Traffic jams, granular flow, and soliton selection

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The flow of traffic on a long section of road without entrances or exits can be modeled by continuum equations similar to those describing fluid flow. In a certain range of traffic density, steady flow becomes unstable against the growth of a cluster, or "phantom" traffic jam, which moves at a slower speed than the otherwise homogeneous flow. We show that near the onset of this instability, traffic flow is described by a perturbed Korteweg–de Vries (KdV) equation. The traffic jam can be identified with a soliton solution of the KdV equation. The perturbation terms select a unique member of the continuous family of KdV solitons. These results may also apply to the dynamics of granular relaxation.

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I. INTRODUCTION

The flow of traffic along a limited-access highway is similar in many respects to the flow of a classical fluid [1-3]. For instance, in the absence of entrances and exits, the total number of vehicles on the road is conserved. This leads to a continuity equation which relates the local density $\rho(x,t)$ of traffic to the local average speed $v(x,t)$,

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho v)}{\partial x}=0 . \tag{1}$$

The average traffic speed v obeys an evolution equation similar to the Navier-Stokes equation, but with some distinctive characteristics. The standard model [1–3] assumes an equation of the form

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{1}{\tau}[V(\rho)-v]-c_0^2\frac{\partial L(\rho)}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 v}{\partial x^2} . \tag{2}$$

The right side of this equation embraces three factors affecting traffic speed. The final term models "viscosity" or "diffusion," a presumed tendency to adjust one's speed to that of the surrounding traffic. The second term is an anticipation factor: drivers slow down at the sight of an increase in traffic density ahead. The (dimensionless) function $L(\rho)$ should then be monotonically increasing. It is usually [1–3] taken to be $\ln\rho$, in which case $c_0^2\rho$ plays the role of a pressure; however, we will not commit ourselves to this choice. The first term expresses the tendency of traffic at a given density ρ to relax to some natural average speed $V(\rho)$. At low densities this speed is determined by such things as road conditions and speed limits, and is only weakly dependent on ρ. At high densities, $V(\rho)$ approaches zero, and so is again weakly dependent on ρ. At intermediate densities it drops off rapidly, largely due to the fact that higher traffic density makes it more difficult for faster drivers to overtake slower drivers. Thus we expect $V(\rho)$ to be a decreasing function, with a small derivative at low and high values of ρ.

Two recent observations [3,4] have brought to our attention the need for a rigorous nonlinear analysis of the traffic equations. The first is that of a clustering of cars, seen in numerical simulations [3]. Homogeneous traffic flow can be unstable, with localized regions of high density and low average velocity spontaneously appearing. These pulses preserve their shape and move with constant speed; they correspond to the "phantom" traffic jams which appear on highways for no apparent reason. The other observation has to do with the flow of granular media. It has recently been recognized [4] that traffic equations might be utilized to study the dynamic relaxation of granular particles in a one-dimensional tube under repeated tapping. Underlying this approach is the assumption that voids in the tube migrate upon tapping toward the top of the tube and accumulate there, resulting in a reduction of the height. In this picture, voids behave like cars on a highway, so the governing equations for voids are precisely the traffic equations. The rate of height reduction is then proportional to the flux of voids past a fixed point in space. Numerically, it was found that beyond the onset of instability the arrival of voids at the top is not continuous. Rather, localized clusters of voids arrive periodically, causing the height to decrease discontinuously. Such discrete reduction has been termed stick-slip relaxation [4].

We note that traffic equations essentially describe the flow of conserved but fluctuating physical quantities, such as cars on a highway or voids in a granular medium. They might equally well describe the migration of people, animals, or electronic messages. Hence the appearance of localized pulses in an initially homogeneous flow is an interesting discovery which requires further investigation.

The model of traffic flow embodied in Eqs. (1) and (2) admits a simple steady-state solution representing uniform, homogeneous flow, namely

$$\rho(x,t)=\rho_h, \quad v(x,t)=V(\rho_h)\equiv v_h . \tag{3}$$

If, as we suppose, $V(\rho)$ is monotonically decreasing, then there is a unique homogeneous steady state for each possible value of ρ_h. At intermediate densities, however, this state is unstable [2,3]. A fluctuation in the local density of traffic produces a local concentration of vehicles which are moving more slowly than v_h because of the higher local density. Their presence slows the faster-moving vehicles behind which catch up to them, further increasing the density of the slow-moving cluster. But the local "pressure," $c_0^2\rho$, also increases, tending to push vehicles away from one another. (This is reminiscent of the clustering instability in granular systems presented in [5].) Eventually the competition between these two effects may create a self-sustaining cluster which moves backwards relative to the overall traffic flow as vehicles behind it must slow down to avoid a collision, while vehicles on its leading edge are able to speed up due to the lower traffic density ahead of them.

