Topology and nematic ordering. II. Observable critical behavior

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This paper is the second in a pair treating a lattice model for nematic media. In addition to the familiar isotropic (I) and nematically ordered (N) phases, the phase diagram established in the previous paper (paper I) contains a new, topologically ordered phase (T) occurring at large suppression of topological defects and weak nematic interactions. This paper (paper II) is concerned with the experimental signatures of the proposed phase diagram. Specific heat, light scattering, and magnetic susceptibility near both the N-T and I-T transitions are studied and critical behavior is determined. The singular dependences of the Frank constants K_1, K_2, K_3 and the dielectric tensor anisotropy $\Delta\epsilon$ on temperature as $T \to T_{NT}^-$ are also found.

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I. INTRODUCTION AND RESULTS

In the preceding paper [1] (hereafter referred to as paper I), we showed that the nematic-to-isotropic transition need not proceed through a single first-order phase transition, contrary to long-held belief [2]. In fact, we have shown that this disordering can proceed by a pair of continuous transitions, with a novel intermediate phase possessing nontrivial topological order. The phase diagram discussed in paper I (Fig. 3 there) is expected to be generic for nematic materials since the gauge theory from which it is derived depends only on the symmetry of nematic media. The present paper is devoted to the calculation of characteristic signatures of the phase transitions into and out of the topologically ordered but physically isotropic phase T discovered in paper I. The quantities we study are the specific heat, light scattering, and magnetic susceptibility near each transition. We also calculate the singular temperature dependences of the Frank constants K_1, K_2, and K_3 as well as the dielectric tensor anisotropy $\Delta\epsilon$.

An unusual feature of our model is the presence of two distinct, physically isotropic phases T and I, which are separated by a continuous phase transition. Unlike the familiar transition between liquid and gas (which are also both isotropic fluids), there is no latent heat and no critical end point between T and I; they cannot be smoothly interpolated. The qualitative distinction between these two phases, both of which are nematically disordered,

is subtle: in the topologically ordered phase there are few configurations with long nematic defects and the locally coarse grained director field can be unambiguously converted into a nonsingular vector field throughout the fluid. The clearest demonstration of this order would be the measurement of the energy per unit length of a defect that is externally imposed by boundary conditions. Such a direct demonstration of topological order, however, is likely to be quite difficult.

Although one cannot use local probes to distinguish T from I deep within these phases, the transition *between* them has very characteristic and measurable properties. Our prediction is that under certain conditions, one should be able to observe critical behavior by all conventional means (specific heat singularities, light scattering, etc.) between two isotropic phases. Observation of such critical behavior would support our scenario. Verification of the scaling laws derived below would confirm the universality classes identified in paper I.

A less dramatic but also surprising prediction of paper I is the existence of a continuous phase transition between the nematically ordered and the topologically ordered states. According to conventional wisdom based on Landau theory, the development of nematic order from any isotropic state should occur as a first-order phase transition. As we have shown in paper I, this need not be the case if the physically isotropic phase has topological order, since then the transition can be in the universality class of the three-dimensional Heisenberg model. Detailed predictions based on our model are presented below.

For convenient reference, we summarize our results here. Detailed derivations of each of the results 1–9 are

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found in later sections. Unless otherwise indicated, reduced temperature relative to a particular transition is denoted by $t \equiv (T - T_c)/T_c$ and the corresponding correlation length is denoted by ξ. Subscripts $+$ and $-$ denote quantities pertaining above $(t > 0)$ and below $(t < 0)$ T_c. "Above" always refers to the more disordered phase, so that the isotropic phase is above the topologically ordered phase, which in turn is above the nematically ordered state.

1. Specific heat. The singular specific heat near the continuous transitions into and out of the topologically ordered phase varies as

$$C(T) = A_{\pm}|t|^{-\alpha}. \tag{1.1}$$

Near the I-T transition, the exponent α takes the (positive) three-dimensional Ising model value [3] $\alpha_I = 0.1085 \pm 0.0075$, which implies a divergent specific heat. The amplitude ratio A_+/A_- is inverted relative to the usual Ising spin model, a result of the duality transformation discussed in paper I. Near the T-N transition, α takes the three-dimensional Heisenberg value [3] $\alpha_H = -0.130 \pm 0.021$, resulting in a cusp. The amplitude ratio A_+/A_- takes the corresponding universal value.

2. Polarized light scattering near the I-T transition. Near the I-T transition, the light scattering intensity $I_{xx}(\mathbf{q}, t)$ for incoming and scattered light with parallel polarizations obeys the scaling law

$$I_{xx}(\mathbf{q}, t) = q^{-\alpha_I/\nu_I} f_{\pm}^{ITP}(q\xi) \propto \begin{cases} t^{-\alpha_I}, & q\xi \ll 1 \\ q^{-\alpha_I/\nu_I}, & q\xi \gg 1. \end{cases} \tag{1.2}$$

Here α_I and ν_I are the three-dimensional Ising specific heat and correlation length exponents [3] $\alpha_I = 0.1085 \pm 0.0075$, and $\nu_I = (2 - \alpha_I)/3 = 0.6305 \pm 0.0025$, respectively. (The Ising correlation length ξ diverges as $t^{-\nu_I}$.)

3. Depolarized light scattering near the I-T transition. Near the I-T transition, the light scattering intensity $I_{xy}(\mathbf{q}, t)$ for incoming and scattered beams with perpendicular polarizations shows only a weak nonanalyticity at $\mathbf{q} = 0$. That is, $I_{xy}(\mathbf{q}, t)$ can be decomposed into an analytic and a singular part [4]

$$I_{xy}^{\text{sing}}(\mathbf{q}, t) = q^{(1-\alpha_I)/\nu_I} f_{\pm}^{ITD}(q\xi)$$

$$\propto \begin{cases} t^{1-\alpha_I}, & q\xi \ll 1 \\ q^{(1-\alpha_I)/\nu_I}, & q\xi \gg 1, \end{cases} \tag{1.3}$$

where $f_{\pm}^{ITD}$ is a universal function. Note that the analytic contributions to the scattering dominate the total scattering as $|\mathbf{q}| \to 0$ and $t \to 0$.

The exponent in (1.3) is $(1 - \alpha_I)/\nu_I = 1.414 \pm 0.006$, which is greater than unity. The singularity is therefore merely a divergence in the *second* derivative $\partial^2 I_{xy}/\partial q^2$ of the scattering intensity. This singularity is not a promising signature of the I-T transition since it is so weak.

