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Thermodynamic approach to deterministic diffusion of mixed enhanced-dispersive type

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The thermodynamic approach is applied to diffusion generated by an intermittent map on a grid of unit cells. The associated reduced map has two types of intermittent fixed points, one responsible for enhancing the diffusion, the other responsible for dispersing the diffusion. The dependence of the global characterization of the diffusion is investigated as a function of a pair of intermittency exponents that describe the strength of the intermittencies.

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In order to arrive at quantitative overall statements on the nature of the transport [1–4], traditionally the approach of the random walk is used [1,3,4]. For the problem of mixed enhanced-dispersive diffusion, however, only few results have been obtained so far. Apart from a random walk approach [5], we are only aware of results obtained by an application of fractional Fokker-Planck equations [6]. We show that the thermodynamic approach [7–10] in its bivariate variant [11–15] yields a straightforward solution of the problem [16–19].

While the mathematical properties of the thermodynamic approach of hyperbolic systems are well known [7], the extension of this approach to nonhyperbolic systems poses some problems of a mathematical nature that are connected with the phenomenon of phase transitions [12–15,20–22]. Recently, it has been shown that the diffusional behavior of one-dimensional maps can be cast in all essential aspects by means of a diffusion-related free energy [16–19] or by the associated entropy function [23–25]. While this description furnishes in principle the transport coefficients of all orders [17], often one is only interested in an overall characterization of the transport. Under the absence of a possible drift (we will consider only this case), this characterization is furnished by the diffusion coefficient D. In an expansion of the diffusion $t \rightarrow x(t)$ into moments, D appears as the second-order coefficient and we have

$$\langle x^2(t) \rangle = D t^{\alpha=1} \tag{1}$$

(in the case of a discrete iterative map we may replace t by k). As has been shown earlier [16,26], this rather rough characterization of diffusion can be obtained from the more refined description by means of a diffusion-related free energy F_d by taking derivatives as follows [16–19,26]:

$$D = \frac{1}{2} \frac{\partial^2}{\partial q^2} F_d(q,\beta)|_{q=0,\beta=1} \,. \tag{2}$$

Alternatively, the diffusion-related entropy function can be used [24]. Equation (2), of course, applies only if the derivatives yield finite values. This is always true if the associated free energy is analytic (i.e., if we have "normal" diffusion) or if the free energy is not analytic, but D is nonzero and finite ("regular" diffusive behavior). For the so-called enhanced diffusional behavior, due to a nonanalyticity of the free energy, D diverges if evaluated according to Eq. (2) [17,19]. Therefore, the mean-squared average is then described by higher powers of t and we have $\langle r^2(t) \rangle \sim t^{\alpha}$, where now $\alpha > 1$. For the dispersive diffusional behavior, on the contrary, the diffusion coefficient vanishes (in this case, a higher-order transport coefficient diverges).

From first sight, as a topic of interest the combined enhanced-dispersive diffusional behavior may appear as a little bit exotic. However, already the standard example for anomalous diffusion, the two-dimensional Chirikov-Taylor standard map [27–30], shows this feature. Here, the "stickiness" of Kol'mogorov-Arnol'd-Moser tori associated with transport or absence of transport may lead to enhanced or dispersive diffusion [31,32]. Furthermore, recent experiments on fluid dynamics [33] indicate that such a behavior is indeed relevant for experimental situations. A one-dimensional example of such a system that is able to generate diffusion of mixed enhanced-dispersive type is shown in Fig. 1(a). From a neighborhood of the fixed points of the associated reduced map [see Fig. 1(b)], the trajectories are driven away according to different intermittency exponents u,v, respectively. The two exponents give rise to a two-parameter family of diffusive maps. With the help of this family one may explore the whole range of possible diffusional behavior that can be induced via classical intermittency (for a more general definition of intermittency compare [9]).

