Generalization of the Planck radiation law and application to the cosmic microwave background radiation

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Within the framework of the recently introduced nonextensive statistical mechanics, we generalize the Planck law for the blackbody radiation. The generalized thermal spectrum contains an additional parameter q, for which $q=1$ in the Planck limit. For the cosmic microwave background radiation, we find a 95% confidence limit of $|q-1|<3.6\times10^{-5}$ from the data of Mather *et al.* [Astrophys. J. **420**, 439 (1994)] that were obtained with the *Cosmic Background Explorer* satellite, under the assumption that the internal reference of the apparatus has a Planck spectrum.

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The need for *nonextensive* thermodynamics has been well known for decades in cosmology, gravitation, and astrophysics (e.g., black holes, superstrings, three-dimensional gravitational N-body problem, dark matter; see [1–3] and references therein); moreover, it is expected to be a common feature whenever the (linear) size of the system is smaller than or comparable to the range of the relevant interactions between the elements of the system. Typically, this situation occurs whenever we have *long-range* interactions (for instance, such that the associated attractive potential decreases with distance with a power law whose exponent is *smaller* than the dimensionality d of the system) [4–6]. Very recently, formal nonextensive thermodynamics and statistical mechanics have become available ([7,8]; see also [9–14]), which contain an extra parameter q, the entropy index. ($q=1$ reproduces the standard, extensive, thermodynamics.) This formalism has been successfully applied to Newtonian gravitation [15,16], Levy-type anomalous diffusion [17], correlated-type anomalous diffusion [18], and optimization techniques [19], among others. It is within this (generalized) framework that we intend to discuss the blackbody radiation. To be more specific, it is legitimate to think that the present cosmic background radiation might be (even with the common assumption of thermal equilibrium) slightly different from Planck's blackbody law due to long-range gravitational influence. This influence could be a small long-range memory of times when matter and light were still strongly coupled or it could be due to more complex phenomena. To check the validity of a theory (the Planck law, in our present case) we must [20], locate it in a more general framework, herein represented by the fact that q is allowed to differ from unity.

The formalism starts by postulating [7] a generalized form for the entropy, namely,

$$S_q \equiv k\frac{1-\sum_s p_s^q}{q-1} \quad (q\in\mathbb{R}) , \tag{1}$$

where $\{p_s\}$ are the probabilities of the microscopic states and k is a dimensional positive constant. In the $q\to1$ limit, S_q recovers the well known Shannon form $-k_B\sum_s p_s \ln p_s$, from which standard statistical mechanics and thermodynamics follow. Let us reproduce here a property *(pseudoadditivity)*, which is relevant in the present context. If Σ and Σ' are two *independent* systems (in the sense that $p_{s,s'}^{\Sigma\cup\Sigma'}=p_s^{\Sigma}p_{s'}^{\Sigma'}$) then

$$\frac{S_q^{\Sigma\cup\Sigma'}}{k}=\frac{S_q^{\Sigma}}{k}+\frac{S_q^{\Sigma'}}{k}+(1-q)\frac{S_q^{\Sigma}}{k}\frac{S_q^{\Sigma'}}{k} . \tag{2}$$

In other words, $(1-q)$ is a *measure of the lack of extensivity* of the system. For the canonical ensemble, Eq. (1) yields the following equilibrium density operator [7,8],

$$\hat{\rho}=[\hat{1}-(1-q)\beta\hat{\mathcal{H}}]^{1/(1-q)}/Z_q , \tag{3}$$

where $\beta\equiv1/kT$ is the Lagrange parameter associated with the thermostat, $\hat{\mathcal{H}}$ is the Hamiltonian, and the generalized partition function is given by

$$Z_q \equiv \mathrm{Tr}[\hat{1}-(1-q)\beta\hat{\mathcal{H}}]^{1/(1-q)} . \tag{4}$$

In the $q\to1$ limit we recover the Boltzmann-Gibbs distribution $\hat{\rho}\propto\exp(-\beta\hat{\mathcal{H}})$. It can be shown [8] that $\forall q$, $1/T=\partial S_q/\partial U_q$, $U_q\equiv\mathrm{Tr}\hat{\rho}^q\hat{\mathcal{H}}=-(\partial/\partial\beta)(Z_q^{1-q}-1)/(1-q)$, and $F_q\equiv U_q-TS_q=-(1/\beta)(Z_q^{1-q}-1)/(1-q)$. The relevant mean value associated with any observable $\hat{O}$ is given [6,8,10,11,14] by

$$\langle \hat{O}\rangle_q \equiv \mathrm{Tr}\hat{\rho}^q\hat{O}=\langle\hat{\rho}^{q-1}\hat{O}\rangle_1 . \tag{5}$$