4. Depolarized light scattering near the N-T transition. Near the N-T transition, the depolarized light scattering intensities $I_{xz}(\mathbf{q}, t)$ and $I_{yz}(\mathbf{q}, t)$ obey the scaling laws

$$I_{iz}(\mathbf{q}, t) = q^{2\eta_H - 1} f_{\pm}^{NTD}(q\xi), \qquad i = x, y. \tag{1.4}$$

(The incoming polarization $\hat{z}$ is along the mean director.) The very small anomalous dimension of the spin for the three-dimensional Heisenberg model is $\eta_H = 0.02 \pm 0.01$. Matching the form (1.4) onto the known small-wavenumber behavior of I_{iz} deep in the nematic and isotropic phases (for this purpose, T and I behave identically) implies

$$I_{iz}(\mathbf{q}, t) \propto \begin{cases} q^{-2} t^{\nu_H (1 + 2\eta_H)}, & t < 0, \ q\xi \ll 1 \\ q^{2\eta_H - 1}, & q\xi \gg 1 \\ t^{-\nu_H (1 - 2\eta_H)}, & t > 0, \ q\xi \ll 1. \end{cases} \tag{1.5}$$

Approaching the transition from the nematic side $(t < 0)$, this result implies a relationship between the critical behavior of the Frank constants and the dielectric anisotropy $\Delta\epsilon$ since

$$I_{xz} = \frac{c(\Delta\epsilon)^2}{K_1 q_x^2 + K_3 q_z^2} \tag{1.6}$$

and

$$I_{yz} = \frac{c(\Delta\epsilon)^2}{K_2 q_x^2 + K_3 q_z^2}, \tag{1.7}$$

with some constant c (we have taken $q_y = 0$). The form (1.5) then implies both that (i) the Frank constants are asymptotically equal upon approaching T_{NT} (due to the isotropy of the scaling law) and that (ii) there is a particular relation between the exponents governing the vanishing of the Frank constants and of $\Delta\epsilon$ at the N-T transition. These are, indeed, our next two conclusions.

5. Frank constants. The Frank constants approach a common value and vanish as

$$K_i \sim |t|^{\nu_H} \qquad \text{as} \quad T \to T_{NT}^- \tag{1.8}$$

at the N-T transition. This can be tested by direct measurements of the Frank constants either by light scattering (see result 4) or by Fréedericksz instability measurements (see result 8).

The approach of the "splay" Frank constant K_1 to equality with the "bend" and "twist" constants K_2 and K_3, however, is very slow. Specifically, we expect

$$\frac{K_1 - K_{2,3}}{K_{2,3}} \propto \frac{1}{|\ln(|t|)|}. \tag{1.9}$$

Experimentally, this behavior may look very much like K_1 being proportional to $K_{2,3}$ with a constant prefactor of order unity.

6. Dielectric anisotropy near the N-T transition. The dielectric anisotropy tensor $\Delta\epsilon$ vanishes like

$$\Delta\epsilon \sim |t|^{2\beta_H} \qquad \text{as} \quad T \to T_{NT}^-, \tag{1.10}$$

where [3] $\beta_H = 0.368 \pm 0.004$ is the Heisenberg order parameter exponent. This result for $\Delta\epsilon$ can be tested either through light scattering as mentioned above or by simple optical birefringence measurements. Combining our previous results (results 4 and 5) for the Frank constants

and I_{iz} gives

$$I_{iz}(\mathbf{q}, t < 0) \propto q^{-2}|t|^{4\beta_H - \nu_H}. \tag{1.11}$$

With the exact relation $2\beta = \nu(d - 2 + \eta)$ that is derived by combining the Fisher scaling law and the Rushbrooke scaling law [5], we find Eq. (1.5) for $q\xi \ll 1$ (and $d = 3$).

7. *Magnetic response near the N-T transition.* Near the N-T transition, the magnetic response of the system is given by

$$M_\alpha = \chi_I H_\alpha + \Delta\chi(t, H)\left(\hat{n}_\alpha \hat{n}_\beta - \frac{1}{3}\delta_{\alpha\beta}\right)H_\beta, \tag{1.12}$$

where α denotes Cartesian components, $\mathbf{n}$ is the nematic director (always along $\mathbf{H}$ in the topologically ordered phase), and the isotropic part of the susceptibility χ_I is analytic through the (continuous) N-T transition. The *anisotropic* part of the susceptibility $\Delta\chi$, on the other hand, obeys

$$\Delta\chi(t, H) = A|t|^{2\beta_H} f_\pm^\chi\left(\frac{H}{H_c|t|^{\phi_x}}\right), \tag{1.13}$$

where A and H_c are nonuniversal (i.e., system-dependent) constants. The exponent is $\phi_x = \nu y_\tau/2 = 0.62 \pm 0.01$, where y_τ is the renormalization-group eigenvalue of the "spin-tensor interaction [5]" for the ($n = 3$) Heisenberg model. To second order in the $\epsilon = 4 - d$ expansion, $y_\tau \approx 1.77 \pm 0.01$. The scaling functions $f_\pm^\chi(x)$ obey

$$f_+^\chi(x) \to x^2 \quad \text{as} \quad x \to 0,$$
$$f_-^\chi(x) \to \text{const} \quad \text{as} \quad x \to 0, \tag{1.14}$$
$$f_+^\chi(x) \to f_-^\chi(x) \propto x^{1/\delta_H - 1} \quad \text{as} \quad x \to \infty,$$

where $\delta_H^{-1} - 1 = 2(1 + \eta_H)(y_\tau^H)^{-1} = 1.15 \pm 0.01$. The result (1.15) for f_+^χ implies that the nonlinear susceptibility diverges as the N-T transition is approached from the isotropic (T) side as

$$\frac{\partial^3 M}{\partial H^3} \propto |t|^{\gamma_2} \quad \text{for} \quad \frac{H}{H_c|t|^{\phi_x}} \ll 1, \tag{1.15}$$

with

$$\gamma_2 = 2(\beta_H - \phi_\chi) = -0.52 \pm 0.01. \tag{1.16}$$

Recall that in this topologically ordered but isotropic phase, $\mathbf{M}$ is always along the applied field $\mathbf{H}$. From (1.13) and (1.15), one deduces a nonanalytic (though nondivergent) temperature dependence of the *linear* susceptibility as the N-T transition is approached from the nematic side,

$$\frac{\partial M}{\partial H} = \chi_I + \frac{2}{3}\Delta\chi = a + b|t|^{2\beta_H}, \tag{1.17}$$

where a and b are constants. Increasing the magnetic field so that $H \ll H_c|t|^{\phi_x}$, the singular temperature dependence (1.15) of the nonlinear susceptibility becomes a singular magnetic field dependence

$$\frac{\partial^3 M}{\partial H^3} \propto H^{\delta_H^{-1} - 3}, \quad \frac{H}{H_c|t|^{\phi_x}} \gg 1, \tag{1.18}$$

which is readily seen to cross over smoothly to the low-field ($\frac{H}{H_c|t|^{\phi_x}} \ll 1$) result at $H \sim H_c|t|^{\phi_x}$.

