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Colored-noise problem: A Markovian interpolation procedure

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We analyze the colored-noise problem from the point of view of consistent Markovian approximations. We extend the "unified colored-noise approximation" of P. Hänggi *et al.* through its interpretation as an interpolation procedure between the zero correlation time limit (white noise) and the infinite correlation time limit. We consider other interpolating functions, obtaining stationary probability distributions and the mean first passage time (associated with the lowest nonzero eigenvalue) and compare with exact numerical results. The potential of this scheme to represent adequately the results of the colored-noise problem through a convenient choice of the interpolating function is also discussed.

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I. INTRODUCTION

The study of dynamical systems perturbed by noise is recurrent in many contexts of physics and other sciences. In the theory of nonequilibrium systems especially, where the macrovariables obey some nonlinear equations of motion, noise plays an important role. In fact, the system can overcome potential barriers and reach different macrostates only because of the presence of noise.

In particular, in the context of more realistic models of physical systems, the consideration of noise sources with finite correlation time (i.e., colored noise) has become a subject of current study. For example, in describing the static and dynamical properties of dye lasers it is usual to model the phenomena in terms of stochastic equations, where, besides the standard internal white noise, the system is driven by an external colored-noise [1]. In a different context, the effect of time correlations in the fluctuations has also been considered in models of gene selection [2]. This interest, together with the absence of exact analytical results, has stressed the need of analyzing from new points of view the colored-noise problem. Some recent papers and reviews [3–6] offer a view of the state of the art.

Some authors have focused their efforts on the obtention of Markovian approximations, trying to capture the essential features of the original non-Markovian problem. One particular case is the "unified colored-noise approximation" (UCNA) of Hänggi and collaborators [7]. The aim of this approximation can be understood in the following way. The original formulation of the problem is in terms of a non-Markovian stochastic differential equation in the relevant variable. However, this problem can be transformed into a Markovian one by extending the number of variables (and equations). The UCNA consists of an adiabatic elimination procedure that allows us to reduce this extended problem to an "effective" Markovian one in the original variable space. The ultimate goal of these procedures is the obtention of a consistent single variable Fokker-Planck approximation for the probability distribution of the original variable. The UCNA approximation has been justified as a reliable Markovian approximation by means of path integral techniques [8].

In this paper, we present an extension of the UCNA, through its interpretation as an interpolation procedure between the white-noise limit and the infinite correlation time limit. Such an interpretation can be more easily seen using a path integral description of the problem. The advantages of this procedure consist in the possibility of devising the interpolating function that best fits a particular set of experimental data, and in this way accurately predicting other relevant functions.

The paper is organized as follows: in Sec. II we introduce our interpolation scheme; in Sec. III we present results of the stationary probability distribution and mean first passage time for a bistable potential, obtained with a particular family of interpolating functions; and finally in Sec. IV we draw our conclusions.

II. INTERPOLATING SCHEME

In the simplest form, the problem of colored noise can be introduced considering a relevant macrovariable $q(t)$ that satisfies a stochastic differential equation of the form

$$\dot{q}(t) = f[q(t)] + \epsilon(t), \tag{2.1}$$

where $f[q(t)]$ represents a deterministic force and $\epsilon(t)$ indicates the noise. For the particular case of the Ornstein-Uhlenbeck process, the noise is Gaussian with zero mean, and correlation

$$\langle \epsilon(t)\epsilon(t')\rangle = \frac{D}{\tau}\exp\left(-\frac{\mid t-t'\mid}{\tau}\right), \tag{2.2}$$

where D denotes the noise intensity, and τ is the correlation time.

It is possible to rewrite Eq. (2.1) as a pair of equations with a white-noise term acting on one of them,

$$\dot{q}(t) = f[q(t)] + \epsilon(t), \tag{2.3}$$

and

$$\dot{\epsilon}(t) = -\epsilon(t)\tau^{-1} + D^{-1/2}\tau^{-1}\xi(t), \tag{2.4}$$

where $\xi(t)$ is the white-noise source $[\langle\xi(t)\rangle = 0, \langle\xi(t)\xi(t')\rangle = 2\delta(t-t')]$.