In the $\beta(1-q)\to0$ limit (which we focus on from now on), Eq. (4) *asymptotically* becomes

$$Z_q = \mathrm{Tr}\exp\left\{\frac{1}{1-q}\ln[\hat{1}-(1-q)\beta\hat{\mathcal{H}}]\right\} \sim Z_{\mathrm{BG}}\{1-\tfrac{1}{2}(1-q)\beta^2\langle\hat{\mathcal{H}}^2\rangle_{\mathrm{BG}}\}\ , \tag{6}$$

where BG stands for Boltzmann-Gibbs and where we have retained only the $(1-q)$ correction to the leading term. Also, by using Eq. (3), we can consistently rewrite Eq. (5) as follows

$$\langle\hat{O}\rangle_q = Z_q^{1-q}\left\langle\frac{\hat{O}}{\hat{1}-(1-q)\beta\hat{\mathcal{H}}}\right\rangle_1 \sim Z_q^{1-q}\langle\hat{O}\rangle_{\mathrm{BG}}\left\{1+(1-q)\beta\left[\frac{\langle\hat{O}\hat{\mathcal{H}}\rangle_{\mathrm{BG}}}{\langle\hat{O}\rangle_{\mathrm{BG}}}+\frac{\beta}{2}\left[\langle\hat{\mathcal{H}}^2\rangle_{\mathrm{BG}}-\frac{\langle\hat{O}\hat{\mathcal{H}}^2\rangle_{\mathrm{BG}}}{\langle\hat{O}\rangle_{\mathrm{BG}}}\right]\right]\right\}$$
$$\sim Z_{\mathrm{BG}}^{1-q}\langle\hat{O}\rangle_{\mathrm{BG}}\left\{1+(1-q)\beta\left[\frac{\langle\hat{O}\hat{\mathcal{H}}\rangle_{\mathrm{BG}}}{\langle\hat{O}\rangle_{\mathrm{BG}}}+\frac{\beta}{2}\left[\langle\hat{\mathcal{H}}^2\rangle_{\mathrm{BG}}-\frac{\langle\hat{O}\hat{\mathcal{H}}^2\rangle_{\mathrm{BG}}}{\langle\hat{O}\rangle_{\mathrm{BG}}}\right]\right]\right\}\ , \tag{7}$$

where we used Eq. (6) in the last step.

Let us now consider the system $\hat{\mathcal{H}}=h\nu\hat{n}$, ν being the (photon) frequency, and $\hat{n}$ being the bosonic particle-number operator. By choosing $\hat{O}\equiv\hat{n}$, Eq. (7) becomes

$$\langle\hat{n}\rangle_q \sim \langle\hat{n}\rangle_{\mathrm{BG}}Z_{\mathrm{BG}}^{q-1}\left\{1+(1-q)x\left[\frac{\langle\hat{n}^2\rangle_{\mathrm{BG}}}{\langle\hat{n}\rangle_{\mathrm{BG}}}+x\left[\langle\hat{n}^2\rangle_{\mathrm{BG}}-\frac{\langle\hat{n}^3\rangle_{\mathrm{BG}}}{\langle\hat{n}\rangle_{\mathrm{BG}}}\right]\right]\right\}\ , \tag{8}$$

with $x\equiv\beta h\nu$, and

$$Z_{\mathrm{BG}}=\sum_{n=0}^{\infty}e^{-\beta x}=(1-e^{-x})^{-1}\ , \tag{9}$$

$$\langle\hat{n}\rangle_{\mathrm{BG}}=\frac{e^{-x}}{1-e^{-x}}=\frac{1}{e^x-1}\ , \tag{10}$$

$$\langle\hat{n}^2\rangle_{\mathrm{BG}}=\frac{e^{-x}+e^{-2x}}{(1-e^{-x})^2}\ , \tag{11}$$

$$\langle\hat{n}^3\rangle_{\mathrm{BG}}=\frac{e^{-x}+4e^{-2x}+e^{-3x}}{(1-e^{-x})^3}\ . \tag{12}$$