8. *Fréedericksz instability.* The Fréedericksz instability [2] provides a means of measuring the ratio of the anisotropic susceptibility $\Delta\chi$ to the Frank constants K_i. When a nematic is confined between parallel plates that have been treated to create boundary conditions favoring a particular orientation of the director, it will assume that orientation even in the presence of an applied magnetic field $\mathbf{H}$ that favors a different orientation, as long as that field is sufficiently small. Usually experimentalists choose this field to be perpendicular to that favored by boundary conditions. When the strength of the applied field exceeds a threshold H_F, however, the director rotates away from the alignment favored by the boundaries to that favored by the field. This is the Fréedericksz instability. The standard result for the threshold field is

$$H_F = \frac{1}{L}\sqrt{\frac{K_i}{\Delta\chi}}O(1). \tag{1.19}$$

The precise geometry of the experiment determines which Frank constant K_i appears, as well as the $O(1)$ factor. Inserting the known scaling forms

$$K_i(t, H) = |t|^{\nu_H} f_\pm^i\left(\frac{H}{H_c|t|^{\phi_x}}\right) \tag{1.20}$$

and Eq. (1.13) for $\Delta\chi$ into (1.19), we obtain

$$H_F = L^{-1 + \frac{\eta_H}{2}} f_\pm^F(L/\xi_H), \tag{1.21}$$

where the scaling function $f_\pm^F(x)$ has the asymptotic forms

$$f_-^F(x) \to x^{-\eta/2} \quad \text{as} \quad x \to \infty,$$
$$f_+^F(x) \to c \quad \text{as} \quad x \to 0,$$
$$f_+^F(x) \to c'e^{-c''x} \quad \text{as} \quad x \to \infty, \tag{1.22}$$

for some constants c, c', and c''. These relations imply that

$$H_F \propto \begin{cases} |t|^{-\frac{\eta\nu}{2}}L^{-1} \propto |t|^{-0.007} & \text{for } t < 0, L \gg \xi \\ L^{-1 + \frac{\eta}{2}} = L^{(-0.99 \pm 0.01)} & \text{for } L \ll \xi \\ e^{-\frac{L}{\xi}} \propto e^{-L|t|^\nu} \propto e^{-L|t|^{0.7}} & \text{for } t > 0, L \gg \xi. \end{cases} \tag{1.23}$$

Unfortunately, the smallness of η_H makes the dependence of H_F on the temperature (and on the anomalous length scale in the critical regime) extremely weak in the nematic phase and hence difficult to discern experimentally. In practice, one expects to observe an H_F that is nearly constant with increasing temperature, until it begins to drop in the unusual stretched exponential manner indicated in (1.23). The observation of either this stretched exponential or the apparent temperature independence of H_F below T_{NT} would confirm our theory.

The constancy of H_F below T_{NT} should be contrasted

with the predictions of Landau theory in the case of a weakly first-order transition. These predictions are

$$\Delta\chi(T) \propto Q_0(T) \propto \sqrt{|t|},$$
$$K_i(T) \propto Q_0^2 \propto |t|. \tag{1.24}$$

Given these scalings, (1.19) would imply that $H_F \propto |t|^{1/4}$, instead of (1.23).

9. Magnetic susceptibility. The elusive *I-T* transition can also be detected by magnetic susceptibility measurements, albeit with more difficulty. We find that the nonlinear susceptibility has a $|t|^{1-\alpha_I}$ singularity at the transition

$$\frac{\partial^3 M}{\partial H^3} \sim |t|^{1-\alpha_I}. \tag{1.25}$$

The rest of this paper is devoted to deriving these conclusions. The results quoted above for the specific heat require no further discussion, as they follow directly and immediately from what has already been shown (in paper I) about the partition function, which determines all thermodynamic functions. The other results involve various correlation functions.

II. POLARIZED LIGHT SCATTERING NEAR THE *I-T* TRANSITION

Consider first light scattering near the *I-T* transition. In general, light scattering in an isotropic material is given by [6] $I_{xx}(\mathbf{q}) \propto \langle|\epsilon(\mathbf{q})|^2\rangle$. This scattering is caused by local fluctuations in the (isotropic) dielectric constant ϵ, which in turn are caused by density fluctuations of the various components of the material. For simplicity we consider here a situation with just one such density ρ; including more does not alter the conclusions. Expanding the density dependence of ϵ to linear order about the average density, we obtain $I_{xx}(\mathbf{q}) \propto \langle|\delta\rho(\mathbf{q})|^2\rangle$. The calculation of the polarized light scattering near the *I-T* transition thus reduces to determining the fluctuations of the (nonordering) density ρ. To calculate these correlations, we must know how density fluctuations enter the Hamiltonian.

Recall that the basic Hamiltonian defining our theory [Eq. (2.1) of paper I] is

$$-\beta\mathcal{H} = J\sum_{\langle i,j\rangle} U_{ij}\mathbf{S}_i \cdot \mathbf{S}_j + K\sum_{\{ijkl\}} U_{ij}U_{jk}U_{kl}U_{li}, \tag{2.1}$$

where the spins $\mathbf{S}_i$ are three-dimensional unit vectors on the sites of a lattice (cubic for convenience), the variables $U_{ij} = \pm1$ are associated with links (ij) between nearest-neighbor sites, and the second sum in the Hamiltonian runs over elementary plaquettes $ijkl$.

We showed in paper I that the spins are irrelevant at small J, so they may be neglected when one incorporates density-fluctuation effects into an effective renormalized lattice gauge Hamiltonian. There are no symmetry restrictions upon the dependence of the Hamiltonian on ρ. For long wavelengths, the dominant dependence is

upon the value of ρ itself and spatial gradients can be neglected. A Hamiltonian with the appropriate symmetry-allowed coupling is

$$\mathcal{H} = \sum \{V[\rho_\mu(x)] - [K_R + \gamma\delta\rho_\mu(x)]U\}, \tag{2.2}$$

where U abbreviates the renormalized plaquette term from Eq. (2.1) above. [Recall that U measures defect density, so that Eq. (2.2) describes a coupling between defect density and fluid density.] In the lattice model, the local density variables ρ_μ are associated with the plaquettes, so we denote this dependence simply by the directional index μ. (Each plaquette is then labeled by its normal vector.) K_R is the renormalized defect stiffness obtained after integrating out the spins as described in paper I, γ is the coupling between ρ and the disclination density, and $\delta\rho_\mu = \rho_\mu(x) - \rho_0$.

For the moment, all we need to know about $V(\rho)$ is that it is a local, *smooth* function through the transition, since ρ is nonordering, and it has a minimum *near* (but not at) the equilibrium density ρ_0. A duality transformation can be applied to the model implied by Hamiltonian (2.2). As explained in paper I, the duality transformation is completely local in the couplings so the partition function becomes

$$Z = \int D\rho \sum_{\{\sigma_\alpha\}} \exp\left\{\sum_\mu[-V_R(\delta\rho_\mu)\right.$$
$$\left. +J_\mu(\delta\rho_\mu)\sigma_\alpha\sigma_{\alpha+\mu}]\right\}, \tag{2.3}$$

with a "renormalized" potential V_R and density-dependent spin coupling J given by

$$V_R(\rho_\mu) = V(\rho_\mu) - (K_R + \gamma\delta\rho_\mu) - 1/2\ln(1 - e^{-4K(r_\mu)}),$$
$$J(\delta\rho_\mu) = 1/2\ln\coth(K_R + \gamma\delta\rho_\mu). \tag{2.4}$$