The UCNA results are obtained by differentiating Eq. (2.3) and replacing in it by Eq. (2.4), setting $\ddot{q} = 0$ through an adiabatic elimination scheme, and making a scaling of the time variable according to $\underline{t} = t\tau^{-1/2}$. The result is a multiplicative Markovian process described by

$$\dot{q}(t) = f(q)\gamma(q,\tau)^{-1} + D^{1/2}\tau^{-1/4}\gamma(q,\tau)^{-1}\Gamma(\underline{t}), \tag{2.5}$$

where $\gamma(q,\tau) = [1 - \tau f'(q)]\tau^{-1/2}$, and $\Gamma(\underline{t})$ is a white noise (the prime denotes differentiation with respect to q).

The Fokker-Planck equation (FPE) associated with the Langevin-like equation (2.5) has the form

$$\frac{\partial P(q,\tau,\underline{t})}{\partial \underline{t}} = -\frac{\partial}{\partial q}\{[f(q)\gamma(q,\tau)^{-1} - D\tau^{-1/2}\gamma'(q,\tau)\gamma(q,\tau)^{-3}]P(q,\tau,\underline{t})\} + D\frac{\partial^2}{\partial q^2}\{[\gamma(q,\tau)^{-2}\tau^{-1/2}]P(q,\tau,\underline{t})\}. \tag{2.6}$$

The results indicated in Eqs. (2.5) and (2.6) configure the UCNA [7].

We now return to Eqs. (2.3) and (2.4). It is possible to find the exact behavior for this equation in two limits: $\tau \to 0$ (white noise) and $\tau \to \infty$. The results for each case are the following.

(i) $\tau \to 0$: the equation reduces to

$$\dot{q}(t) = f[q(t)] + D^{1/2}\xi(t), \tag{2.7}$$

with the associated FPE

$$\frac{\partial[P_0(q,t)]}{\partial t} = -\frac{\partial}{\partial q}\{f[q(t)]P_0(q,t)\} + D\frac{\partial^2}{\partial q^2}[P_0(q,t)]. \tag{2.8}$$

(ii) $\tau \to \infty$: following a procedure similar to the one used for the UCNA we get

$$\dot{q}(t) = -f[q(t)]\{\tau f'[q(t)]\}^{-1} - D^{1/2}\{\tau f'[q(t)]\}^{-1}\xi(t), \tag{2.9}$$

and the associated FPE

$$\frac{\partial P_\infty(q,t)}{\partial t} = -\frac{\partial}{\partial q}(\{-f(q)[\tau f'(q)]^{-1} - D[\tau^2 f'^3(q)]^{-1}f''(q)\}P_\infty(q,t)) + D\frac{\partial^2}{\partial q^2}\{[\tau f'(q)]^{-2}P_\infty(q,t)\}. \tag{2.10}$$

From a *path-integral* point of view [9], the *Lagrangians* associated with each of the Langevin equations [(2.7) or (2.9)] or FPE's [(2.8) or (2.10)] are the following.

(ii) $\tau \to 0$,

$$\mathcal{L}_0(q,\dot{q}) = \frac{1}{4D}[\dot{q} - f(q)]^2 + \frac{1}{2}f'(q). \tag{2.11}$$

(ii) $\tau \to \infty$,

$$\mathcal{L}_\infty(q,\dot{q}) = \frac{1}{4D}[\tau f'(q)\dot{q} + f(q)]^2 - \frac{1}{2\tau}. \tag{2.12}$$

Similarly, the Lagrangian associated with the UCNA [Eqs. (2.5) and (2.6)] has the form

$$\mathcal{L}_{UCNA}(q,\dot{q}) = \frac{1}{4D}\{[1 - \tau f'(q)]\dot{q} - f(q)\}^2 + \frac{1}{2}[1 - \tau f'(q)]^{-1}f'(q). \tag{2.13}$$

