partial comparison (for the main resonance and symmetric oscillations) of the results from Secs. II and III with those expected from classical perturbation theory. (The details of the stability analysis of the symmetric oscillations are contained in an Appendix.) Finally, Sec. V includes a summary of the results and conclusions.

II. MELNIKOV'S ANALYSIS

The MM [5,6] is perhaps the only analytical technique currently available to provide a criterion for the occurrence of homoclinic chaos in a dynamical system, and it is today considered a standard method. Although the predictions from MM are both limited (only valid for motions based at points sufficiently near the separatrix) and approximate (the MM is a first-order perturbative method), they are of great interest due to the general scarcity of such analytical results in the theory of chaos. Since it has been described many times by distinct authors [1,5–9], I do not discuss it in detail here, but analyze the results previously obtained from it.

The dimensionless equation to be studied consists of a damped, harmonically driven pendulum with a PM of the sine term

$$\ddot{x} = -[1+\xi\cos(\Omega t+\theta)]\sin x - \delta\dot{x} + \gamma\cos(\omega t) , \qquad (1)$$

where x is the angular coordinate and Ω, ξ, and θ are the normalized frequency, amplitude, and initial phase, respectively, of the PM ($\xi \ll 1$) which has a suppressory effect on the chaotic dynamics of the remaining system ($\delta, \gamma \ll 1$) [10]. δ, γ, and ω are the usual normalized parameters: damping coefficient, driving term amplitude, and frequency, respectively. The physical meaning of such a PM is clear: the simple pendulum is mounted on a vertically oscillating point of suspension (see, e.g., [11]).

The application of MM to Eq. (1) gives us [1]

$$M^{\pm}(t_0) = -C \pm A\cos(\omega t_0) + B\sin(\Omega t_0+\theta) , \qquad (2)$$

*Corresponding address.

with

$$C=8\delta \ ,$$
$$A=2\pi\gamma\,\mathrm{sech}(\pi\omega/2) \ , \tag{3}$$
$$B=2\pi\xi\Omega^2\mathrm{csch}(\pi\Omega/2) \ ,$$

where the positive (negative) sign refers to the top (bottom) homoclinic orbit (of the underlying conservative system):

$$x_0(t)=\pm(4\arctan e^{t}-\pi), \quad \dot{x}_0(t)=\pm 2\,\mathrm{sech}t \ . \tag{3'}$$

As is well known [6], the Melnikov function (2) measures the distance between the perturbed stable and unstable manifolds in the Poincaré section at t_0. If $M^{\pm}(t_0)$ has a simple zero, then a homoclinic bifurcation occurs, signifying the *possibility* of chaotic behavior. Observe that the PM introduces an asymmetry between the homoclinic orbit with $\dot{x}_0<0$ and that with $\dot{x}_0>0$. As will be demonstrated below, this *asymmetry* gives rise to *two* distinct sets of phase differences between the cosines in (1) that are suitable for the regularization of the dynamics, corresponding to (the orbits based near) the two homoclinic trajectories.

Suppose that for $\xi=0$ we are in a chaotic state for which the associated Melnikov function $M_0^{\pm}(t_0) = -C\pm A\cos(\omega t_0)$ has simple zeros, i.e.,

$$A-C\equiv d\geq 0 \ , \tag{4}$$

where the equals sign corresponds to the case of tangency between the stable and unstable manifolds. If we now let the PM act on the system ($\xi\neq 0$) such that $B\geq d$, i.e., $A-B-C\leq 0$, this relationship represents a necessary condition for $M^{\pm}(t_0)$ to always have the same sign, specifically $M^{\pm}(t_0)<0$, which is

$$\xi > \left[1-\frac{C}{A}\right]R \ , \tag{5}$$

with

$$R=\left[\frac{\gamma}{\Omega^2}\right]\frac{\sinh(\pi\Omega/2)}{\cosh(\pi\omega/2)} \ . \tag{6}$$

For general Ω and θ ($0\leq\theta\leq 2\pi$), we shall see that this condition is not sufficient to assure the negativity of $M^{\pm}(t_0)$. In order to obtain such a sufficient condition, we shall first need five lemmas. For the sake of clarity, we shall treat separately the cases $M^{\pm}(t_0)$.

A. Motion near the upper homoclinic orbit

Lemma I. Let Ω/ω be irrational. Then there is some $\bar{t}_0$ such $A\cos(\omega\bar{t}_0)+B\sin(\Omega\bar{t}_0+\theta)>A-B$.

