Transition to nonchaotic behavior in a Brownian-type motion

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A theoretical and numerical analysis of the transition from chaotic to nonchaotic behavior in an ensemble of particles with different initial conditions which move according to Newton's equations in a bounding potential and are driven by an identical sequence of random forces [see S. Fahy and D. R. Hamann, Phys. Rev. Lett. **69**, 761 (1992)] is presented. The threshold values of the parameters for transition from chaotic to nonchaotic behavior are defined on the basis of the map for distances between the particles and differences of velocity. Numerical analysis is fulfilled for one-dimensional Duffing $V(x) = x^4 - x^2$ and $V(x) = x^4$ potentials.

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Recently an interesting transition from chaotic to nonchaotic behavior in randomly driven systems has been discovered [1]. When an ensemble of particles with different initial conditions which are bounded in a fixed external potential are driven by an identical sequence of random forces, the ensemble of trajectories may become identical at long times. Fahy and Hamann [1] considered a particle of mass m moving according to Newton's equations in a potential $V(x)$, except that at regular time intervals τ the particle is stopped and its velocity is reset to a random value chosen from a Maxwell distribution with temperature T. It should be stressed that for every particle of the ensemble it was given an identical, randomly chosen velocity at the start of each step of time length τ. This motion is in many respects similar to Brownian motion of the particles at a temperature T and is frequently used in Monte Carlo simulations [2]. However, if the time interval τ between stops is lower than a threshold value τ_c, the final trajectories of the particles are independent of the initial conditions; all trajectories become point by point identical in time. Although the trajectory is highly erratic and random, the system is not chaotic.

A similar effect may also be observed in a more general and realistic (from the physical point of view) case, i.e., when mixing at time intervals τ some part α of the old velocity $\mathbf{v}^{old}$ with random velocity $\mathbf{v}^{ran}$ to get a new starting velocity $\mathbf{v}^{new} = \alpha \mathbf{v}^{old} + \mathbf{v}^{ran}$ [3]. Here a threshold value τ_c depends on α.

Note that the observed effect resembles a phase transition but does not depend crucially on the dimension of the space in which the particles move. Besides its intrinsic interest, this phenomenon has some importance for Monte Carlo simulations [1] and, probably, can influence some processes, e.g., clustering of particles.

In Ref. [1], except for the discovery in the computer simulations of the transition from chaotic to nonchaotic behavior, the analytical estimation of the average rate of contraction of the distance between initially close together identically driven particles (for a one-dimensional

potential, very short time intervals τ between stops and $\alpha = 0$) is given. See Ref. [4], in which a similar problem was analyzed and it was found that trajectories of a noisy logistic map for various initial conditions may become point by point identical in time if the noise is sufficiently strong and independent of the initial condition. Moreover, using the Langevin equation, it was shown quite generally that nonlinear systems coupled through identical strong noises may have the same trajectory in the long time limit. However, the derivation of dependence of the threshold noise on the parameters of the system [1] is absent.

It is the purpose of this paper to present a theoretical and numerical analysis of the transition from chaotic to nonchaotic behavior in randomly driven one-dimensional systems, i.e., the dependencies of the threshold value τ_c on the parameters of the problem, including α, and convergence rates of the trajectories. Generalization of the method for two- and three-dimensional systems is rather straightforward (see below). It should be noted that in the numerical calculations of the mean-square distance between particles we observe the intermittency route to chaos when $\tau \to \tau_c$: mean-square distance as a function of time alternates randomly between almost clustered phases and irregular bursts of the dispersed particles. Moreover, we find a transition from chaotic to nonchaotic behavior not only for the regular time intervals τ but for the randomly changing intervals between stops, i.e., for random τ_i (where i is the number of the step) as well. Our theoretical analysis is based on the mapping form of the equations of motion for the distance between the particles and the difference of the velocity of the particles.