To quantify this picture, we assume that traffic is initially in a state which differs infinitesimally from the homogeneous flow of Eq. (3). We decompose this flow into a linear combination of Fourier modes, each of which grows or decays with its own growth rate. Thus we write

$$\begin{aligned}\rho(x,t)&=\rho_h+\sum_k \hat{\rho}_k \exp(ikx+\sigma_k t),\\ v(x,t)&=v_h+\sum_k \hat{v}_k \exp(ikx+\sigma_k t)\ ,\end{aligned} \tag{4}$$

substitute these expressions into (1) and (2), and linearize in $\hat{\rho}_k$ and $\hat{v}_k$. We find that each linear growth rate σ_k must satisfy the quadratic equation

$$\begin{aligned}0=(\sigma_k+iv_h k)^2+&\left[\frac{1}{\tau}+\mu k^2\right](\sigma_k+iv_h k)\\ &+c_0^2\rho_h L'(\rho_h)k^2+i\frac{\rho_h V'(\rho_h)}{\tau}k\ .\end{aligned} \tag{5}$$

Here primes represent derivatives with respect to ρ. It is not difficult to show (using, e.g., the Nyquist criterion) that both roots of this quadratic have negative real parts provided

$$\rho_h|V'(\rho_h)|^2<c_0^2L'(\rho_h)(1+\mu\tau k^2)^2\ , \tag{6}$$

while otherwise one root has a positive real part. The flow is stable against *all* infinitesimal perturbations for

$$\rho_h|V'(\rho_h)|^2<c_0^2L'(\rho_h)\ . \tag{7}$$

Figure 1 shows the two sides of this inequality for the parameter values used by Kerner and Konhäuser [3]. There is an intermediate range of density, $\rho_{c1}<\rho<\rho_{c2}$, in which $V(\rho)$ is so sensitive to changes in ρ that homogeneous flow is unstable. From (6) we see that the instability first appears at small wave numbers k near either ρ_c.

Writing the solution of (5) as a power series in k, we find

$$\begin{aligned}\mathrm{Re}(\sigma_k)&\approx\rho_h\tau[\rho_h|V'(\rho_h)|^2-c_0^2L'(\rho_h)]k^2-O(k^4)\ ,\\ \mathrm{Im}(\sigma_k)&\approx-[v_h+\rho_h V'(\rho_h)]k+O(k^3)\end{aligned} \tag{8}$$

for small k. From this we see that the real part of the growth rate is negative for small k provided (7) is satisfied, while it turns positive when (7) is violated. From the imaginary part of σ_k, we see that the critical disturbance travels with a speed

$$c(\rho_h)=V(\rho_h)+\rho_h V'(\rho_h)\ , \tag{9}$$

which is slower than the steady-state traffic speed $v_h=V(\rho_h)$, since V' is negative.