Lemma II. Let $q\Omega=p\omega$ for some positive integers p and q. Then a t_0^* exists such that $\cos(\omega t_0^*) = -\sin(\Omega t_0^*+\theta)=1$ if and only if

$$\frac{p}{q}=\frac{2m+3/2-\theta/\pi}{2n} \ , \tag{7}$$

for some integers m and n.

Remarks. Note that a requirement for Eq. (7) to be fulfilled for some integers m and n is $\theta=m_1\pi/m_2$, $m_{1,2}$ integers. For the particular case ($\omega=\Omega$, $\theta=0$) considered in Ref. [1], lemma II implies that it is not possible to find integers m,n fulfilling Eq. (7). This permits one to explain why in that reference it is reported that the PM favors chaos instead of its suppression.

Lemma III. Let $f(t;p,q)=[1-\cos(pt/q)]/(1-\cos t)$, t real, p and q integers. Then f is finite if and only if $q=1$. One also has that $0\leq f(t;p,1)\leq p^2$.

It is obvious that for Eq. (5) to also be a sufficient condition for $M^{+}(t_0)$ to be negative for all t_0, one must have

$$A-B\geq A\cos(\omega t_0)+B\sin(\Omega t_0+\theta) \ . \tag{8}$$

We now look for the values of Ω, ω, and θ permitting Eq. (8) to be fulfilled for all t_0. From Lemma I, a resonance condition is required: $p\omega=q\Omega$. In such a situation, Lemma II provides a condition for Eq. (8) to be satisfied for a infinity of t_0 values. Thus, let us suppose that p, q, and θ verify Eq. (7). One can then rewrite Eq. (8) in the form

$$\frac{A}{B}\geq\frac{1-\cos(pt/q)}{1-\cos t} \ , \tag{9}$$

with $t\equiv\omega t_0-2n\pi$. Finally, if $q=1$, Lemma III provides a condition for Eq. (9) to be fulfilled for all t:

$$\xi\leq\frac{R}{p^2} \ , \tag{10}$$

with R given by (6).

B. Motion near the lower homoclinic orbit

For this case we need two additional lemmas.

Lemma IV. Let Ω/ω be irrational. Then there is some $\bar{t}_0$ such that $B\sin(\Omega\bar{t}_0+\theta)-A\cos(\omega\bar{t}_0)>A-B$.

Lemma V. Let $q\Omega=p\omega$ for some positive integers p and q. Then a t_0^* exists such that $\cos(\omega t_0^*) =\sin(\Omega t_0^*+\theta)=-1$ if and only if

$$\frac{p}{q}=\frac{2m+3/2-\theta/\pi}{2n+1} \ , \tag{11}$$

for some integers m and n. The same Remarks hold as for Lemma II.

In this case, for Eq. (5) to be also a sufficient condition for $M^{-}(t_0)$ to be negative for all t_0, one must have

$$A-B\geq B\sin(\Omega t_0+\theta)-A\cos(\omega t_0) \ . \tag{12}$$

We then look for the values of ω, Ω, and θ permitting Eq. (12) to be fulfilled for all t_0. From Lemma IV, the resonance condition is again required: $p\omega=q\Omega$. With that, Lemma V gives a condition for Eq. (12) to be verified for a infinity of t_0 values. Let us suppose that p, q, and θ satisfy Eq. (11). It is then possible to rewrite Eq. (12) as

$$\frac{A}{B}\geq\frac{1-\cos(p\tau/q)}{1-\cos\tau} \ , \tag{13}$$

where $\tau\equiv\omega t_0-(2n+1)\pi$. Lastly, with $q=1$, Lemma III provides the condition (10) for Eq. (13) to be fulfilled for all τ.

The proofs of the lemmas are quite straightforward, so they will not be included here.

C. Suppression theorem (ST)

In brief, we have the following theorem: Let $\Omega = p\omega$, p an integer, such that, for $M^{+}(t_0)$ $[M^{-}(t_0)]$, $p = (2m + 3/2 - \theta/\pi)/(2n)$ $[p = (2m + 3/2 - \theta/\pi)/(2n+1)]$ is satisfied for some integers m and n. Then $M^{\pm}(t_0)$ always has the same sign, i.e., $M^{\pm}(t_0) < 0$, if and only if the following condition is fulfilled:

$$\xi_{\min} < \xi \le \xi_{\max} ,$$
$$\xi_{\min} = \left[1 - \frac{C}{A}\right] R , \tag{14}$$
$$\xi_{\max} = \frac{R}{p^2} .$$