As the starting point for the thermodynamic investigation of such systems, the bivariate diffusion-related partition function of the system is needed (see, e.g., [14,18]). In this partition function, in addition to the length scales of the usual thermodynamic formalism, which measure the dynamical instability of pieces of orbits, a second measure is used to count the number of identical unit cells the orbit has passed (counting jumps to the right as positive and jumps to the left as negative). While usually in the partition function all possible pieces of orbits of a fixed length appear, a glance at Fig. 1 shows that in the examples we are interested in, typically pieces of orbits labeled by the same symbols E, e.g., with decreasing indices, appear. This suggests the use of a grand-

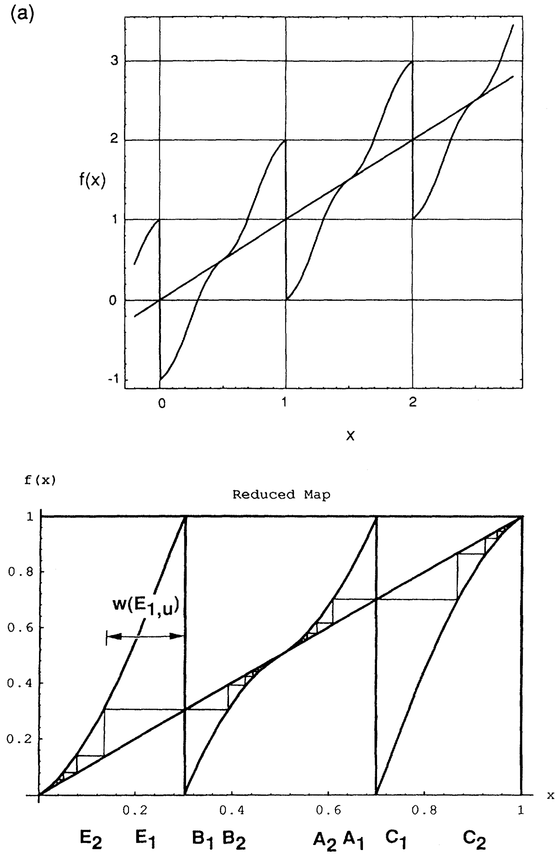

FIG. 1. (a) One-dimensional map f on a grid of unit cells. Note that there are two points per unit cell at which the map has a marginal slope. These points account for the intermittent dynamical behavior of the map. The exponents u,v, which describe the departure of the branches from the marginal stabilities [here $f_1(u,x)=(1+\epsilon)x+a_u x^u$ for $0\le x\le \frac{1}{4}$ and $f_2(v,x)=(1+\epsilon)x-a_v(\frac{1}{2}-x)^v$ for $\frac{1}{4}\le x\le \frac{1}{2}$, where a_u,a_v are constants and ϵ is small], determine the diffusive behavior of the map. Depending on the value of the "intermittency exponents" u and v, respectively, the map is able to generate regular, enhanced, and dispersive diffusion, respectively (see the text). (b) Reduced map associated with (a). In the reduced map, the partition of the interval generated by the map is indicated by horizontal lines. Each partition element is labeled by the indexed symbols A, B, C, and E, respectively. Whereas the capital letter describes on which of the four "channels" the partition element is situated, the index indicates how many iterations are needed to leave the channel. A piece of orbit can then be described in a unique way by the associated sequence of symbols $E_3,E_2,E_1,A_5,A_4,\ldots,A_1$, for example ("symbolic encoding of the orbit"). Through Eq. (3) a simplified model equipped with a complete grammar is described (see the text).

canonical-like partition function [18,19], where the strategy is to piece together orbits from so-called fundamental orbits of smaller lengths [34]. In this way, a much higher level of the partition function can be investigated. As a consequence, let us therefore start from the fundamental orbits of a typical system. In this case, the partition sum is composed from all fundamental orbits of lengths k in the following way (compare, e.g., Refs. [18,19,24,25]):

$$1=\sum_{k=1}^{\infty} z^k[w(A_{k,v})^{\beta}e^{+q}+w(B_{k,v})^{\beta}e^{-q}] + \sum_{k=1}^{\infty} z^k[w(C_{k,u})^{\beta}e^{+q(k-1)}+w(E_{k,u})^{\beta}e^{-q(k-1)}] . \tag{3}$$