Considering simultaneously (2.11) and (2.12), it is clear that, if we have a function $\theta[\tau f'(q)]$ fulfilling the limit conditions

$$\lim_{\tau\to 0}\theta[\tau f'(q)] = 1 \tag{2.14}$$

and

$$\lim_{\tau\to\infty}\theta[\tau f'(q)] = -[\tau f'(q)]^{-1}, \tag{2.15}$$

we could define an *interpolating Lagrangian* according to

$$\mathcal{L}_I(q,\dot{q}) = \frac{1}{4D}\left\{\frac{\dot{q}}{\theta[\tau f'(q)]} - f(q)\right\}^2 + \frac{1}{2}\theta[\tau f'(q)]f'(q). \tag{2.16}$$

This Lagrangian $\mathcal{L}_I$ in the limits $\tau \to 0$ or $\tau \to \infty$ coincides with $\mathcal{L}_0(q,\dot{q})$ or $\mathcal{L}_\infty(q,\dot{q})$, respectively. The corresponding FPE is

$$\frac{\partial[P_I(q,t)]}{\partial t} = -\frac{\partial}{\partial q}(\{f(q)\theta[\tau f'(q)] + D\theta[\tau f'(q)]\theta'[\tau f'(q)]\}P_I(q,t)) + D\frac{\partial^2}{\partial q^2}\{\theta^2[\tau f'(q)]P_I(q,t)\}. \tag{2.17}$$

According to this discussion, the UCNA can be interpreted as an interpolation in the sense indicated by (2.16) and (2.17), where

$$\theta[\tau f'(q)] = [1 - \tau f'(q)]^{-1}. \tag{2.18}$$

This alternative point of view introduces new possibilities for finding better Markovian approximations to the colored-noise problem. This can be achieved through the definition of an interpolating function that is different from that of the UCNA, one that is better suited to describe the dynamics of the system.

In the next section we present an example of a particular family of interpolating functions that, in some limit, reduces to the UCNA.

III. STATIONARY DISTRIBUTIONS AND FIRST PASSAGE TIME

In order to analyze the possibilities of the interpolation scheme, we will consider the problem of diffusion in a bistable potential driven by colored noise. We choose the very well known symmetric potential given by the expression

$$V(q) = -\frac{1}{2}aq^2 + \frac{1}{4}bq^4. \tag{3.1}$$

We introduce the dimensionless variables $q \to (bq^2/a)^{1/2}$, $\epsilon \to (b\epsilon^2/a^3)^{1/2}$, $t \to at$, $D \to bD/a^2$; and consider $a = 1$, $b = 1$.

Remembering that the deterministic force is

$$f(q) = -\frac{dV(q)}{dq}, \tag{3.2}$$

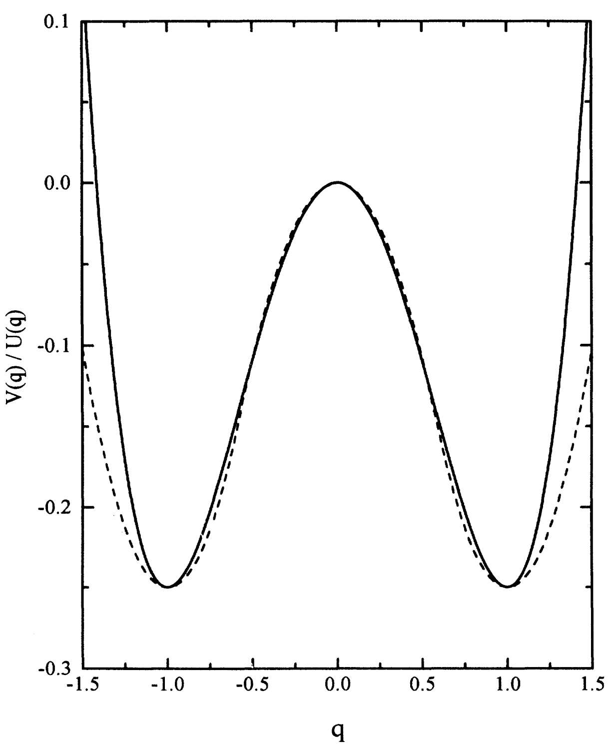

FIG. 1. Bistable symmetric potential. Exact Eq. (3.1) (solid line), approximate Eq. (3.5) (dotted line).