Let us now calculate the $q=1$ blackbody radiation law assuming a $d+1$ space-time and $(d-1)$ transverse modes for the electromagnetic field (with the spectrum given by $2\pi\nu=c|\mathbf{k}|$, where $\mathbf{k}$ is the d-dimensional wave vector). By following along the standard lines we obtain for the *photon energy density per nit volume*

$$D_1(\nu)=\frac{\pi^{d/2}(d-1)dh\nu^d}{\Gamma(d/2+1)c^d(e^{h\nu/k_BT}-1)}\ . \tag{13}$$

For the $3+1$ space-time this expression recovers Planck's law

$$D_1(\nu)=\frac{8\pi h\nu^3}{c^3(e^{h\nu/k_BT}-1)}\equiv D_{\mathrm{Planck}}(\nu)\ . \tag{14}$$

Although everything that follows could be done for a $d+1$ space-time, we shall from now on focus on the standard $d=3$ case. So, if we maintain the one-photon approach, Eq. (8) implies, for $q\simeq 1$,

$$D_q(\nu)\sim D_{\mathrm{Planck}}(\nu)(1-e^{-x})^{q-1}\left\{1+(1-q)x\left[\frac{1+e^{-x}}{1-e^{-x}}-\frac{x}{2}\frac{1+3e^{-x}}{(1-e^{-x})^2}\right]\right\}\ ; \tag{15}$$

hence

$$\frac{D_q(\nu)h^2c^3}{8\pi(k_BT)^3}\sim\frac{x^3}{e^x-1}(1-e^{-x})^{q-1}\left\{1+(1-q)x\left[\frac{1+e^{-x}}{1-e^{-x}}-\frac{x}{2}\frac{1+3e^{-x}}{(1-e^{-x})^2}\right]\right\}\ , \tag{16}$$

which generalizes *Planck's law* and is illustrated in Fig. 1. In the $h\nu\ll k_BT$ region we have

$$D_q(\nu)\sim\frac{8\pi(k_BT)^3}{h^2c^3}\left[\frac{h\nu}{k_BT}\right]^{1+q}\ , \tag{17}$$

which generalizes the *Rayleigh-Jeans law.* For $h\nu|1-q|\gg k_BT$, the $(1-q)$ correction diverges, hence the present expansion is not valid; in other words, the high-frequency tail should be calculated separately. For $q\approx 1$, the divergent region has no practical importance, because of the suppression of the exponential factor. Nevertheless, the natural variable indeed is $h\nu/k_BT$, consequently the *total emitted power per unit surface*

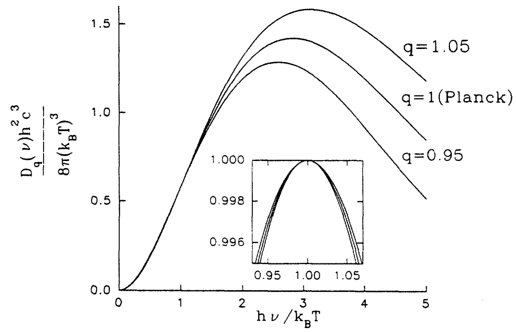

FIG. 1. Blackbody photon energy density per unit volume (D_q) within extensive ($q=1$; Planck's law) and slightly nonextensive ($q=0.95$ and $q=1.05$) statistical mechanics. The inset presents $Dq(\nu)/Dq(\nu_M)$ vs ν/ν_M (the top, middle, and bottom curves respectively correspond to $q=1.05$, $q=1$, and $q=0.95$) where the curvature effect is exhibited.

$P_q \propto \int_0^\infty d\nu D_q(\nu)$ is given by

$$P_q \sim \sigma_q T^4 . \tag{18}$$

Therefore, the *Stefan-Boltzmann law* remains the usual one, but with a q-dependent prefactor [$\sigma_1=\pi^2k_B^4/60\hbar^3c^2 \simeq 5.67\times10^{-12}$ W/cm^{2}K^4 is the Stefan constant; $(d\sigma_q/dq)_{q=1}>0$]. A similar type of behavior was obtained within the quantum group formalism (q_G blackbody [21]), where we introduced q_G to avoid confusion with q). A second immediate consequence of $h\nu/k_BT$ being the natural variable for all values of q is that the *Wien shift law* also preserves its form with a q-dependent prefactor. More precisely, a straightforward expansion of $D_q(\nu)$ [Eq. (16)] in the neighborhood of ν_M [location of the maximum of $D_q(\nu)$] yields

$$h\nu_M/k_BT \sim a_1+a_2(q-1) , \tag{19}$$

with $a_1 \simeq 2.821\,439$ and $a_2 \simeq 5.171\,965$. The comparison of this result with the corresponding one in quantum groups [21] shows an important difference: at fixed temperature, $\nu_M(q>1)>\nu_M(q=1)>\nu_M(q<1)$, whereas $\nu_M(q_G)=\nu_M(1/q_G)$ and $\nu_M(q_G<1)<\nu_M(q_G=1)$.