In this model, fluctuations in ρ are driven entirely by those of the Ising spins that are dual to the gauge field. More precisely, they are proportional to the energy-energy correlations of the Ising model obeying the scaling law (1.2). To see this, expand V_R to second order and $J(\delta\rho)$ to first order in powers of $\delta\rho$. The result for the Hamiltonian is

$$H \approx H_{\text{Ising}}(\rho_0) + h\sum\delta\rho_\mu$$
$$-\gamma_R\sum\delta\rho_\mu\sigma_\alpha\sigma_{\alpha+\mu} + A/2\sum(\delta\rho)^2 \tag{2.5}$$

with

$$h = V_R'(\rho_0), \quad A = V_R''(\rho_0), \quad \gamma_R = J'(\rho_0), \tag{2.6}$$

where primes indicate derivatives. The calculations required to find expressions for $\langle\delta\rho_\mu\rangle$ and $\langle\delta\rho_\mu\delta\rho_{\mu'}\rangle$ are easy in this approximation since they are Gaussian integrals over $\delta\rho$ (fluctuations on different sites are even decoupled). Specifically,

$$-A\langle\delta\rho_\mu\rangle = h - \gamma_R\langle\sigma_\alpha\sigma_{\alpha+\mu}\rangle. \tag{2.7}$$

Since this is zero by definition, $h = \gamma_R\langle\sigma_\alpha\sigma_{\alpha+\mu}\rangle$. Using

this expression for h,

$$A^2 \langle \delta \rho_\mu \delta \rho_{\mu'} \rangle = \gamma_R^2 \langle \sigma_\alpha \sigma_{\alpha+\mu}; \sigma_{\alpha'} \sigma_{\alpha'+\mu'} \rangle + A \delta_{\mu\mu'}, \quad (2.8)$$

where $\langle x; y \rangle \equiv \langle xy \rangle - \langle x \rangle \langle y \rangle$ is the truncated correlation function. Aside from the (zero-range) second term, the right-hand side of this equation is simply the energy-energy correlation function for the three-dimensional Ising model, the well-known behavior of which now implies Eq. (1.2) of result 2.

III. DEPOLARIZED SCATTERING NEAR THE N-T TRANSITION

Next we turn to depolarized light scattering near the the N-T transition. The standard form for such scattering in an *anisotropic* medium is [6]

$$I_{ij}(\mathbf{q}) \propto \langle \epsilon_{ij}(\mathbf{q}) \epsilon_{ij}(-\mathbf{q}) \rangle, \quad (3.1)$$

where $\epsilon_{ij}(\mathbf{q})$ is the Fourier transform of the position-dependent local dielectric tensor and the summation convention has been suspended on the right-hand side.

For small nematic order the dielectric tensor can be expanded in the tensor nematic order parameter

$$Q_{ij}(\mathbf{r}) = S_i(\mathbf{r}) S_j(\mathbf{r}) - \delta_{ij}/3 \quad (3.2)$$

as

$$\epsilon_{ij} = \epsilon_0 \delta_{ij} + g Q_{ij}, \quad (3.3)$$

where g is a microscopic parameter and is therefore nonsingular at the transition. The correlation function

$$C_{ij}(\mathbf{r}) \equiv \langle \epsilon_{ij}(\mathbf{r}) \epsilon_{ij}(\mathbf{0}) \rangle \quad (3.4)$$

for $i \neq j = z$ can therefore be expressed in terms of Heisenberg spins of our model as

$$C_{iz}(\mathbf{r}) \propto \langle \mathbf{S}_i(\mathbf{r}) \mathbf{S}_z(\mathbf{r}) \mathbf{S}_i(\mathbf{0}) \mathbf{S}_z(\mathbf{0}) \rangle. \quad (3.5)$$

In paper I we have shown that near the N-T transition the system is described by an effective Heisenberg model, so that we can apply a renormalization-group transformation to relate $C_{iz}(\mathbf{r})$ to the same function in the Heisenberg model with different parameters:

$$\begin{aligned} C_{iz}(\mathbf{r}, t) &= b^{-2(d-2+\eta_H)} C_{iz}(b^{-1}\mathbf{r}, b^{1/\nu_H} t) \\ &= r^{-2(d-2+\eta_H)} g_\pm(r/\xi). \end{aligned} \quad (3.6)$$

The second equality follows upon making the choice $b = r/a$, where a is a microscopic length and $\xi = \xi_0 |t|^{-\nu_H}$ is the three-dimensional Heisenberg correlation length, noting the isotropy of the Heisenberg model and making the definition

$$g(x) \equiv C_{iz}[a, x^{1/\nu_H}(\xi_0/a)^{1/\nu_H}] a^{2(d-2+\eta_H)}. \quad (3.7)$$

Fourier transformation (and the substitution $d = 3$) leads directly to Eq. (1.4) (result 4), with

$$f_\pm^{NTD}(q\xi) = \Gamma \int d^3 R \frac{g_\pm(R/q\xi)}{R^{2(1+\eta)}} e^{iR_z}. \quad (3.8)$$

To complete the description of the critical scattering near the N-T transition requires matching the scaling form (1.4) onto known long-wavelength behavior in the N and T phases to obtain the asymptotic forms of $f_\pm$.

Consider first the topologically ordered (T) phase ($t > 0$). No Goldstone modes are present since there is no broken symmetry. The fluctuations of the dielectric tensor (hence light scattering) must therefore be *finite* as $|\mathbf{q}| \to 0$ at fixed temperature. Consistency with the scaling law demands $f_+(x) \propto x^{1-2\eta_H}$ as $x \to 0$. The $q \to 0$ scattering therefore diverges upon approaching the transition from the isotropic side

$$I_{iz}(q \to 0, t) \propto \xi^{1-2\eta_H} \propto t^{-\nu(1-2\eta_H)}, \quad q\xi \ll 1. \quad (3.9)$$

The requirement of finite scattering at fixed q in the critical regime ($q\xi \gg 1$) implies that f_+ and f_- tend to the same constant as $x \to \infty$; this means

$$I_{iz}(q) \propto q^{-1+2\eta_H} \quad \text{for } q\xi \gg 1. \quad (3.10)$$

On the nematic side of the transition ($t < 0$), scattering is dominated by Goldstone modes — fluctuations in the orientation of the director. The resulting long-wavelength behaviors of Eqs. (1.6) and (1.7) are standard results [6]. In these equations, the anisotropy of the dielectric tensor $\Delta \epsilon$ is given by

$$\langle \epsilon_{ij} \rangle \equiv \epsilon_0 \delta_{ij} + \Delta \epsilon \left(n_i n_j - \frac{1}{3} \delta_{ij} \right), \quad (3.11)$$

the mean director $\mathbf{n}$ lies along $\hat{\mathbf{z}}$, and $\mathbf{q} \cdot \hat{\mathbf{y}} = 0$. If the scaling results are to coincide with Eqs. (1.6) and (1.7) in the $q \to 0$ limit, then $f_-(x) \propto x^{-(1+2\eta)}$ as $x \to 0$. As a result,

$$I_{iz}(q) \sim \xi^{-(1+2\eta_H)} q^{-2} \quad (3.12)$$

$$\propto |t|^{\nu(1+2\eta_H)} q^{-2} \quad \text{as } q\xi \to 0, \ t \to 0 -. \quad (3.13)$$

This is the result announced in Eq. (1.5) of result 4. Implications for the critical behavior of the Frank constants are also carried by Eq. (3.13) combined with Eqs. (1.6) and (1.7). Specifically,

$$\frac{(\Delta \epsilon)^2}{K_i} \propto |t|^{\nu_H(1+2\eta_H)}, \quad i = 1, 2, 3. \quad (3.14)$$

Clearly all three Frank constants have the same critical behavior.