Remarks. First, we can test the ST theoretically by considering the limiting case $\delta = 0$ (no damping). From Lemma II (Lemma V) and Eq. (14), one has $\Omega = \omega$, $\theta = 3\pi/2$ $(\theta = \pi/2)$, and $\xi = R$ as a sufficient and necessary condition for eliminating stochasticity. [But this is the obvious result arising from a direct analysis of Eq. (2) with $\delta = 0$, i.e., having $M^{\pm}(t_0) = 0$ for all t_0.] Second, for a given set of parameters satisfying the ST hypothesis, as the resonance order p is increased, the allowed interval $]\xi_{\min}, \xi_{\max}]$ for suppression shrinks quickly (as $1/p^2$). In Fig. 1 is plotted the width $\Delta\xi(\omega) = \xi_{\max} - \xi_{\min}$ versus ω for $\delta, \gamma = \text{const}$, showing the existence of a minimum frequency for each resonance. Note that the minimum $\omega_{\min}$ is lower as the resonance order is increased. The asymptotic behavior $\Delta\xi(\omega \to 0, \infty) = \infty$ means that chaotic motion is not possible in such limits. Third, the ST imposes having $\theta = 3\pi/2$ as the unique value (for *all* the resonances) in order to tame chaos when we consider orbits based near the upper homoclinic orbit. This is a consequence of Lemma II for $q = 1$. The striking result is that one obtains different suppressory θ values for distinct resonances ($\theta = \pi/2$ for $\Omega = m\omega$, m an odd integer, and $\theta = 3\pi/2$ for $\Omega = n\omega$, n an even integer) in the case of trajectories based near the lower homoclinic orbit. This arises from Lemma V for $q = 1$. Observe that since distinct θ values imply different initial conditions for the whole system (1), the above results give us information about the *size* and *symmetry* of the basins of attraction of the regularized orbits. In fact, as we will see in Sec. III, the permitted θ values are compatible with the natural symmetries of the system (1).

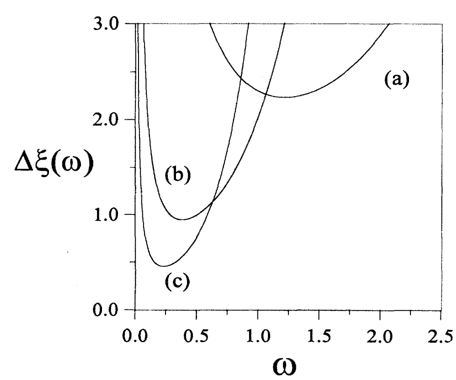

FIG. 1. Function $\Delta\xi(\omega) = \xi_{\max} - \xi_{\min}$ vs ω for $\delta = \text{const}$, $\gamma = \text{const}$, and $\Omega = p\omega$. (a) $p = 1$. (b) $p = 2$. (c) $p = 3$. ξ and ω are dimensionless variables.

III. NATURAL SYMMETRIES

The damped, harmonically forced pendulum ($\xi = 0$) has a natural symmetry [12,13] with respect to the transformation $(x \to -x, t \to t + \pi/\omega)$, i.e., if $[x(t), \dot{x}(t)]$ is a solution of Eq. (1) with $\xi = 0$, then so is $[-x(t + \pi/\omega), -\dot{x}(t + \pi/\omega)]$. This means that nonsymmetric stationary solutions always occur in pairs. As was mentioned in Sec. II, for $\xi \neq 0$ and arbitrary θ, this symmetry is generally broken. Note that the reason for that breaking is

$$\cos(\Omega t + \theta) \neq \cos[\Omega(t + \pi/\omega) + \theta] , \tag{15}$$

for arbitrary ω, Ω, and θ. With a resonance condition $\Omega = p\omega$, the survival of the above symmetry implies

$$\cos(p\omega t + \theta) = (-1)^p \cos(p\omega t + \theta) . \tag{16}$$

It is clear that this is only the case for p an even integer. For p an odd integer, we have the new transformation $(x \to -x, t \to t + \pi/\omega, \theta \to \theta \pm \pi)$, i.e., if $[x(t), \dot{x}(t)]$ is a solution for a value θ, then so is $[-x(t + \pi/\omega), -\dot{x}(t + \pi/\omega)]$ for $\theta \pm \pi$. Thus, this is the origin of the differences between the corresponding (at the same resonance order) permitted θ values (from the ST) for the two homoclinic orbits (see the third remark to ST). Similar results have been found for the DH oscillator with a parametric modulation of the cubic term [14].