Let us consider a particle of mass m moving in a one-dimensional potential $V(x)$ which confines particles to a finite region. At time intervals τ_i the particle is partially stopped and its velocity is reset to a new starting velocity $v_i = \alpha v_i^{old} + v_i^{ran}$. Between the stops the particle moves according to Newton's equations

$$\frac{dx}{dt} = v, \qquad \frac{d^2x}{dt^2} = -\frac{1}{m}\frac{dV}{dx}. \qquad (1)$$

When two particles initially at points x_0 and x'_0 are started with velocities v_0 and v'_0 and are driven by an

identical sequence of random velocities v_i^{ran} at the same time intervals τ_i, coordinates and their velocities may accidentally draw closer to one another. The convergence of the two trajectories to the single final trajectory will depend on the evolution with time of the small variances of the distance $\Delta x_i = x_i' - x_i$ and velocity $\Delta v_i = v_i' - v_i$. Moreover, we investigate a transition from chaotic to nonchaotic behavior. Generally, such a transition may be detected from analysis of behavior of the neighboring trajectories and it is described by the Lyapunov characteristic exponents and KS metric entropy of the flow of trajectories in a given region of phase space [5–9]. We will follow this method. Comparisons of the results of such analysis with investigations by the direct numerical simulations confirm the suitability of our theoretical approach.

From formal solutions $x = x(x_i, v_i, t)$ and $v = v(x_i, v_i, t)$ of Eqs. (1) with initial conditions $x = x_i$ and $v = v_i$ at $t = 0$ there follows an equation for $\Delta x(t)$ and $\Delta v(t)$ at a time moment t:

$$\begin{pmatrix} \Delta x(t) \\ \Delta v(t) \end{pmatrix} = \mathbf{T}(\alpha; x_i, v_i, t) \begin{pmatrix} \Delta x_i \\ \Delta v_i \end{pmatrix} , \tag{2}$$

where the matrix $\mathbf{T}$ is of the form

$$\mathbf{T} = \begin{pmatrix} T_{xx} & \alpha T_{xv} \\ T_{vx} & \alpha T_{vv} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x(x_i, v_i, t)}{\partial x_i} & \alpha \dfrac{\partial x(x_i, v_i, t)}{\partial v_i} \\ \dfrac{\partial v(x_i, v_i, t)}{\partial x_i} & \alpha \dfrac{\partial v(x_i, v_i, t)}{\partial v_i} \end{pmatrix} . \tag{3}$$

Note that a similar method of investigation is used in the theory of transition to chaos in classical systems [5,6]. However, the motion in the form (2) and (3) is represented as the non-area-preserving tangent map, while classical dynamics of the conservative systems may be represented by the area-preserving maps.

According to Eqs. (1) and (3) matrix elements T_{xx} and T_{xv} satisfy the equation

$$\frac{d^2 T_x}{dt^2} = -\frac{1}{m} \frac{d^2 V}{dx^2}\bigg|_{x=x(x_i, v_i, t)} T_x \tag{4}$$

while $T_{vx} = \dot{T}_{xx}$, $T_{vv} = \dot{T}_{xv}$, and the initial conditions at $t = 0$ are

$$T_{xx}(x_i, v_i, 0) = T_{vv} = 1, \quad T_{xv} = T_{vx} = 0 ,$$
$$\dot{T}_{xx}(x_i, v_i, 0) = \dot{T}_{vv} = 0, \quad \dot{T}_{xv} = 1, \tag{5}$$
$$\dot{T}_{vx} = -\frac{1}{m} \frac{d^2 V}{dx^2}\bigg|_{x=x_i} .$$

Therefore the dynamics of the distance between the particles Δx and the difference of the velocity Δv may be represented by the non-area-preserving mapping form of the equations of motion

$$\begin{pmatrix} \Delta x_{i+1} \\ \Delta v_{i+1} \end{pmatrix} = \mathbf{T}(\alpha; x_i, v_i, \tau_i) \begin{pmatrix} \Delta x_i \\ \Delta v_i \end{pmatrix} . \tag{6}$$

In general, the intervals between the resets of the velocity τ_i may be depending on the number of step i.

Further analysis of the model may be based on the general theory of the dynamics of classical systems represented as maps [5–9]. Thus for $\alpha = 0$ the Lyapunov exponent is defined as

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\tau_i} \ln |T_{xx}(x_i, v_i, \tau_i)| \tag{7}$$

and may be easily evaluated numerically.

For $\alpha = 1$ the map (6) is area-preserving and $\det \mathbf{T}(1; x_i, v_i, \tau_i) = 1$, while in general $\det \mathbf{T} = \alpha$, $\mathrm{Tr}\mathbf{T} = T_{xx} + \alpha T_{vv}$, and the eigenvalues $\mu_{1,2}$ of the $\mathbf{T}$ matrix are given by the equation

$$\mu^2 - \mu \mathrm{Tr}\mathbf{T} + \det \mathbf{T} = 0 , \tag{8}$$

which yields

$$\mu_{1,2} = \frac{1}{2}[T_{xx} + \alpha T_{vv} \mp \sqrt{(T_{xx} + \alpha T_{vv})^2 - 4\alpha}]. \tag{9}$$

So the eigenvalues come in a reciprocal pair, $\mu_1 \mu_2 = \alpha$. For $(T_{xx} + \alpha T_{vv})^2 - 4\alpha < 0$ the eigenvalues form a complex conjugate pair with $|\mu_1| = |\mu_2| = \sqrt{\alpha}$, otherwise the eigenvalues are real.