We verify that, for $\nu \simeq \nu_M$,

$$\frac{D_q(\nu)}{D_q(\nu_M)} \sim 1-B\left[\frac{\nu}{\nu_M}-1\right]^2 , \tag{20}$$

where

$$B \sim b_1-b_2(q-1) , \tag{21}$$

with $b_1 \simeq 1.232\,159$ and $b_2 \simeq 2.690\,571$, and

$$D_q(\nu_M)\frac{c^3h^2}{8\pi(k_BT)^3} \sim d_1+d_2(q-1) , \tag{22}$$

with $d_1 \simeq 1.421\,435$ and $d_2 \simeq 2.933\,276$.

If we have (in arbitrary units) the experimental photon energy density $D^{\text{expt}}(\nu)$ of a blackbody radiation, we can, in the neighborhood of its maximum, fit it with

$$D^{\text{expt}}(\nu) \sim D_M^{\text{expt}}\left[1-B^{\text{expt}}\left[\frac{\nu}{\nu_M^{\text{expt}}}-1\right]^2\right] , \tag{23}$$

thus obtaining D_M^{expt}, B^{expt}, and ν_M^{expt}. From Eq. (21) we have

$$q-1 \simeq \frac{b_1-B^{\text{expt}}}{b_2} . \tag{24}$$

If, within the experimental error, we could guarantee that this quantity is different from zero (i.e., if $B^{\text{expt}} \neq 1.232\,159$), *it would be legitimate to think about a possible unfamiliar influence of gravitation on light.* The associated temperature can be obtained by placing Eq. (24) into Eq. (19), which yields

$$T \simeq \frac{h\nu_M^{\text{expt}}b_2}{k_B(b_2a_1+a_2b_1-a_2B^{\text{expt}})} . \tag{25}$$

Summarizing, we have extended to $q \simeq 1$ (i.e., to slightly nonextensive thermodynamics) Planck's law (hence Rayleigh-Jeans, Stefan-Boltzmann, and Wien shift laws) for the radiation of a blackbody. The main results are as follows: (i) In the low-frequency region ($h\nu \ll k_BT$), the photon energy density D_q is proportional to ν^{1+q}, hence, the slope in a $\log D_q$ versus $\log \nu$ plot provides q. (ii) The location (ν_M) of the maximum of $D_q(\nu)$ and the associated curvature provide [through Eqs. (24) and (25)] q and T. (iii) The behavior of $D_q(\nu)$ for fixed $h\nu/k_BT$ is very different from that obtained within quantum groups [21]: for increasing q, D_q *monotonically increases* (except for $h\nu/k_BT$ below roughly unity, where it very slightly decreases) and crosses Planck's law for $q=1$, whereas, for increasing q_G, Dq_G becomes *maximal* (Planck's law) at $q_G=1$.

We apply the generalized thermal spectrum [Eq. (16)] to the cosmic microwave background radiation to test for deviations from Planck's law. The above remark (i) could be used in the low-frequency region, but more precise data exist at higher frequencies. The most accurate data are from the FIRAS (far-infrared absolute spectrophotometer) instrument on the COBE (Cosmic Background Explorer) satellite [22]. A plot (Fig. 2) of the brightness referenced to a Planck spectrum at a temperature of 2.72584 K ($\pm$0.005 K systematic error) shows significant deviations in the data from a Planck spectrum (the χ^2 of the fit with a Planck spectrum is a factor of 4 greater than the number of degrees of freedom), but the deviation may be due entirely to instrumental effects, which are especially large at low frequencies [22]. Mather *et al.* [22] find upper limits on spectral distortions that arise from additional input of energy into the cosmic background. Here we ask whether a generalized thermal spectrum can account for the data.