Computer simulations of the system described by Eq. (1) showed overall good agreement between the numerical results and the theoretical predictions, even when the initial conditions and the perturbation amplitudes do not fit reasonably the MM requirements [6,8,9]. In Fig. 2 is plotted an illustrative example for the limiting Hamiltonian case ($\delta = 0$). The initial conditions were $(x_0 = 0, \dot{x}_0)$ and $(x_0 = 0, -\dot{x}_0)$ for the cases (a), (b), (c) and (d), (e), (f), respectively. Figure 3 shows an example in a situation clearly outside the perturbative requirements. For the set of parameters employed (see the caption to Fig. 3), the prediction of the ST was $(\xi_{\max} + \xi_{\min})/2 = 0.739\,78$. The initial conditions were $(x_0 = 0, \dot{x}_0)$ and $(x_0 = 0, -\dot{x}_0)$ for the cases (a), (b) and (c), (d), respectively. Observe the symmetry transformation between the chaotic orbits of cases (a) and (c), and between the periodic attractors of cases (b) and (d). The power spectra corresponding to the velocity series of the cases displayed in Figs. 3(a) and 3(b) is presented in Figs. 4(a) and 4(b), respectively.

IV. SYMMETRIC OSCILLATING STATES

As was pointed out by Kautz and Macfarlane [15], the type of chaos—in the example of a rf-biased Josephson junction—for which the onset is accurately predicted by the MM is described as phase-locked chaos [16–19]. This chaotic behavior is of a weak type (low positive Lyapunov exponent), the displacement maintains synchrony with the forcing, and its noise spectrum approaches zero at low frequencies. Thus, one might observe the suppression of this so-called "chaos without diffusion" when the requirements of the ST are satisfied. The simplest regular motion one can expect to find in this way would be phase-locked symmetric oscillations, possibly resulting from the reverse of the route: symmetric oscillations→symmetry breaking→period doubling→phase-locked chaos [17,20,21]. From classical perturbation theory, one can expect that the suitable θ values—in order to obtain symmetric oscillations—will be very close to those predicted from the ST. For the purpose of simplicity, only the case of the main resonance will be considered here. Observe that if the normalized frequency $\omega=\Omega<1$ (i.e., if the natural frequency of the unperturbed pendulum is greater than that of the two modulations), there will be some oscillation amplitude for which the period of the conservative system matches the com-