We now verify both Eq. (3.14) and the critical behavior ($K_i \sim |t|^{\nu_H}$) of the Frank constants by an independent derivation. If we take an expectation value and express Q_{ij} in terms of spins [see Eq. (3.2)], we find

$$\langle Q_{ij} \rangle \propto |t|^{2\beta} (n_i n_j - \delta_{ij}/3). \quad (3.15)$$

Comparing with the definition of $\Delta \epsilon$, we deduce $\Delta \epsilon \propto t^{2\beta_H}$.

Some information about the Frank constants can be

obtained from generalized Josephson relations [7]. Imagine imposing a twist on the boundary conditions for a nematic confined to an $L \times L \times L$ volume and compare to one with periodic boundary conditions. The (Frank) free-energy difference is $\Delta F(L) = (K_2/2)L\theta^2$, where θ is the twist angle. From scaling, one expects $\Delta F = g(L/\xi)$ near a continuous transition. Combining these two equations,

$$K_2 \propto \xi^{-1} \propto |t|^{\nu_H}. \tag{3.16}$$

Very similar arguments that use boundary conditions that impose a bend or splay distortion imply $K_1, K_3 \propto |t|^\nu$ as well. Gathering these results and the scaling of $\Delta\epsilon$ we find

$$\frac{(\Delta\epsilon)^2}{K_i} \propto |t|^{4\beta_H - \nu_H}. \tag{3.17}$$

Application of the exact scaling relation $2\beta = \nu(d - 2 + \eta) = \nu(1 + \eta)$ (the last equality following from $d = 3$) recovers the result Eq. (3.14).

IV. CORRECTIONS TO SCALING FOR FRANK CONSTANTS

Let us next consider the corrections to scaling for the Frank elastic constants. In the context of an $|\mathbf{S}|^4$ soft-spin theory [5], the starting point is a Landau free energy expressed in terms of a vector-valued field $\mathbf{S}$ with fluctuating magnitude and direction. The term of lowest order in gradients is uniquely of the form $f(|\mathbf{S}|^2)(\nabla\mathbf{S})^2$, as long as the invariance under global rotations of $\mathbf{S}$ is unbroken (compare the discussion in paper I). In real nematics, this invariance is broken and only the invariance under combined identical rotations in internal and real space survives.

The Landau-Ginzburg free energy then contains three types of gradient terms

$$\mathcal{F}_{\text{grad}} = K_1(\nabla \cdot \mathbf{S})^2 + K_2|\mathbf{S} \cdot (\nabla \times \mathbf{S})|^2$$
$$+K_3|\mathbf{S} \times (\nabla \times \mathbf{S})|^2, \tag{4.1}$$

where K_1, K_2, and K_3 are the Frank constants for splay, bend, and twist, respectively. In an $\epsilon = 4 - d$ expansion, the renormalization-group eigenvalues of these operators at the $n = 3$ Wilson-Fisher fixed point are $\lambda_{2,3} = -2 + O(\epsilon)$, as can be seen by power counting (four powers of $\mathbf{S}$ and two gradients). Since these eigenvalues are large and negative, the difference between K_2 and K_3 vanishes rapidly as the transition is approached. The standard form for the correction to scaling is

$$K_i(t) = K_0|t|^\nu(1 + \Delta_i|t|^{\lambda_i\nu}), \quad i = 1, 2, 3, \tag{4.2}$$

with λ_2 and λ_3 given above.

The difference between K_2 and K_3 therefore vanishes rapidly, but K_1 is another story. The eigenvalue of the operator $(\nabla \cdot \mathbf{S})^2$ is [8] $\lambda_1 = -\epsilon^2/108 + O(\epsilon^3)$, so this Frank splay constant vanishes very slowly under renormalization. By way of illustration, for $\epsilon = 1$ and $\Delta_1 = 1$,

K_1 still differs by 50% from the value $K_0|t|^\nu$ taken by K_2 and K_3 at the ridiculously tiny reduced temperature $|t| \approx 2^{-100/\nu} \approx 10^{-43}$.

In fact, λ_1 is so small that for all reasonable values of Δ_1, the renormalization-group flows of $\Delta K_1 \equiv (K_1 - K_{2,3})$ are presumably dominated by a nonlinear term proportional to $(\Delta K_1)^2$ (which to our knowledge has never been calculated). The full recursion relation for ΔK_1 is then of the form

$$\frac{d\Delta K_1}{d\ell} = \lambda_1\Delta K_1 - c(\Delta K_1)^2, \tag{4.3}$$

where c is a constant (again, to our knowledge, never calculated). Due to the smallness of λ_1, Eq. (4.3) is valid outside an extremely narrow range of temperatures in which $\Delta K_1 \ll \lambda_1$. Assuming $c > 0$, i.e., the fixed point $\Delta K_1 = 0$ is globally, as well as locally, stable, and neglecting the linear term, we conclude that ΔK_1 vanishes as $1/\ell$ for all experimentally observable temperatures.

If we choose ℓ such that $e^\ell \xi = 1$, we find $\ell \propto |\ln|t||^{-1}$. This in turn implies that $(K_1 - K_{2,3})/K_{2,3}$ vanishes like $|\ln(|t|)|^{-1}$, as claimed in result 5. This dependence is so weak that in many situations it will simply look as if the correction term is independent of t. In this case, we can write $K_1(t) = K_0|t|^\nu(1 + \Delta_1)$ for all practical purposes. That is, K_1 will vanish with the same exponent as K_2 and K_3 but with a prefactor of (presumably) order one. This means that for $\mathbf{q} \cdot \hat{\mathbf{y}} = 0$, $I_{xz}(\mathbf{q})$ will remain *anisotropic*:

$$I_{xz}(\mathbf{q}) = \frac{c(\Delta\epsilon)^2 K_0^{-1}|t|^{-\nu_H}}{(1 + \Delta_1)q_x^2 + q_z^2}. \tag{4.4}$$

The singular character of the anisotropy should be unobservable for all practical purposes. $I_{yz}(\mathbf{q})$ is still expected to become isotropic. Note that the anisotropy of I_{xz} does not alter the scaling law $I_{xz} \propto |t|^{\nu(1+2\eta)}q^{-2}$, but rather introduces a (nonsingular) anisotropic prefactor.