In the formula, u,v are the exponents of the branches of the map that lead to the enhancement and to the dispersion of the diffusive motion, respectively (cf. Fig. 1). A symmetric map that excludes a possible drift is obtained if we require $w(A_{k,v})=w(B_{k,v})$ and $w(C_{k,u})=w(E_{k,u})$. In accordance with the example presented in Fig. 1, the elements $w(E_{k,u})$ are chosen to scale as $w(E_{k,u})\sim[k^{1/(u-1)}-(k+1)^{1/(u-1)}]$ and analogously for the other symbols. Note that this power-law dependence of the partition elements on index k is used to generate the intermittent behavior. Furthermore, as can be derived from Fig. 1, for such a system the length scale (or "the dynamical weight" [34] or the k-step Lyapunov exponent) associated with a periodic orbit of symbolic encoding $E\ldots E$ (k times, starting at E_k, ending at E_1) is determined directly by the width $w(E_{k,u})$. We use this property to arrive at the particularly simple structure of Eq. (3). Another property that we assume in order to keep our model as simple as possible concerns the grammar [34] of the system: Eq. (3) assumes a trivial grammar, i.e., chains of symbols $S_k\ldots S_1$, where $S\in\{A,B,C,E\}$, are allowed to combine freely, without restrictions. Note that our prototype shown in Fig. 1 does not satisfy this property (the transition from A to B is not allowed). While such a modification of the grammar does indeed change measures such as the value of the diffusion coefficient (where it exists) and the value of the topological entropy, for the more robust diffusion exponent this fact is without influence [26]. However, our focus of interest is the analysis of the generic situation, which is not restricted to one-dimensional phenomena, and not the analysis of a specific one-dimensional map. Equation (3) generally admits different solutions. From the largest solution $z(q,\beta)$ follows the diffusion-related free energy F_d as $F_d(q,\beta)=\ln[z(q,\beta)]$ [14,15]. Using Eq. (2), our result is

$$D\sim\frac{\sum_{k=1}^{\bar{k}} w_{k,v}+\sum_{k=1}^{\bar{k}} k^2 w_{k,u}}{\sum_{k=1}^{\bar{k}} k w_{k,u}+\sum_{k=1}^{\bar{k}} k w_{k,v}} \quad \text{for } \bar{k}\to\infty . \tag{4}$$

From this equation, we may derive the diffusional behavior of a system, depending on the values of the intermittency exponents u,v. Let us first observe that for each summand in the denominator, the interpretation of an average staying time can be given. As long as this time is finite for v, the enhancing aspect alone determines whether the diffusion is regular or anomalous. As it is easy to see from the corresponding expressions of the un-

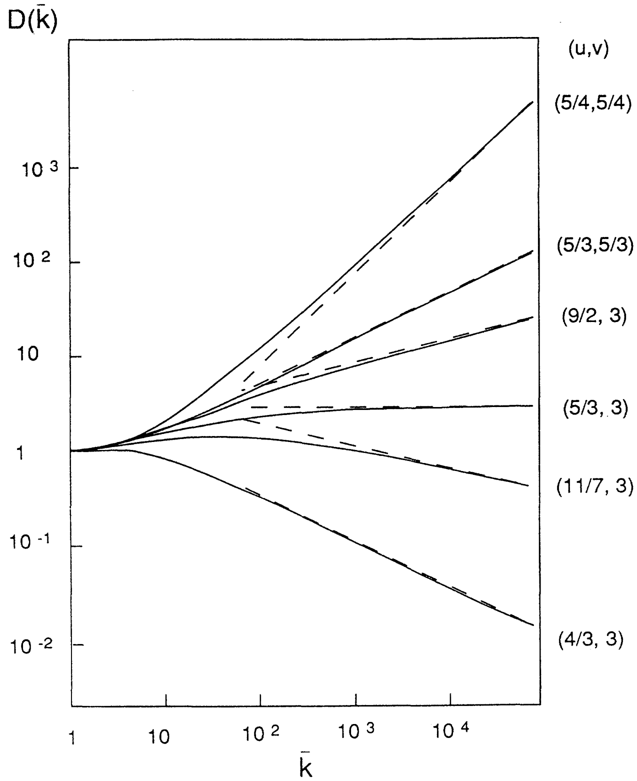

FIG. 2. Behavior of $D(\bar{k})$ for six pairs of exponents, according to Eq. (4) (solid lines) and asymptotically (broken lines).