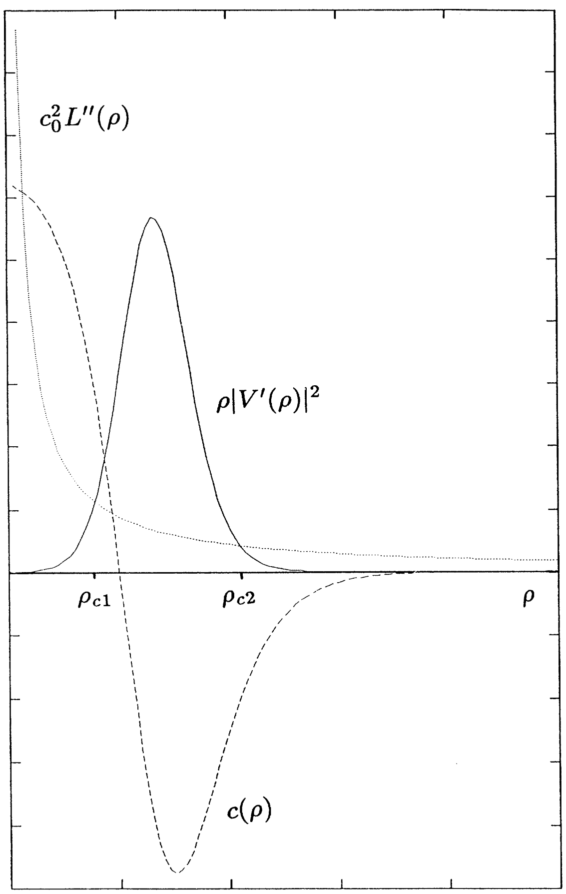

FIG. 1. The two terms $\rho|V'(\rho)|^2$ (solid curve) and $c_0^2L''(\rho)$ (dotted curve) in the stability criterion (7). The intersections of these two curves locate the critical densities ρ_{c1} and ρ_{c2}. The dashed curve is the velocity $c(\rho)$ from Eq. (9). Note that c' is negative at ρ_{c1} and positive at ρ_{c2}. All curves are calculated for the functions $V'(\rho)=-21.025\,\mathrm{sech}^2[(\rho-0.25)/0.12]$ and $L(\rho)=\ln\rho$ and the value $c_0=2.48445$ used in Ref. [3].

II. LONG-WAVELENGTH ANALYSIS

Suppose the density of traffic is in a range for which the stability criterion (7) is violated, but only slightly. We quantify this by writing

$$\begin{aligned}(\rho_c+\delta\rho)&|V'(\rho_c+\delta\rho)|^2-c_0^2L'(\rho_c+\delta\rho)\\ &\approx[|V'(\rho_c)|^2+2\rho_c V'(\rho_c)V''(\rho_c)-c_0^2L''(\rho_c)]\delta\rho\\ &\equiv\alpha\epsilon^2\ ,\end{aligned} \tag{10}$$

where ϵ is an arbitrary small parameter. A positive value of α means that steady flow is unstable; from Fig. 1 we see that this occurs for positive $\delta\rho$ near ρ_{c1} and for negative $\delta\rho$ near ρ_{c2}. From the linear stability result (6) we see that perturbations with wave numbers of order ϵ will grow. The leading order time dependence of these perturbations then comes from the term in σ_k which is linear in k, which implies that the growing perturbations are traveling waves with speed c. The next leading time dependence comes from the k^3 term, so it occurs on time scales of order ϵ^{-3}. Finally, we expect that an amplitude equation would balance the linear growth term of order $\epsilon^4 A$ with a stabilizing nonlinear term of order A^3; thus we expect the disturbance to saturate at a size of order ϵ^2.

We implement these scalings by writing

$$\begin{aligned} \rho(x,t) &= \rho_h + \epsilon^2 \hat{\rho}(\epsilon x, \epsilon^3 t) , \\ v(x,t) &= v_h + \epsilon^2 \hat{v}(\epsilon x, \epsilon^3 t) . \end{aligned} \tag{11}$$

Inserting this ansatz into the governing equations (1) and (2), transforming into a frame of reference moving with the speed $c(\rho_h)$, and expanding systematically in powers of ϵ, we find that the velocity and density perturbations are related by

$$\hat{v} = -|V'|\hat{\rho} + O(\epsilon^2) , \tag{12}$$

and that the density perturbation satisfies the dynamical equation

$$\frac{\partial \hat{\rho}}{\partial T} - \mu\tau\rho_h |V'| \frac{\partial^3 \hat{\rho}}{\partial X^3} + [2V' + \rho_h V'']\hat{\rho}\frac{\partial \hat{\rho}}{\partial X} = -\epsilon\tau\rho_h \frac{\partial^2}{\partial X^2}\left[2\mu\tau|V'|^2 \frac{\partial^2 \hat{\rho}}{\partial X^2} + \alpha\hat{\rho} + \tfrac{1}{2}(|V'|^2 - 2\rho_h|V'|V'' - c_0^2 L'')\hat{\rho}^2\right] + \cdots , \tag{13}$$

where $X = \epsilon x$, $T = \epsilon^3 t$, and all derivatives are evaluated at ρ_h. The leading order in this equation is the Korteweg–de Vries equation, which is a classic integrable nonlinear evolution equation which is known to lead to solitons [6].

The initial value problem for the Korteweg–de Vries (KdV) equation can be solved by the inverse scattering transform [6], with the result that localized initial conditions tend to resolve into a train of solitons. Moreover, perturbation theories based on inverse scattering ideas are also available [7]. For our purpose, however, a less sophisticated method suffices. We concentrate on the simple one-soliton solution, which corresponds to a single, isolated traffic jam. The one-soliton solution of the leading order of (13) is

$$\hat{\rho} = S(X,T;k) \equiv A\,\mathrm{sech}^2 k(X+uT) , \tag{14}$$

where k is a free parameter and

$$u = 4\mu\tau|V'|k^2, \quad A = -\frac{12\mu\tau|V'|}{2V' + \rho_h V''}k^2 . \tag{15}$$

Note that the denominator in the expression for A is the ρ derivative of the velocity $c(\rho)$ of the perturbation given by (9). If this is negative, then the soliton represents a local increase in traffic density, with a concomitant decrease in speed—a traffic jam. On the other hand, if it is positive then it represents a local rarefaction of traffic density and an increase of speed. The function $c(\rho)$ is plotted in Fig. 1; for the parameter values used there, traffic jams appear near ρ_{c1} and rarefactions near ρ_{c2}.