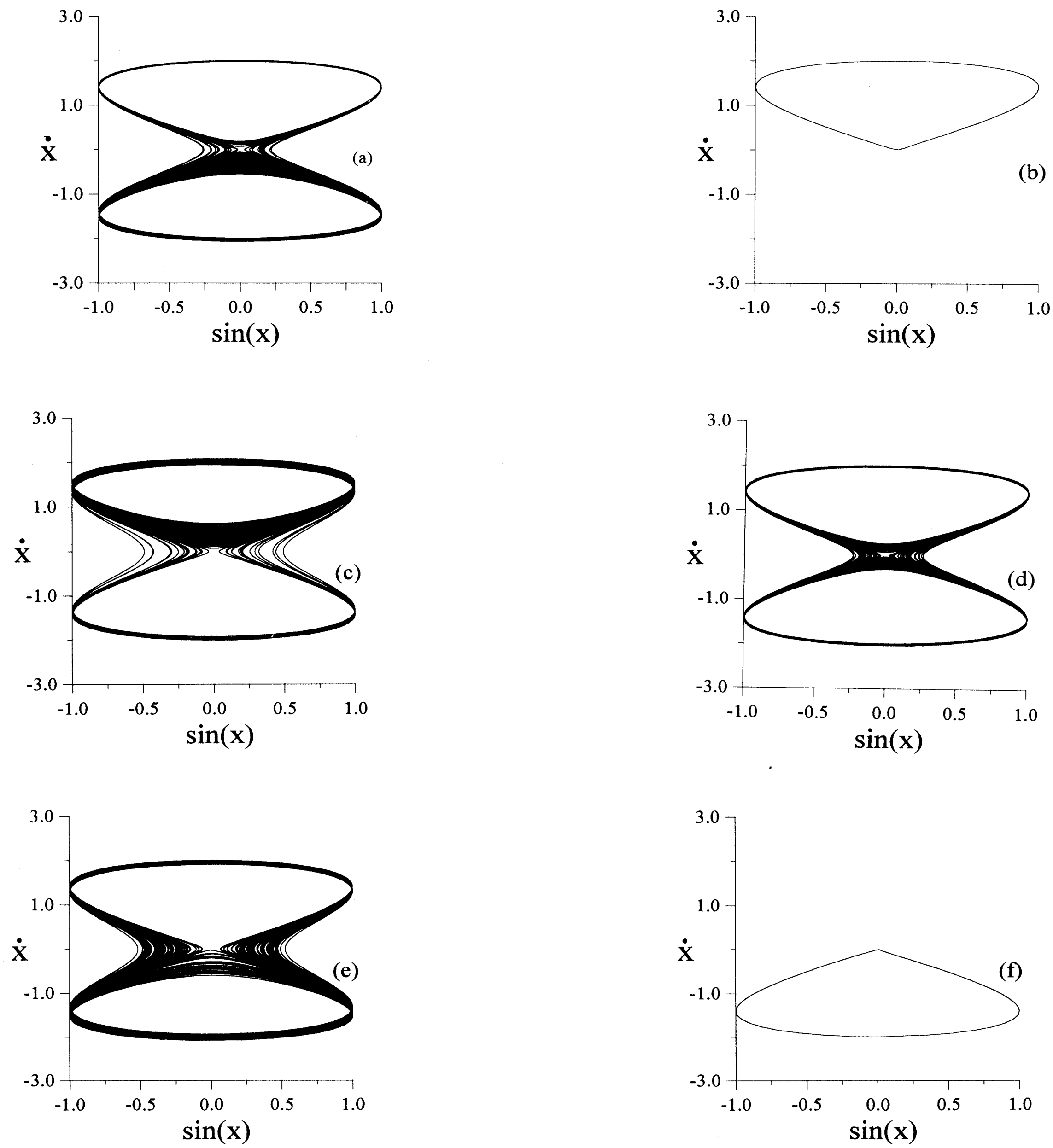

FIG. 2. Phase-space portraits for the parameters $\delta=0$, $\gamma=0.01$, $\omega=1.0$. Cases (a), (d) correspond to $\xi=0$. For the remaining cases $\xi=(\xi_{\min}+\xi_{\max})/2$ and $\theta=3\pi/2$ [$(b),(e)$], $\theta=\pi/2$ [$(c),(f)$]. x and $\dot{x}$ are dimensionless variables.

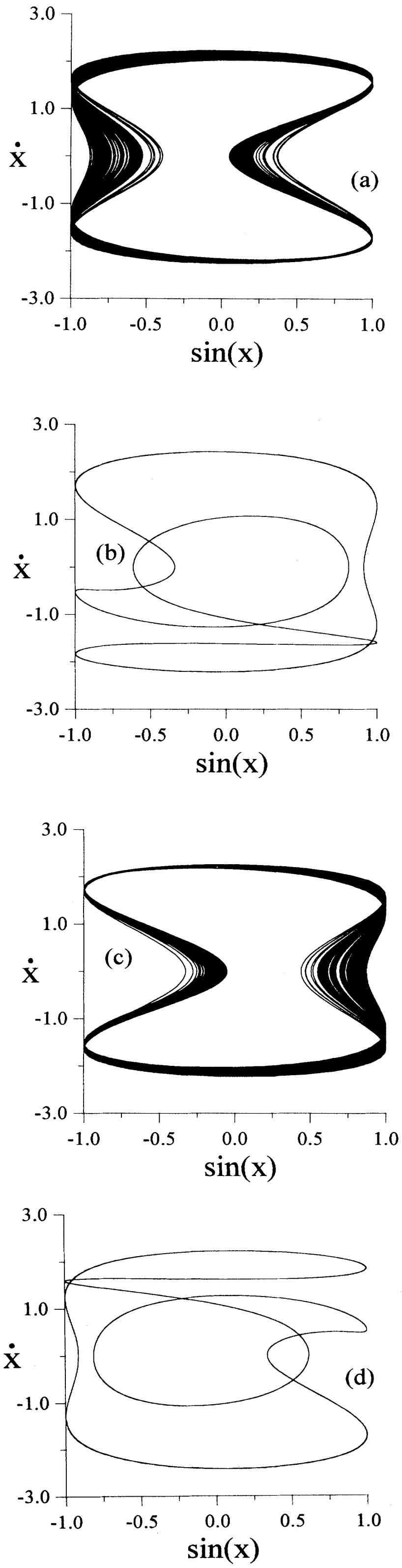

FIG. 3. Phase-space portraits for the parameters $\delta=0.25$, $\gamma=0.68$, $\Omega=\omega=0.67$ and initial conditions $(x_0=0, \dot{x}_0>0)$ $[(x_0=0, -\dot{x}_0)]$ for cases (a), (b) [(c), (d)]. Cases (a), (c), [(b), (d)] correspond to $\xi=0$ $[\xi=(\xi_{\min}+\xi_{\max})/2]$. (b) $\theta=3\pi/2$. (d) $\theta=\pi/2$. x and $\dot{x}$ are dimensionless variables.