V. MAGNETIC SUSCEPTIBILITY NEAR THE *N-T* TRANSITION

The application of a magnetic field $\mathbf{H}$ adds to the Hamiltonian a term

$$\mathcal{H}_{\text{mag}} = -\Delta\chi_0 \sum_i (\mathbf{H} \cdot \mathbf{S}_i)^2, \tag{5.1}$$

where the *molecular* diamagnetic anisotropy $\Delta\chi_0$ is assumed to be a smooth function of temperature, in the spirit of modern critical phenomenology. This standard expression Eq. (5.1) for the nematic magnetic energy is the lowest-order [9] term in $\mathbf{H}$ that respects the gauge symmetry ($\mathbf{S}_i \to \sigma_i\mathbf{S}_i$, $U_{ij} \to \sigma_i\sigma_j U_{ij}$). The magnetization $\mathbf{M}$ induced by the applied magnetic field can be deduced via the standard thermodynamic relation

$$M_\alpha = \frac{1}{N}\frac{\partial F}{\partial H_\alpha} = 2\Delta\chi_0\langle S_i^\alpha S_i^\beta\rangle H_\beta. \tag{5.2}$$

Now we know that

$$\langle S_i^\alpha S_i^\beta \rangle = Q_0(n_\alpha n_\beta - \delta_{\alpha\beta}/3) + \delta_{\alpha\beta}/3, \qquad (5.3)$$

where $Q_0(T,H)$ is the magnitude of the nematic tensor order parameter. This immediately implies Eq. (1.12), with

$$\chi_I = (2/3)\Delta\chi_0 \qquad (5.4)$$

and

$$\Delta\chi = 2\Delta\chi_0 Q_0(T,H). \qquad (5.5)$$

Equation (5.4) ensures that χ_I is perfectly smooth through T_{NT} since $\Delta\chi_0$ is; this follows from the fact that $|\mathbf{S}|^2 = 1$, regardless of temperature. Hence, taking the trace over α and β in Eq. (5.3) fixes the coefficient of $\delta_{\alpha\beta}$ at *exactly* 1/3, independent of temperature (the first term is traceless by construction and does not affect this result). This coefficient in turn leads to the *exact* relation (5.4) between χ_I and $\Delta\chi_0$, which in turn implies analyticity of χ_I.

We must still determine the scaling of $Q_0(T,H)$. Since Q_0 involves correlations of two $\mathbf{S}$ fields, its renormalization group eigenvalue is simply (see the Appendix) $-2\beta_H/\nu_H$. Therefore, the renormalization-group transformation near the $d = 3$ Heisenberg critical fixed point can be used to relate $Q_0(T,H)$ to Q_0 at rescaled temperature and field as

$$Q_0(T,H) = b^{-2\beta_H/\nu_H} Q_0(b^{1/\nu_H}t, b^{y_h}H), \qquad (5.6)$$

where y_h is the renormalization-group eigenvalue of H. This eigenvalue can be determined by rewriting the magnetic Hamiltonian as

$$\mathcal{H}_{\mathrm{mag}} = \Delta\chi_0 \sum_i H_\alpha H_\beta (S_i^\alpha S_i^\beta - 1/3|\mathbf{S}|^2\delta_{\alpha\beta}), \qquad (5.7)$$

which differs from (5.1) only by a constant (independent of $\mathbf{S}$), owing to $|\mathbf{S}|^2 = 1$. In this form, $\mathcal{H}_{\mathrm{mag}}$ is simply the "spin tensor interaction" of Ref. [5]. The renormalization-group eigenvalue of this perturbation has the $\epsilon = 4 - d$ expansion [8]

$$y_\tau = 2\left(1 - \frac{\epsilon}{n+8} + \frac{n^2 - 18n - 88}{4(n+8)^2}\epsilon^2\right) + O(\epsilon^3)$$
$$\simeq 1.77 \pm 0.01, \qquad (5.8)$$

where in the second, approximate, equality we have set $d = 3$ and $n = 3$. The quoted error is typical of second order in ϵ expressions for $d = 3$, $O(n)$ models. Since the coefficient of the spin tensor perturbation (5.7) is H^2, the renormalization-group eigenvalue of H is $y_h = y_\tau/2 \simeq 0.89 \pm 0.01$. With this eigenvalue in hand, we can now obtain the scaling of Q_0. Choosing $b = |t|^{-\nu_H}$ in Eq. (5.6) and using (5.5) to relate Q_0 to $\Delta\chi$, we immediately obtain Eq. (1.13) for $\Delta\chi$ (result 7), with

$$f_I(x) = Q_0(t = \pm 1, H = x). \qquad (5.9)$$

To obtain the asymptotic form (1.15) of f_+, note that deep within the disordered phase the susceptibility of

Q_0 to H^2 (the coefficient of the magnetic perturbation) should be finite. Hence $Q_0 \propto H^2$ in that regime. (Q_0 of course vanishes as $H \to 0$ in the disordered phase, so this H^2 term is the leading order term.) In the ordered phase, Q_0 remains nonzero as $H \to 0$, which implies the asymptotic form (1.15) for f_-. Finally, requiring that $Q_0(t,H)$ remain finite and nonzero as $t \to 0$ for fixed H implies the final asymptotic form in Eq. (1.15) for $f_\pm(x)$ in the critical ($x \to \infty$) regime. All the other results Eqs. (1.15)–(1.18) quoted for the magnetic susceptibility now follow straightforwardly from the scaling law Eq. (1.13) and the asymptotic expression in Eq. (1.15) for f_+.

VI. FRÉEDERICKSZ INSTABILITY

Regarding the Fréedericksz instability, we need only derive the scaling form (1.20) for K_i and the asymptotic forms (1.22). The former can be done in a manner precisely parallel to the argument given for Q_0. The difference is that the Frank constants have renormalization-group eigenvalue 1, as can be seen by requiring that the zero-field Josephson relation $K_i \propto |t|^\nu$ holds.

The first asymptotic form in Eq. (1.22) then follows from the standard result $H_F \propto L^{-1}$, which holds when nematic order is well developed. The second expression in Eq. (1.22) follows by requiring that H_F remain finite and nonzero in the critical regime. In the disordered phase, boundary effects surely fall off exponentially with the distance from the boundary for distances large compared to the correlation length. This gives the last (exponential) form in Eq. (1.22).

VII. MAGNETIC SUSCEPTIBILITY NEAR THE *I-T* TRANSITION

Finally, we consider the magnetic susceptibility and *depolarized* light scattering near the *I-T* transition. All of the manipulations up to Eq. (5.5) are the same here as near the *N-T* transition. The differences all arise in the scaling behavior of $Q_0(t,H)$. This can be obtained by noting that, since there is no *spontaneous* nematic ordering anywhere near the *I-T* transition (except at the multicritical point), the nematic director will necessarily be along the applied field $\mathbf{H}$. (Without loss of generality, we will consider $\mathbf{H}$ along the z direction.) In that case,

$$\langle S_i^\alpha S_i^\beta \rangle = Q_0(\delta_{\alpha z}\delta_{\beta z} - \delta_{\alpha\beta}/3) + \delta_{\alpha\beta}/3, \qquad (7.1)$$

whence

$$Q_0 = \frac{3}{2}[\langle (S_i^z)^2 \rangle - 1/3]. \qquad (7.2)$$

Since the magnetic Hamiltonian can be rewritten

$$\mathcal{H}_{\mathrm{mag}} = \Delta\chi_0 H^2 \sum_i [(S_i^z)^2 - 1/3] \qquad (7.3)$$

by adding a trivial constant, it follows that

$$Q_0 = -\frac{3}{2N}\frac{\partial F}{\partial(h^2)}, \qquad (7.4)$$

where N is the number of sites in the system and $h = H\sqrt{\Delta\chi_0/k_B T}$.