Among the randomly driven systems only dynamics of the harmonic oscillator with regular time intervals τ for the reset of velocity may be described exactly. In this case the $\mathbf{T}$ matrix is independent of the starting coordinate x_i and velocity v_i, i.e.,

$$\mathbf{T} = \begin{pmatrix} \cos \omega \tau & (\alpha/\omega) \sin \omega \tau \\ -\omega \sin \omega \tau & \alpha \cos \omega \tau \end{pmatrix} . \tag{10}$$

If $\alpha < 1$ and $\cos \omega \tau \neq \mp 1$ modules of both eigenvalues of the $\mathbf{T}$ matrix are $|\mu_{1,2}| < 1$. So there is a contraction of the distance in the phase space between the particles in each step. The transition matrix for n steps $\mathbf{T}_n = \mathbf{T}^n$ may be calculated by means of diagonalization of the matrix (10). The eigenvalues of the $\mathbf{T}^n$ matrix are $\mu_{1,2}^n$ and the Lyapunov exponents are simply $\lambda_{1,2} = \frac{1}{\tau} \ln |\mu_{1,2}|$.

Generally, the mapping $\mathbf{T}(\alpha; x_i, v_i, \tau_i)$ in (6) depends on the starting coordinates x_i and v_i. Therefore calculation of the mapping for n steps, $\mathbf{T}_n = \mathbf{T}(\alpha; x_{i+n-1}, v_{i+n-1}, \tau_{i+n-1}) \cdot \mathbf{T}(\alpha; x_{i+n-2}, v_{i+n-2}, \tau_{i+n-2}) \cdots \mathbf{T}(\alpha; x_i, v_i, \tau_i)$, and of the corresponding eigenvalues are complicated problems. So we will evaluate the averaged quantities

$$\sigma_{1,2} = \left\langle \frac{1}{\tau_i} \ln |\mu_{1,2}| \right\rangle$$
$$= \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\tau_i} \ln |\mu_{1,2}(x_i, v_i, \tau_i)| , \tag{11}$$

which are analogous to the averaged Lyapunov exponent (7), characterize the rate of the exponential increase of the separation of the two initially adjacent points, and are related with the KS entropy of the system [5,6]. Comparisons of the threshold values τ_c from the direct numerical simulations with those from the criterion

$$\sigma_{largest} = 0 \tag{12}$$

indicate the usefulness of the quantities (11) for analysis

of transition from nonchaotic to chaotic behavior of the systems.

In Fig. 1 we show typical results of the direct numerical calculations of the mean-square distance between pairs of identically driven particles in the Duffing potential $V(x) = x^4 - x^2$ vs number of steps for different time intervals τ between stops and $\alpha = 0$. The averaged characteristic exponents as functions of time between stops, τ, calculated according to Eq. (7) for Duffing and $V(x) = x^4$ potentials are shown in Fig. 2. Regions of τ with the positive and negative characteristic exponents correspond, respectively, to the chaotic and nonchaotic behaviors. Such a conclusion coincides with the conclusions from the direct numerical investigation of the systems. In Fig. 3 we show dependencies of the threshold values for transition from chaotic to nonchaotic behavior of the time τ_c between resets of the velocity to $v_i^{new} = \alpha v_i^{old} + v_i^{ran}$ on the parameter α calculated according to criterion (11) and (12).

For the relatively short time intervals one can expand the position and velocity of the particles in the powers of time t. Using equations of motion (1) and relations (3) we have expansions for the matrix elements T_{xx} and T_{vv}:

$$T_{xx} = 1 - \frac{1}{2m}\frac{d^2V}{dx_i^2}t^2 - \frac{v_i}{6m}\frac{d^3V}{dx_i^3}t^3 + \cdots ,$$

$$T_{vv} = 1 - \frac{1}{2m}\frac{d^2V}{dx_i^2}t^2 - \frac{v_i}{3m}\frac{d^3V}{dx_i^3}t^3 + \cdots . \tag{13}$$

FIG. 2. Averaged characteristic exponents (Lyapunov exponents multiplied by τ) vs the time τ between stops according to Eq. (7) for (a) Duffing potential and (b) $V(x) = x^4$ potential.