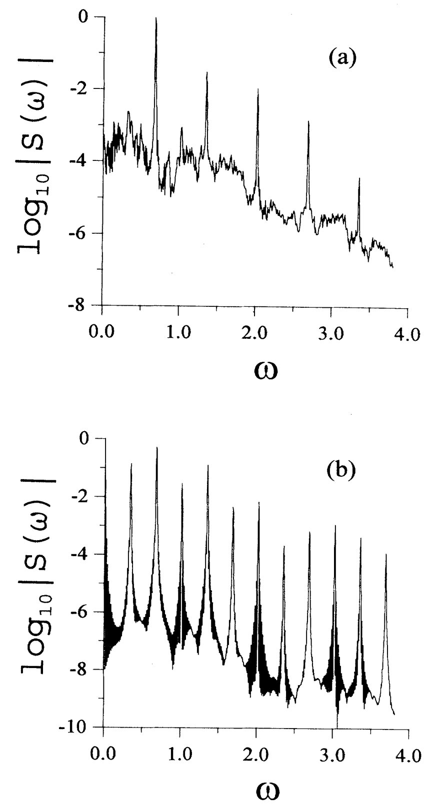

FIG. 4. Power spectra corresponding to the homonymous cases in Fig. 3.

mon period of the two modulations. Thus, as the whole system (1) is weakly damped, we might expect to find resonant motion for certain ranges of amplitudes of the forcing and the PM yielding trajectories with natural periods which *approximate* the common period $2\pi/\omega$. For this reason, we would only expect that the suitable θ values from perturbation theory coincide *exactly* with those from the ST in the Hamiltonian limit ($\delta\to 0$).

In order to test these ideas, I considered solutions to Eq. (1) of the type

$$x=\pm\sum_{n=0}^{\infty}\alpha_{2n+1}\sin[(2n+1)\omega t-\phi_n] \; . \tag{17}$$

The sign + (−) indicates that the orbit is based at a point with $\dot{x}(t=0)\geq 0$ $[\dot{x}(t=0)<0]$ since $0<\phi_n<\pi$, $\alpha_{2n+1}>0$. To first order, one obtains

$$\xi J_1(\alpha)\sin(\phi+\theta)=0 \; , \tag{18}$$

$$\pm\delta\alpha\omega=\gamma\cos\phi \; , \tag{19}$$

$$2J_1(\alpha)-\alpha\omega^2=\gamma\sin\phi \; , \tag{20}$$

where $J_n\equiv J_n(\alpha)$ is a Bessel function of the first kind. It is obvious that, in the limit $\xi\to 0$, we recover the usual nonlinear resonance shape for the damped, harmonically driven pendulum [17]. A first estimate of the error in the approximation (to first order) (17) may be obtained [21] by comparing the frequency of the free oscillations implied by (20) for $\gamma=\delta=\xi=0$,

$$\omega_{f,a}=\sqrt{2J_1(\alpha)/\alpha}\ , \tag{21}$$

with the exact result (in which α is the amplitude of the oscillation)

$$\omega_{f,e}=[(2/\pi)K(\sin\tfrac{1}{2}\alpha)]^{-1}\ , \tag{22}$$

where K is the complete elliptic integral of the first kind. In Ref. [21] Miles reported that the error in $\omega^2_{f,a}$ is less than 5% for $\alpha<2.40$ ($\omega^2_{f,a}>\omega^2_{f,e}$), to which range I shall limit most of the following. Now with $\xi\neq 0$, as $\alpha_0=3.85$ is the smallest zero of J_1, Eq. (18) implies

$$\sin(\phi+\theta)=0\ . \tag{23}$$

By substituting $\cos\phi$ and $\sin\phi$ from Eqs. (19) and (20), respectively, one obtains

$$tg\,\theta=\pm\frac{\alpha\omega^2-2J_1(\alpha)}{\delta\alpha\omega}\ . \tag{24}$$

From Eq. (20) we have $2J_1(\alpha)-\alpha\omega^2\sim\gamma$. Thus, if $\delta\alpha\omega\sim\gamma^2$, we find

$$\theta=\begin{cases}3\pi/2-\gamma, & \text{for } \dot{x}(t=0)\geq 0,\\ \pi/2-\gamma, & \text{for } \dot{x}(t=0)<0.\end{cases} \tag{25}$$

Observe that exact agreement with the requirements from the ST is achieved in the limiting case $\delta=0$, without additional restrictions on α and ω.