Using (7.4), we can obtain Q_0 by familiar manipulations of the partition function. When $\mathcal{H}_{\mathrm{mag}}$ is added to the original gauge theory Hamiltonian, the resulting partition function is

$$Z = \int \prod_i d\Omega_i \sum_{\{U\}} \exp\Big\{ K\sum U(P) + J\sum U_{ij}\mathbf{S}_i\cdot\mathbf{S}_j$$

$$+h^2\sum_i[(S_i^z)^2 - 1/3]\Big\}. \qquad (7.5)$$

As before, we perform a "polymer" expansion of the terms $e^{(JU_{ij}\mathbf{S}_i\cdot\mathbf{S}_j)}$ in the partition function. Extracting a factor

$$Z_D = \Big\langle \exp\Big(h^2\sum_i[(S_i^z)^2 - 1/3]\Big)\Big\rangle_S, \qquad (7.6)$$

which represents completely decoupled spins ($\langle\rangle_S$ indicates normalized expectation with completely uniform spin distribution on the unit sphere), we rewrite the partition function as

$$Z = Z_D Z_G \sum_{\{C_n\}}' \Big\langle\prod_m W(C_m)\Big\rangle_G$$

$$\times \prod_m \frac{\Big\langle \prod_{(ij)\in C_m}(J\mathbf{S}_i\cdot\mathbf{S}_j)\prod e^{h^2[(S_i^z)^2-1/3]}\Big\rangle_S}{\Big\langle \prod_{(ij)\in C_m}\prod e^{h^2[(S_i^z)^2-1/3]}\Big\rangle_S}. \qquad (7.7)$$

This is a sum over disconnected polymers C_n, with factors of $e^{h^2[(S_i^z)^2-1/3]}$ decorating the polymers; the same factors for sites not on polymers have been absorbed into the Z_D out front, which requires the denominator to avoid double counting. Z_D is clearly a smooth function of H. Near the I-T transition the singular contributions to Q_0 come from the fluctuations of the U_{ij}'s. For small J, it is safe to expand in J. The calculation of the effective gauge coupling is similar to that in paper I (Sec. VII D) and involves resumming the leading-order contributions, i.e., the polymers consisting of single disconnected plaquettes. The result is

$$K_{\mathrm{eff}} = K + J^4\frac{\Big\langle\prod(\mathbf{S}_i\cdot\mathbf{S}_j)\prod e^{h^2[(S_i^z)^2-\frac{1}{3}]}\Big\rangle_S}{\Big\langle\prod e^{h^2[(S_i^z)^2-\frac{1}{3}]}\Big\rangle_S}$$

$$+O(J^6). \qquad (7.8)$$

To calculate the lowest-order contributions in H and J from Eq. (7.8), note that the denominator can be replaced by unity. In the numerator, the product of $\mathbf{S}_i\cdot\mathbf{S}_j$ terms depends only upon the *relative* orientation of the spins in the polymer. For the order H^2 term, the absolute orientation of only one spin matters. Integrating over the orientation of this spin first, with relative orientations fixed, and then over the relative orientations, the result is zero since $\langle(S_i^z)^2\rangle_S = 1/3$. Thus [10] the expansion starts at order H^4.

The coefficient of H^4 is easily calculated to lowest order in J. Equation (7.8) becomes

$$K_{\mathrm{eff}} = K + \frac{176}{6075}J^4 H^4 + \cdots. \qquad (7.9)$$

The most important consequence of this analysis is that the generic singularity of the magnetic susceptibility near the I-T transition has the $1-\alpha_I$ singularity reported in Eq. (1.25) of result 9.

All of the critical behavior of Q_0 can be obtained from the field dependence of K_{eff} via Eq. (7.4), which gives

$$Q_0 = -\frac{3}{2N}\frac{\partial F}{\partial K_{\mathrm{eff}}}\frac{\partial K_{\mathrm{eff}}}{\partial(h^2)}$$

$$= -\frac{3h^2}{N}\Big(\frac{k_B T}{\Delta\chi_0}\Big)^2 C(J)\frac{\partial F}{\partial K_{\mathrm{eff}}}$$

$$= (a+b|t|^{1-\alpha})H^2, \qquad (7.10)$$

where in the last equality we have inserted the known singularity of F at the (zero-field) I-T transition, as well as the fact that $\partial F/\partial K_{\mathrm{eff}} \propto \partial F/\partial T$. As a bonus, (7.10) also tells us the field dependence of the transition temperature T_{IT}

$$T_{IT}(H) = T_{IT}(0) + \mathrm{const}\times H^4. \qquad (7.11)$$

VIII. DEPOLARIZED SCATTERING NEAR THE I-T TRANSITION

This hard-won knowledge of the singular behavior of Q_0 near T_{IT} enables us also to determine the critical behavior of the depolarized light scattering $I_{xz}(\mathbf{q},t)$ near T_{IT} (result 3). As shown earlier, its Fourier transform $I_{xz}(\mathbf{r},t)$ obeys

$$I_{ij}(\mathbf{q}) \propto \langle Q_{ij}(\mathbf{r})Q_{ij}(\mathbf{0})\rangle \equiv C_{ij}(\mathbf{r}). \qquad (8.1)$$

We can now obtain the behavior of $C_{ij}(\mathbf{r})$ from scaling

$$C_{ij}(\mathbf{r},t) = b^{2X_Q}C_{ij}(b^{-1}r,b^{1/\nu}t), \qquad (8.2)$$

where X_Q is the renormalization-group eigenvalue of Q_{ij} and b the rescaling factor. We can extract X_Q from the results just obtained for the singular part of the nonlinear susceptibility. Scaling implies

$$Q_0^{\mathrm{sing}}(t,h) = b^{X_Q}Q_0^{\mathrm{sing}}(b^{1/\nu}t,b^{\lambda_h}h), \qquad (8.3)$$

where Q_0^{sing} is the singular part of the magnitude of

the nematic order and λ_h is the (as yet unknown) renormalization-group eigenvalue of H at the Ising gauge critical point. Choosing $b = |t|^{-\nu}$, we obtain

$$Q_0^{\mathrm{sing}}(t, h) = |t|^{-\nu X_Q} f^{ITD}\left(\frac{h}{|t|^{\nu\lambda_h}}\right), \qquad (8.4)$$

where

$$f^{ITD}(x) \equiv Q_0^{\mathrm{sing}}(1, x). \qquad (8.5)$$

For this to be consistent with our earlier result (7.10) for Q_0, we must have $f(x) \propto x^2$ at small x and also $|t|^{-\nu(X_Q + 2\lambda_h)} = |t|^{1-\alpha}$, which implies