We can use expressions (13) for the analytical evaluation of the average rate of contraction of the distance between trajectories for small τ. So, the result of expansion of Eq. (7) according to Eq. (13) in powers of τ is

$$\lambda\tau = -\frac{1}{2m}\left\langle\frac{d^2V}{dx_i^2}\right\rangle\tau^2 + \cdots ,$$
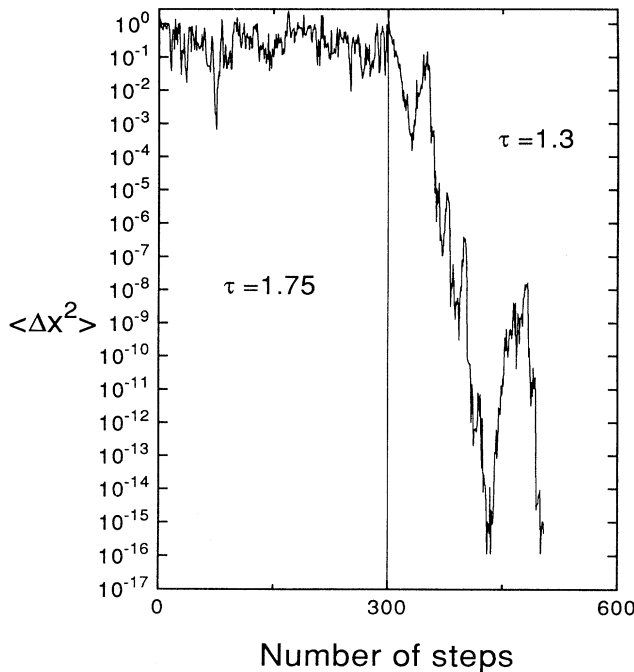
where the angle brackets denote the average with respect

FIG. 1. The mean-square distance $\langle\Delta x^2\rangle$ between pairs of identically driven particles in the Duffing potential $V(x) = x^4 - x^2$ vs number of steps for different time intervals τ between stops. Five hundred independent simulations with initial conditions from the Gaussian distribution of variance $\sigma^2 = 1$ are averaged. The time τ between stops was $\tau = 1.75$ for the first 300 steps and $\tau = 1.3$ for the next steps.
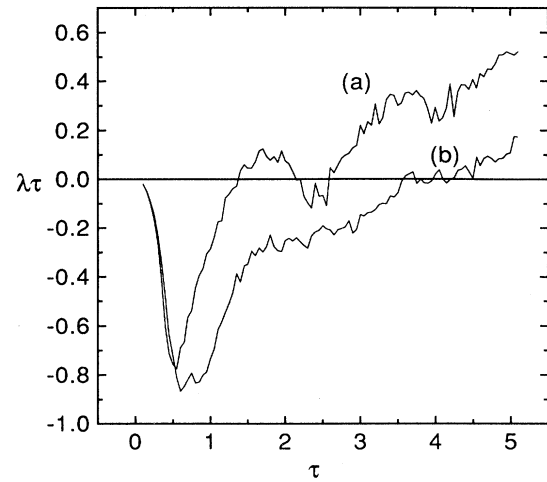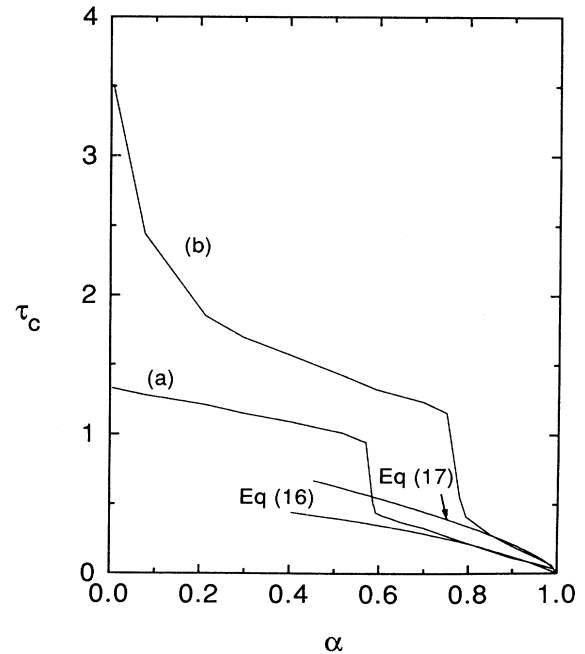