Eq. (2.1) adopts the form

$$\dot{q} = q - q^3 + \epsilon(t). \tag{3.3}$$

We consider the family of interpolating functions given by

$$\theta[\tau f'(q)] = \frac{1 - c[\tau f'(q)]^{n-1}}{1 + c[\tau f'(q)]^n}, \tag{3.4}$$

which fulfills the requirements imposed on the limits $\tau \to 0$ and $\tau \to \infty$. In the case where $c = -1$ and $n = 2$, it reduces to the UCNA function [Eq. (2.18)]. In our example, we will only consider $n = 2$ and different values of c ranging from -1 to 1.

In order to exploit the numerical procedure introduced in Refs. [10–12], it is convenient to simplify the form of the potential, making an approximation in order to calculate the value of the interpolating function. We will consider a second order approximation of the original potential (3.1) (see Fig. 1):

$$U(q) = \begin{cases} \frac{1}{4}\left[\frac{\sqrt{3}}{\sqrt{3}-1}(q+1)^2 - 1\right], & q \le -\frac{1}{\sqrt{3}} \\ -\frac{1}{4}\left[\sqrt{3}q^2\right], & -\frac{1}{\sqrt{3}} < q < \frac{1}{\sqrt{3}} \\ \frac{1}{4}\left[\frac{\sqrt{3}}{\sqrt{3}-1}(q-1)^2 - 1\right], & q \ge \frac{1}{\sqrt{3}}. \end{cases} \tag{3.5}$$

With this approximation, the interpolating function becomes

$$\theta(\tau) = \begin{cases} \theta_1(\tau), & \mid q \mid \ge \frac{1}{\sqrt{3}} \\ \theta_2(\tau), & \mid q \mid < \frac{1}{\sqrt{3}} \end{cases} \tag{3.6}$$

with

$$\theta_1(\tau) = \frac{1 - c\left[\frac{\sqrt{3}\tau}{2(1-\sqrt{3})}\right]}{1 + c\left[\frac{\sqrt{3}\tau}{2(1-\sqrt{3})}\right]^2} \tag{3.7}$$

and

$$\theta_2(\tau) = \frac{1 - c\left[\frac{\sqrt{3}\tau}{2}\right]}{1 + c\left[\frac{\sqrt{3}\tau}{2}\right]^2}. \tag{3.8}$$

In the following, we present the results obtained for the stationary probability distribution (SPD), and the mean first passage time (MFPT).

No exact expression of the stationary distribution is known so far in the colored-noise problem. However, with the interpolation scheme, an approximation can be obtained. In fact, the SPD can easily be found for the FPE (2.17):

$$P_{st}(q,\tau) = \frac{N}{\theta[\tau f'(q)]} \exp\left[\frac{1}{D}\int^q \frac{f(\zeta)}{\theta[\tau f'(\zeta)]} d\zeta\right]. \tag{3.9}$$

With the interpolating function [Eq. (3.6)] and integrating, we obtain

$$P_{st}(q,\tau) = \begin{cases} \frac{N}{2D\theta_2^2(\tau)} \exp\left[-\frac{V(q)}{D\theta_1(\tau)} + \frac{5}{36D}\left(\frac{1}{\theta_2} - \frac{1}{\theta_1}\right)\right], & \mid q \mid \geq \frac{1}{\sqrt{3}} \\ \frac{N}{2D\theta_2^2(\tau)} \exp\left[-\frac{V(q)}{D\theta_2(\tau)}\right], & \mid q \mid < \frac{1}{\sqrt{3}} \end{cases} \tag{3.10}$$

where N is a normalization constant.

Calculations of the SPD for different noise intensities (D), correlation times (τ), and different values of c have been done, and compared with the exact numerical results of Ref. [13] and with the UCNA results. In each case, we find good agreement between our results and the numerical ones. As an example, the results of the SPD for the particular $D = 0.1$, $\tau = 0.99$, and $c = 0.44$ are shown in Fig. 2.

As in the case of the UCNA, the interpolation scheme fails to describe the stationary probability distribution in those spatial regions or those particular values of c or τ, which give a negative value of $\theta[\tau f'(q)]$, but this can be controlled with a suitable choice of c.