$$X_Q + 2\lambda_h = \frac{\alpha - 1}{\nu}. \qquad (8.6)$$

To obtain X_Q, we still need λ_h. But since h^2 couples directly to Q_0 (i.e., to $S_z^2 - 1/3$, whose average is Q_0), we know from scaling that $2\lambda_h = d + X_Q$. Making this substitution, we find (with d set to 3)

$$X_Q = \frac{1}{2}\left(\frac{\alpha - 1}{\nu} - 3\right). \qquad (8.7)$$

Using this eigenvalue in (8.2) and choosing $b = |t|^{-\nu}$, we obtain the scaling form

$$C_{ij}(r, t) = |t|^{3\nu + 1 - \alpha} f_{ij}(r/\xi), \qquad (8.8)$$

where $\xi \propto |t|^{-\nu}$ is the Ising transition correlation length. As the critical point is approached ($t \to 0$), $C_{ij}(r)$ remains finite but nonzero at fixed r, so that the function f_{ij} must scale in such a way as to offset the t dependence of the prefactor in (8.8). This gives the result

$$C_{ij}(r, t = 0) \propto r^{\frac{\alpha - 1}{\nu} - 3}, \qquad (8.9)$$

which, when inserted back into (8.1), implies

$$I_{xz}(q, t = 0) \propto \int \frac{d^3r}{r^{3 - \frac{\alpha - 1}{\nu}}} e^{i\mathbf{q}\cdot\mathbf{r}} = a_2 + b_2|\mathbf{q}|^{\frac{1-\alpha}{\nu}}, \quad (8.10)$$

where a_2 and b_2 are constants and the exponent $(1-\alpha)/\nu$ has the numerical value 1.44. Note that the scattering remains finite at $|\mathbf{q}| = 0$ and lacks even a cusp since the exponent q is greater than one; two derivatives are needed to get a divergence ($\partial^2 I_{xz}/\partial q^2 \sim |\mathbf{q}|^{-0.56}$).

Finally, we argue that a lack of microscopic head-tail symmetry will not alter these results near the I-T transition. It suffices to show that the leading-order contribution to K_{eff} remains $O(H^4)$, even in that case. The H^4 term is nonvanishing [even a system lacking head-tail symmetry will still have a $\Delta\chi_0 \sum (\mathbf{H} \cdot \mathbf{S}_i)^2$ term in its magnetic energy], and we need to show only that lower-order terms continue to vanish. This is an immediate consequence of the absence of nematic order, even in the presence of such a microscopic symmetry breaking. No analytic scalar (rotationally invariant) function can be made that is linear or cubic in $\mathbf{H}$ and the $\mathbf{H}^2$ term vanishes by Eq. (7.4).

IX. CONCLUSION

In this paper, we have analyzed critical behavior of a variety of quantities at the transitions into the new, topologically ordered phase of nematic liquid crystals predicted in our gauge model [1,11]. Characteristic signatures were found for several standard experimental probes of soft condensed-matter, notably specific heat, and optical and magnetic response. We hope that these behaviors will be sought in the laboratory in the near future.

APPENDIX A

In this appendix we calculate the scaling dimension of $Q_0(T)$. Writing the spin as $\mathbf{S} = M(t)\hat{\mathbf{z}} + \delta\mathbf{S}$, M being the magnetization arbitrarily assigned to point along $\hat{\mathbf{z}}$,

$$\langle Q_{ij}\rangle = M^2(t)\delta_{iz}\delta_{jz} + \langle \delta S_i \delta S_j - \delta_{ij}/3\rangle. \qquad (A1)$$

The second term is the expectation value of the connected piece of the operator $\tau = S_i S_j - (1/3)\delta_{ij}$. This expectation vanishes as t^{β_τ} as $t \to 0$. By general renormalization-group theory [5]

$$\beta_\tau = \nu(d - y_\tau), \qquad (A2)$$

where y_τ is the renormalization-group eigenvalue of the operator τ at the relevant (i.e., $n = d = 3$ Wilson-Fisher) fixed point.

Thus the second term in (A1) is asymptotically dominated by the first as $t \to 0$ provided $2\beta = \nu(d+\eta-2) < \beta_\tau$ or, equivalently, if and only if

$$2\beta - \beta_\tau = \nu(y_\tau + \eta - 2) < 0. \qquad (A3)$$

The ϵ expansion for y_τ can be found in Ma's book (Ref. [5], p. 355) and is

$$y_\tau = 2\left(1 - \frac{\epsilon}{n+8} + \frac{n^2 - 18n - 88}{4(n+8)^3}\epsilon^2\right) + O(\epsilon^3), \quad (A4)$$

which, combined with $\eta = \frac{n+2}{2(n+8)}\epsilon^2 + O(\epsilon^3)$ at $n = 3$, gives

$$2\beta - \beta_\tau = \nu\left(-\frac{2\epsilon}{11} - \frac{23\epsilon^2}{5724}\right) + O(\epsilon^3), \qquad (A5)$$

which is indeed negative, as desired. The expectation (A1) is dominated by the connected piece and scales with an exponent 2β.

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[4] We are concerned here with *single* scattering. There are also both polarized and depolarized contributions from *multiple* scattering, even for a simple isotropic fluid [see W. H. Gelbart, Adv. Chem. Phys. **26**, 1, (1974)]. The magnitudes of these effects can be strongly dependent on experimental geometry: the sizes of the sample and the volumes illuminated by the source and visible to the detector. In the case where the density fluctuations are relatively small in magnitude, multiple scattering will be highly suppressed.

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[9] It might appear that in systems *without* microscopic $S \rightarrow -S$ symmetry, one could add a term linear in H, namely, $m_0 \sum_i H \cdot S_i$, where m_0 in a constant permanent molecular dipole moment, which would dominate the quadratic (in H) term (5.1). For this to make sense, we have to rewrite it in a gauge invariant form just as in paper I with the likewise gauge-noninvariant $S_i \cdot S_j$ perturbation. To wit, we introduce additional variables $\sigma_i = \pm 1$ on the sites that keep track of whether S_i points along or opposite the local dipole and transform like S under a gauge transformation so that the combination $\sigma_i S_i$ is *invariant*. Then the perturbation becomes $m_0 \sum_i (\sigma_i H) \cdot S_i$. Since σ's on different sites are decoupled from each other, it is trivial to sum over them, yielding a magnetic perturbation of $\mathcal{H}_{mag}^{eff} = \sum_i \ln [\cosh(m_0 H \cdot S_i)]$, an explicitly even function of each S_i. Expanding to lowest order, we recover the earlier expression (5.1) with $\Delta \chi_0 = \frac{1}{2} m_0^2$.

[10] The absence of a quadratic term is quite clear from a more general perspective. From Eq. (7.10), we see that if K_{eff} did contain an $O(h^2)$ correction, Q_0 would not vanish at $H = 0$, i.e., the system would have spontaneous nematic order. Since such order does not exist in the vicinity of the *I-T* transition, that term must disappear.

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