Going one step further, one can analyze the stability of the solutions $\pm\alpha\sin(\omega t-\phi)$. As is shown in the Appendix, such an analysis leads to a generalized Mathieu's equation. Then the application of Floquet theory gives us (see the Appendix for details)

$$J_0>0\ , \tag{26}$$

$$\delta>[J_2^2-(J_0^2-\omega^2)^2]^{1/2}/\omega\ , \tag{27}$$

$$\xi<(\alpha/J_1)^2[(J_0-\omega^2/4)^2+\omega^2\delta^2/4]^{1/2}\ . \tag{28}$$

The condition (26) implies $\alpha<2.40$ (the smallest zero of J_0), in concordance with the aforementioned assumption. Conditions (27) and (28) provide information about the stability domains in the parameter space. Here, the principal interest is centered on the comparison with the results arising from the ST, so I will mainly concentrate on Eq. (28). In Fig. 5 is plotted $\xi'_{max}\equiv(\alpha/J_1)^2[(J_0-\omega^2/4)^2+\omega^2\delta^2/4]^{1/2}$ versus α for $\delta=0.06$ and several ω values. One sees that the behavior is in agreement with $\Delta\xi(\omega)$ increasing as ω diminishes from ω_{min} (see the second Remark to ST and Fig. 1). Also, for a ξ value above the minimum of a given ω curve, there will be a forbidden interval of α values for stability, as is observed in numerical experiments. Figure 6 shows plots of ξ'_{max} versus ω for several α values. These α curves have a minimum such that $\omega_{min}(\alpha)$ decreases as α increases. Observe that the ξ range for stability increases wit α, i.e., while $\omega_{min}(\alpha)$ decreases. This is coherent with the decrease of the interval of ω values for which $\Delta\xi(\omega)$ is small (see Fig. 1 and the second Remark to ST) as ω_{min} diminishes.

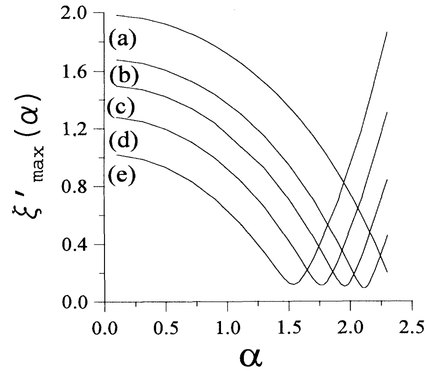

FIG. 5. Function ξ'_{max} (see text) vs α for $\delta=0.06$ and different ω values. (a) $\omega=0.6$. (b) $\omega=0.8$. (c) $\omega=1.0$. (d) $\omega=1.2$. (e) $\omega=1.4$. ξ' and α are dimensionless variables.

V. CONCLUSIONS

In this paper I have studied the suppression of chaos in a damped, harmonically forced pendulum by application of a PM of the sine term. Analytical estimates of the ranges of parameters for stabilization were found by means of the analysis of the results arising from MM. It was shown that some of those analytical results have their explanation in the natural symmetries of the whole system, and hence that the study of such symmetries (if any) could provide valuable information about the suppression of chaos in more complex systems. It was also demonstrated that the requirements for stability of symmetric oscillations are coherent with the predictions from MM. This means that MM is not only a powerful tool for predicting the suppression of chaos but also for obtaining key information about possible regularized motion, in order to discriminate among the possible ones based at a given initial condition.

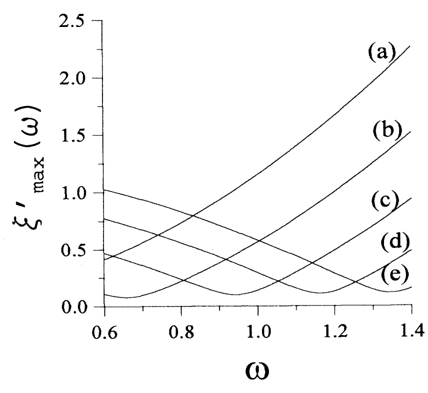

FIG. 6. Function ξ'_{max} (see text) vs ω for $\delta=0.06$ and some α values. (a) $\alpha=2.4$. (b) $\alpha=2.2$. (c) $\alpha=2.0$. (d) $\alpha=1.8$. (e) $\alpha=1.6$. ξ' and ω are dimensionless variables.