In Fig. 3, we plot for the same $c = 0.44$, the results of the interpolation procedure for different values of τ. It is possible to see that the probability in the barrier zone diminishes as expected when the correlation time is increased.

We now proceed with the mean first passage time. The exact expression for the MFPT for a particle to reach the final point x_1, when it started at the initial point x_0 with constant diffusion coefficient D_0, is (Ref. [14])

$$T(x_0 \to x_1, \tau) = \int_{x_0}^{x_1} \frac{dy}{D_0 P_{st}(y)} \int_{-\infty}^{y} P_{st}(z) dz \ . \tag{3.11}$$

In particular, we choose $x_0 = -1$ (the left well) as the initial point, and $x_1 = 0$ as the final point (where the separatrix for small to moderate values of $\frac{D}{\tau}$ is situated).

We have done calculations based on two different methods. On the one hand, we have obtained an approximate value of the MFPT by applying a steepest-descent approximation. On the other hand, we have obtained the corresponding MFPT values by doing numerical calculations Refs. [10–12].

The steepest-descent result, taking into account (3.10) and (3.11), and making a change of variables to have the constant diffusion coefficient (Ref. [15]), is

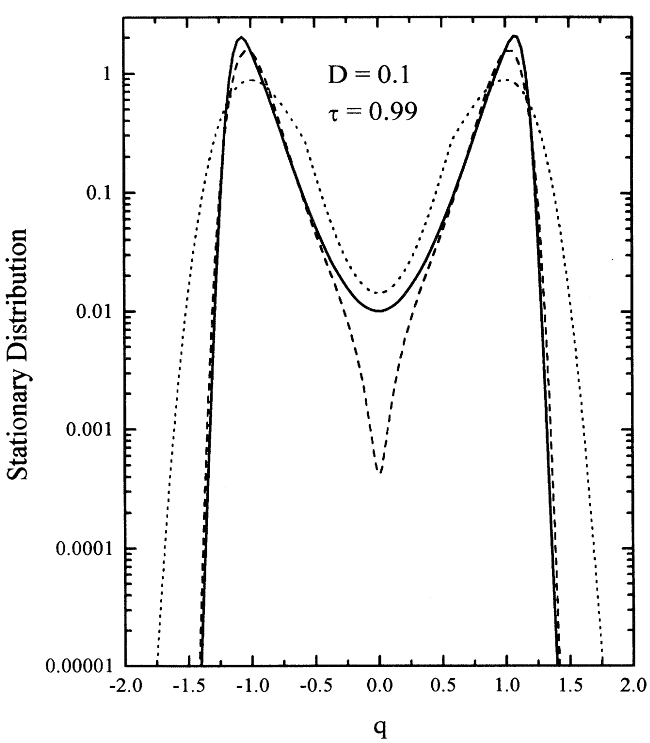

FIG. 2. Stationary distribution for $D = 0.1$ and $\tau = 0.99$. Exact numerical results of Ref. [13] (solid line); UCNA prediction, Ref. [7] (dashed line); interpolation procedure with $c = 0.44$ (dotted line).

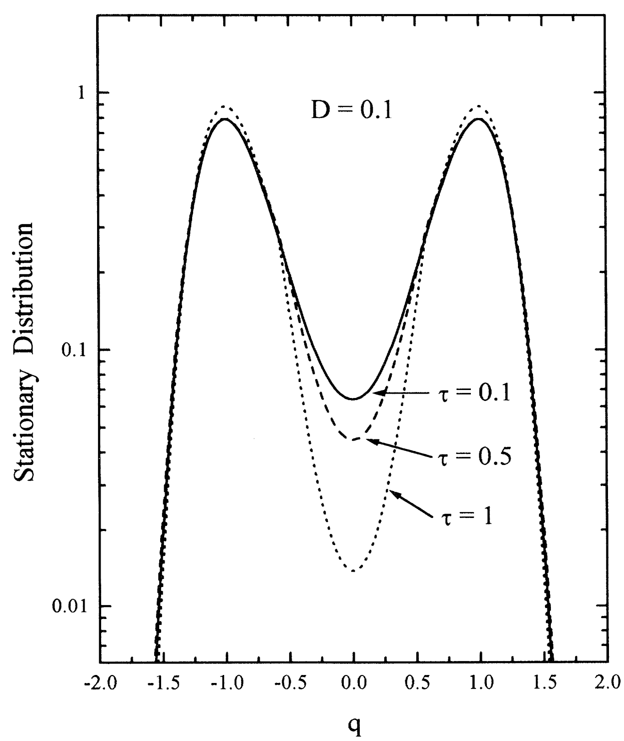

FIG. 3. Stationary distribution for $D = 0.1$ and different values of τ obtained with the interpolation procedure with $c = 0.44$.