FIG. 3. Dependencies of the threshold value for transition to chaotic behavior of the time τ_c between resets of the velocity to $v_i^{new} = \alpha v_i^{old} + v_i^{ran}$ on the parameter α according to criterion (11) and (12) and according to the analytical limiting expressions (16) and (17) for (a) Duffing potential and (b) $V(x) = x^4$ potential.

to x_i. After the average according to the Boltzmann distribution with $k_B T = m = 1$ (instead of the average with respect to x_i from random trajectories) we have $\lambda \tau \simeq -2.2\tau^2$ and $\lambda \tau \simeq -2.0\tau^2$ for the Duffing potential and $V(x) = x^4$ potential, respectively. For $\tau \leq 0.5$ these evaluations of the rate coincide with the numerical calculations in Fig. 2.

Moreover, expressions (13) may be used for the description of the systems with $\alpha \to 1$ and for short time intervals τ. Introducing the notation $\alpha = 1 - 2\varepsilon$ ($\varepsilon \to 0$) one can have from Eqs. (9) and (11) the approximate expressions

$$\mu_{1,2} \simeq 1 - \varepsilon \mp \sqrt{T_{xx} + T_{vv} - 2 + \varepsilon^2} + \cdots, \qquad (14)$$

$$\sigma_{1,2} \simeq \frac{1}{\tau}[-\varepsilon \mp \langle (T_{xx} + T_{vv} - 2 + \varepsilon^2)^{1/2} \rangle], \qquad (15)$$

$$\varepsilon = (1 - \alpha)/2 \ll 1, \quad \tau \leq \tau_c.$$

The angle brackets in Eq. (15) denote the average with respect to x_i and v_i only if $T_{xx} + T_{vv} - 2 + \varepsilon^2 > 0$, otherwise $\mu_{1,2}$ are complex and $\sigma_{1,2} \simeq -\varepsilon/\tau$. The threshold value τ_c of time interval between stops for transition from nonchaotic to chaotic behavior may be evaluated from the criterion (12).

Using the average according to the Boltzmann and Maxwell distributions (instead of the average with respect to x_i and v_i from random trajectories) one can investigate the behavior of the system according to Eqs. (12)–(15). This yields the expressions of the threshold time intervals,

$$\tau_c \simeq 1.44(1 - \alpha)/[1 + 1.6(1 - \alpha)], \qquad (16)$$

$$\tau_c \simeq (1 - \alpha)^{2/3}, \quad 0.01 \leq 1 - \alpha \ll 1 \qquad (17)$$

for the Duffing and $V(x) = x^4$ potentials, respectively.

Here and in the numerical simulations we set $k_B T = m = 1$. It should be noted that the "effective temperature" of the system in such simulations is $T/(1 - \alpha^2)$.

In Fig. 3 the dependencies (16) and (17) are compared with those obtained from the numerical calculations according to Eqs. (4), (9), (11), and (12). Note the dependence of the law of the approach $\tau_c \to 0$ as $\alpha \to 1$ on the shape of potential.

All such numerical analysis and comparisons with the theoretical results indicate the usefulness and fitness of the present theory for the investigation of transition from chaotic to nonchaotic behavior in randomly driven systems. Generalization of the theory to more degrees of freedom is straightforward: for two- or three-dimensional systems Eqs. (1)–(4) and initial conditions (5) for the **T** matrix are essentially the same, only instead of one equation (4) we have the systems of two or three coupled equations and the dimension of the **T** matrix is 4 or 6, respectively. Further analysis is rather similar to that according to Eqs. (6)–(12). The only additional complication is the diagonalization of the multidimensional **T** matrix.

In summary, we have developed a theory for transition to nonchaotic behavior in a Brownian-type motion. Theoretical analysis based on the mapping form of equations of motion for the distance between the particles and the difference of the velocity allows us to simplify the problem of investigation of transition to nonchaotic behavior and results in the expressions for the criteria of the nonchaotic motion. Theoretical results agree well with the direct numerical simulations and indicate the possibilities of generalization of the model, e.g., to more degrees of freedom, for random values of the time intervals between the resets of the velocity, and for systems driven by the random forces.

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