APPENDIX: STABILITY OF SYMMETRIC OSCILLATIONS

In Sec. IV solutions of Eq. (1) of the form

$$x = \pm\alpha \sin(\omega t - \phi) \tag{A1}$$

were investigated, with α, ω, and ϕ related to γ, δ, ξ, and θ by Eqs. (18)–(20), (24). Such solutions will be observed only if they are stable, i.e., only if any small perturbation of X is damped. To this end one can consider [22]

$$X = X_s + \mathrm{Re}[e^{r\tau} P(\tau)] , \tag{A2}$$

where X_s is the symmetric solution to be tested, $\tau \equiv \omega t - \phi$, Re means *the real part of*, r is a small parameter, the real part of which must be nonpositive for stability, and P ($|P| \ll 1$) is periodic in τ. Substituting (A2) into (1), linearizing in P, approximating X_s by (A1), neglecting the third and higher harmonics [as is consistent with (A1)] in the Fourier expansion of $\cos(\alpha \sin\tau)$, taking into account (21), and introducing

$$\eta \equiv (r + \delta/2)/\omega , \tag{A3}$$

$$\beta \equiv [r^2 + \delta r + J_0(\alpha)]/\omega^2 , \tag{A4}$$

$$q_1 \equiv \xi J_1(\alpha)/(\alpha\omega^2) , \tag{A5}$$

$$q_2 \equiv J_2(\alpha)/\omega^2 , \tag{A6}$$

one obtains

$$P'' + 2\eta P' + (\beta + 2q_1 \cos\tau + 2q_2 \cos 2\tau) P = 0 . \tag{A7}$$

Observe that (A7) reduces to Mathieu's equation if β and either q_1 or $q_2 = 0$. Now, taking into account the behavior of the perturbation solution of that equation near one of its stability boundaries in a β, q plane [23], one posits

$$P = P_0 + P_1 + P_2 + \cdots , \tag{A8}$$

$$\beta = \beta_0 + \beta_1 + \beta_2 + \cdots , \tag{A9}$$

where $P_n, \beta_n = O(q^n)$, $q_1, q_2, \eta = O(q)$, $q = \max(q_1, q_2)$.

Substituting (A8) and (A9) into (A7) and equating terms of like order in q, one finds

$$P_0'' + \beta_0 P_0 = 0 , \tag{A10}$$

$$P_1'' + \beta_0 P_1 = -2\eta P_0' - (\beta_1 + 2q_1 \cos\tau + 2q_2 \cos 2\tau) P_0, \ldots . \tag{A11}$$

From the Floquet theory, it is clear that P may be of period 2π or 4π (at the present order of perturbation) and therefore

$$\beta_0 = n^2 , \tag{A12}$$

$$P_0 = A_n \cos n\tau + B_n \sin n\tau , \tag{A13}$$

where n can take the values 0, 1, 1/2. Substituting (A12) and (A13) into (A11), one obtains

$$\begin{aligned} P_1'' + n^2 P_1 = & -2\eta n (B_n C_n - A_n S_n) - \beta_1 (A_n C_n + B_n S_n) \\ & - q_1 [A_n (C_{n-1} + C_{n+1}) + B_n (S_{n+1} + S_{n-1})] \\ & - q_2 [A_n (C_{n-2} + C_{n+2}) \\ & \quad + B_n (S_{n+2} + S_{n-2})] , \end{aligned} \tag{A14}$$

where $C_n \equiv \cos n\tau$ and $S_n \equiv \sin n\tau$. The requirement that P_1 be periodic implies that the coefficients of S_n and C_n in (A14) must vanish (to prevent the secular growth to P_1); requiring the determinant of the resulting homogeneous equations in A_n and B_n to vanish, one obtains

$$\beta_1^2 = \delta_{n1/2} q_1^2 + \delta_{n1} q_2^2 - (2\eta n)^2 , \tag{A15}$$

where δ_{mn} is the Kronecker delta. Combining (A4), (A12) and (A15) in (A9) and neglecting $O(q^2)$, one obtains

$$r^2 + \delta r + L_n = 0 , \tag{A16}$$

where

$$L_n = J_0(\alpha) - \omega^2 (n^2 + \beta_1) . \tag{A17}$$

The necessary and sufficient conditions for stability then are $L_n > 0$ for $n = 0, 1, 1/2$, and one obtains directly Eqs. (26)–(28).

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