$$T(-1 \to 0,\tau) = \pi \left[\frac{\theta_1(\tau)}{(3-\mid\theta_1(\tau)\mid)}\right]^{1/2} \exp\left[\frac{\theta_1(\tau)}{4D} + \frac{5}{36D}\left(\frac{1}{\theta_2(\tau)} - \frac{1}{\theta_1(\tau)}\right)\right] . \tag{3.12}$$

This result, together with our own numerical calculations is compared with the exact numerical results of Ref. [13], for different values of D, and the same values of c obtained for the SPD, in Fig. 4, where we plot the lowest nonzero eigenvalue λ ($\lambda = \frac{1}{T}$). A more detailed graph of the particular case $D = 0.1$ is presented in Fig. 5, where the approximations of Refs. [16,17] are also plotted. All these results show, in general, a very good agreement with the known exact numerical calculations, indicating that the present interpolation scheme offers promising perspectives in this problem.

IV. DISCUSSION

In this paper, we have presented a different point of view in the colored-noise problem. We have introduced an extension of the UCNA scheme through its interpretation as an interpolation procedure between the white-noise and the infinite correlation time limits. We have particularly studied the relevant case of diffusion in a bistable symmetric potential. Among all the possible interpolation functions, we have considered a family of such functions that, for certain values of the parameters reduces to the UCNA function [see Eq. (3.4)].

Our main goal has been to show that this interpretation leads to consistent Markovian approximations to the colored-noise problem, curing some of the defects of the UCNA on describing the dynamics of the process. We have focused on two aspects: the stationary probability distribution and the mean first passage time.

In order to simplify the numerical evaluation, we have approximated the original quartic potential with a piecewise parabolic one. The calculations of stationary distributions show good agreement with the exact numerical results [13], and particularly give a better description of the barrier zone (Fig. 2). The behavior of the stationary distribution is also good when considering variations of the correlation time while keeping the value of D (Fig. 3) fixed. In both figures we see a discontinuity in the first

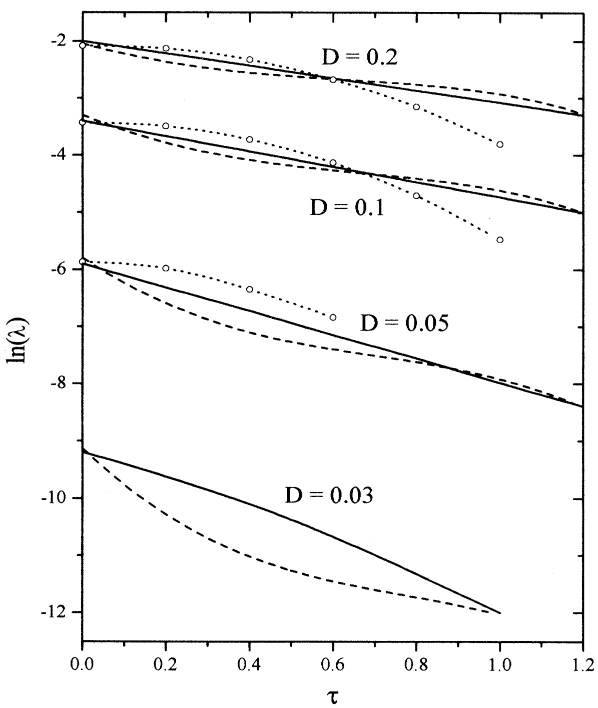

FIG. 4. Lowest nonzero eigenvalue of the Fokker-Planck equation for different values of D, as a function of τ. Exact numerical results, Ref. [13] (solid line); numerical calculations with the interpolation procedure (circles with dotted line); steepest-descent approximation with the interpolation scheme (dashed line). The values of c are $c = 0.52$ ($D = 0.2$), $c = 0.44$ ($D = 0.1$), $c = 0.385$ ($D = 0.05$), and $c = 0.34$ ($D = 0.03$).