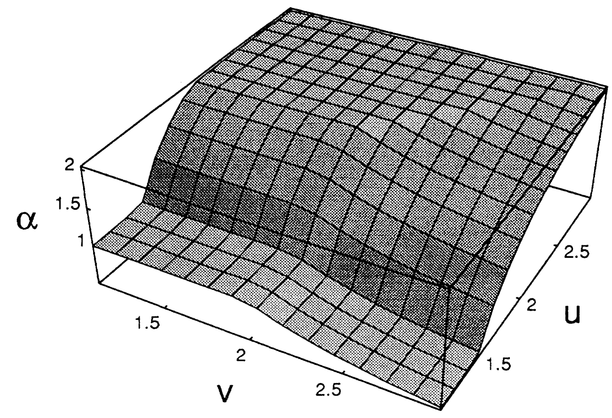

FIG. 3. Dependence of the diffusion exponent α on the intermittency exponents v and u. The plateau at the level $\alpha=1$ corresponds to regular diffusive behavior, $\alpha<1$ to dispersive behavior and $\alpha>1$ to enhanced diffusive behavior.

mixed cases, the mean staying time in the dispersive case diverges if $v \geq 2$ (corresponding to an unnormalizable measure). In the plots of Fig. 2, the behavior of Eq. (4) for increasing times $\bar{k}$ is shown as solid lines. These results have been calculated numerically for different pairs of intermittency exponents in the different characteristic regions of the (u,v) quadrant. Asymptotic results can be obtained analytically by observing that for large $\bar{k}$

$$\sum_{k=1}^{\bar{k}} k^2 w_{k,u} \sim (1+\bar{k})^{-1/(u-1)+2} . \tag{5}$$

From this observation the following overall behavior is obtained: for $v<2$,

$$\langle (x(t))^2 \rangle \sim \begin{cases} t & \text{for } u<\frac{3}{2} \\ t^{3-1/(u-1)} & \text{for } \frac{3}{2}<u<2 \\ t^2 & \text{for } u>2 \,, \end{cases} \tag{6}$$

and for $v>2$ the behavior

$$\langle (x(t))^2 \rangle \sim \begin{cases} t^{1/(v-1)} & \text{for } u<\frac{3}{2} \\ t^{2+1/(v-1)-1/(u-1)} & \text{for } \frac{3}{2}<u<2 \\ t^{2+\min\{1/(u-1),1/(v-1)\}-1/(u-1)} & \text{for } u>2 \end{cases} \tag{7}$$

arises. In Fig. 2, the convergence of Eq. (4) towards the asymptotic behavior can be estimated from a comparison of the solid lines [Eq. (4)] with the broken lines indicating the asymptotic prediction. Furthermore, a direct simulation of the map agrees very well with the predicted behavior. In view of the high partition levels $\bar{k}$ attainable for our model, from a thermodynamic point of view a good degree of agreement can be expected. In Fig. 3, the results of Eqs. (6) and (7) are summarized, where the plot shows the value of the diffusion exponent α as a function of the intermittency exponents v and u. Results equivalent to Eqs. (6) and (7) have recently been derived from a random walk approach [5]. In the latter approach, however, the information about the relationship between the dynamical and the diffusive aspect of the system is lost. In addition to a correct prediction of the behavior of the second moment, the thermodynamic approach offers therefore a more specific characterization of the generating process. This information is contained in the diffusion-related free energy or the associated entropy function, which allows a distinction between individual systems, even if they fall into the same class if seen from Eqs. (6) and (7) [35].

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[35] Note that for the map of Fig. 1 and for the system of Eq. (3) a full thermodynamic analysis yields different topological entropies, while identical diffusional behavior dependent on the parameters u,v is obtained.