Because FIRAS is designed to measure the spectrum of the cosmic background to high precision, it makes *differential* measurements of the radiation and relies on the calibration of its thermometers to tie to absolute standards. FIRAS measures the difference between the cosmic background and an internal reference, whose temperature is adjusted to be approximately 2.7 K to null out

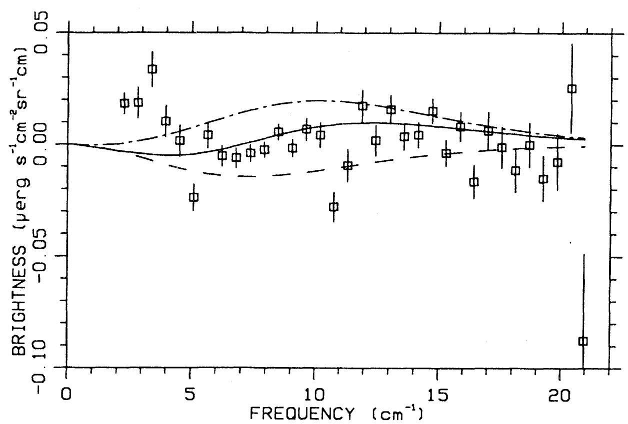

FIG. 2. The brightness of the cosmic microwave background referenced to a Planck spectrum with a temperature of 2.72584 K [22] and a non-Planck spectrum with $q-1=-3.6\times10^{-5}$ and a temperature difference $\delta T=-0.1$ mK (solid line). Also shown are the two terms separately: a spectrum with $q-1=-3.6\times10^{-5}$ and $\delta T=0$ (dot-dashed line) and a spectrum with $q-1=0$ and $\delta T=-0.1$ mK (dashed line).

problems with imperfect emissivity and to limit the dynamic range of the instrument. In addition, an external blackbody with a temperature adjustable from 2 to 25 K is used to calibrate the gain of the instrument [23]. Since FIRAS measures a differential signal and not an absolute one, it is most sensitive in comparing q between the cosmic radiation and the internal thermal reference. To determine q absolutely requires analysis of the measurements used to tie its thermometers to an absolute temperature, or a measurement of q for the spectrum of a laboratory blackbody. A quantity with comparable errors is the temperature: The absolute error in the temperature is 1 part in 300 [22], whereas the random error is 1 part in 50 000. We shall *assume* that the internal reference has a Planck spectrum, since for the reference blackbody nonlocal or edge effects, which give rise to nonextensivity, are likely to be small. With that assumption we shall determine q of the cosmic radiation.

We fit the data for q and the temperature shift δT, which is the difference between the best fitting temperature with $q\neq1$ and $q=1$. To show the true random errors, we removed the factor of 2 increase in the errors that Mather *et al.* [22] applied to account for unexplained errors. We removed the point at a frequency of at 10.8 cm^{-1} because of a vibration in the instrument [22]. (Removing this datum reduced the χ^2 by 20.) We find that

$$\delta T=(-0.3\pm2.5)\times10^{-5}\ \text{K}\ ,\qquad q-1=(-1.0\pm0.7)\times10^{-5}\ , \tag{26}$$

and $\chi^2=108$ with 31 degrees of freedom. The three points at low frequency and the point at 5.1 cm^{-1} account for half of the χ^2. With $q=1$, χ^2 increases by 2. We inflate the errors by 1.87 to force χ^2 to be equal to the number of degrees of freedom and find an upper limit (95% confidence level) of

$$|q-1|<3.6\times10^{-5} \tag{27}$$

with the assumption that the internal reference has a Planck spectrum.

A practical difficulty with determining q from the FIRAS data is that within the data set, q and the temperature are statistically correlated (correlation of 0.929). To a large extent, a change in the temperature of -0.1 mK cancels the deviation from a Planck spectrum of $q-1=-3.6\times10^{-5}$. See Fig. 2.

A non-Planck deviation in the thermal spectrum cannot account for the nonstatistical residuals in the data, since χ^2 improves insignificantly when q is allowed to differ from 1 yet there is hope that the FIRAS data can yield an even more precise measurement of q, if the systematic errors can be understood or when the entire data set is published. If indeed it turned out that $q\neq1$, then we would have an important key for a better comprehension of the interaction between light and matter, which could ultimately enlighten our notions of the structure of space-time.

Note added in proof. A. R. Plastino [Ph.D. thesis, Universidad Nacional de La Plata–Argentina, 1994] and A. R. Plastino, A. Plastino, and H. Vucetich (unpublished) obtained results ($|q-1|\leq0.67\times10^{-4}$ from the experimental value of the Stefan-Boltzmann constant, and $|q-1|\leq5.3\times10^{-4}$ from the FIRAS-COBE 1990–1992 data) which are consistent with the present ones.

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