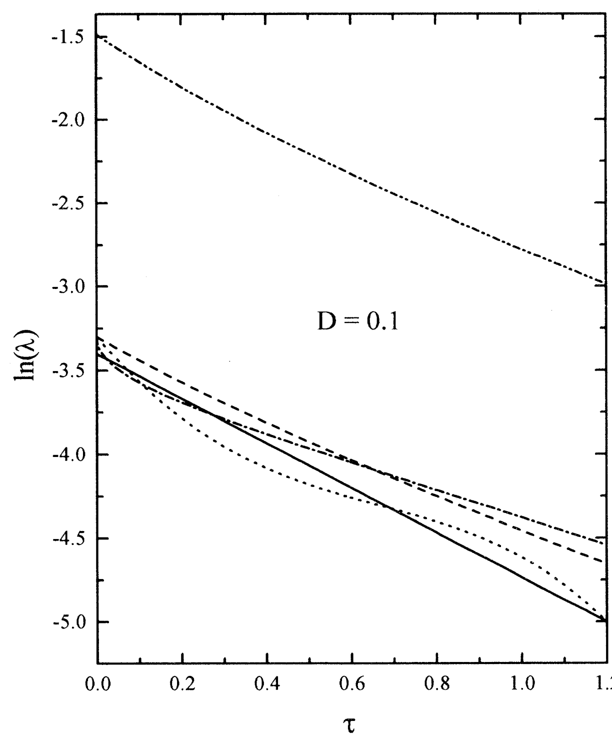

FIG. 5. Lowest nonzero eigenvalue of the Fokker-Planck equation for $D = 0.1$, as a function of τ. Exact numerical results, Ref. [13] (solid line); bridging formulas of Refs. [16] and [17] (dashed line and dashed-dotted line, respectively); UCNA prediction, Ref. [7] (dashed and double-dotted line); steepest-descent calculation within the interpolation scheme (dotted line).

TABLE I. Values of the parameter α for different noise intensities D. The interpolation values are compared with those of Ref. [13].

D	Ref. [13]	Interpolation
0.2	0.21	0.21
0.1	0.13	0.15
0.05	0.10	0.13
0.03	0.07	0.11

derivative of the stationary distribution at the matching points. This is not due to the interpolation procedure, but is a consequence of the second order approximation of the original potential, and can be avoided considering the exact potential.

The results for the mean first passage time also show very good agreement with the exact numerical results [13]. This is due to the improved description of the probability distribution in the barrier zone. Our method compares favorably with other approximations [16,17].

Concerning the lowest nonzero eigenvalue, we have also analyzed the value of the Arrhenius parameter α in $\lambda \propto \exp[-\alpha\tau]$ for $0.2 < \tau < 1.5$, as well as the parameter β in $\lambda(\tau) = \lambda(0)(1-\beta\tau)$ for $\tau \to 0$. The value of α approaches $\alpha = 0.1$ for weak noise and $\beta = 1.5$. With the interpolation procedure, the α values obtained with the steepest-descent approximation are in very good agreement with those of Ref. [13] (see Table I), while the β values are not.

The results we have presented here for the interpolation procedure indicate that this is a promising approach. So far, we have only considered a particular family of interpolating functions that in a certain case reduces to the UCNA. However, there are several other possibilities; for example,

$$\theta[z] = \frac{(1-cz)^{n-1}}{\left(1+c^{2(n-1)}z^{2n}\right)^{1/2}}, \tag{4.1}$$

with $z = \tau f'(q)$.

It is then possible to expect that, applying a certain measure of craftmanship when devising the function, one would be able to obtain much better results for the static (stationary probability distribution) or dynamic description (i.e., first passage time) of the colored-noise problem in terms of a consistent, satisfactory, Markovian approximation. The analysis of the above indicated possibilities [Eq. (4.1)] as well as others not indicated here will be the subject of further work.

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