To account for the effect of the order-ϵ correction on the right hand side of (13), we must allow for the possibility that it could cause a slow change in the parameter k of the unperturbed soliton. Thus we write

$$\hat{\rho} = S(X,T;k(T_1)) + \epsilon\hat{\rho}_3 , \tag{16}$$

where $T_1 \equiv \epsilon T$ is a slow time variable. Substituting this into (13) then leads to

$$\frac{\partial S}{\partial k}\frac{dk}{dT_1} + \left[\frac{\partial}{\partial T} - \mu\tau|V'|\frac{\partial^3}{\partial X^3} + (2V' + \rho_h V'')\frac{\partial}{\partial X}S\right]\hat{\rho}_3 = \mathcal{R} , \tag{17}$$

where $\mathcal{R}$ stands for the order-ϵ term on the right hand side of (13) with $\hat{\rho}$ replaced by S. Multiplying (17) by S, integrating over all X, and averaging over T annihilates the term involving $\hat{\rho}_3$ [8], and so leaves us with an evolution equation for k:

$$\frac{dk}{dT_1} = \frac{8\tau\rho_h}{15}\alpha k^3 - \frac{64}{105}\mu\tau^2\rho_h|V'|\frac{4|V'|^2 + 7\rho_h|V'|V'' + 6c_0^2 L''}{2|V'| - \rho_h V''}k^5 . \tag{18}$$

From this we see that when the coefficient of k^5 is negative and α is negative, k decreases to zero (so that the soliton amplitude vanishes), while for positive α, k tends to the nontrivial zero of the right side of (18). We then find that the velocity of the selected soliton is

$$\frac{7}{2}\frac{2|V'| - \rho_h V''}{4|V'|^2 + 7\rho_h|V'|V'' + 6c_0^2 L''}\alpha , \tag{19}$$

and the amplitude of the density perturbation is

$$\frac{21}{2}\frac{\alpha}{4|V'|^2 + 7\rho_h|V'|V'' + 6c_0^2 L''} . \tag{20}$$

The amplitude of the perturbation of the vehicle velocity, according to (12), is this multiplied by $-|V'|$. It is noteworthy that these results are independent of the diffusion coefficient μ, although the width k of the selected soliton does depend on it (being proportional to $\mu^{-1/2}$).

It is also possible for the coefficient of k^5 in (18) to be positive. In this case there is a subcritical instability. The zero of (18) is then a threshold: k decreases to zero if it initially is less than this (for $\alpha<0$), otherwise it grows. The final form of the traffic jam would be something larger than assumed in the scalings (11) used to derive (15). This is the case for the parameters used by Kerner and Konhäuser; in their work, subcritical instabilities were observed at both ρ_{c1} and ρ_{c2}, leading to traffic jams in the former case and rarefactions in the latter.

We emphasize the role played by the correction term on the right hand side of (13). This destroys the exact integrability of the evolution equation for the traffic density. Without this correction, a continuous range of soliton solutions would exist, and solitons which collided would pass through each other and maintain their separate identities. With it, a unique soliton solution is selected, and multiple solitons could merge. This last point is a subject of ongoing research. We have also found that applying a similar perturbation analysis to the periodic, cnoidal solutions of the KdV equation leads to selection of the amplitude of the solution, but not of the wavelength; thus periodic arrays of traffic jams could arise from (13) with any period.

We now briefly discuss the physical relevance of the existence of this soliton mode in the context of granular dynamics. In an effectively one-dimensional granular assembly, the propagation of disturbance in the form of solitons might be associated with the sudden appearance of local ordering, perhaps hexagonal, which generates a massive void. Such massive voids then move collectively until they disappear at the top layer to the air. The appearance of density waves [9,10] and clogging [11] in a pipe flow of granular materials might be a strong indication of such soliton modes in granular dynamics. Such soliton modes perhaps might be of relevance in understanding unusual sound propagation through granular media [12].

Note added in proof. After submitting this paper we were informed that Hayakawa *et al.* have obtained a similar result [13] in the context of a two-fluid model of vibrating beds.

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