Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process

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(Received 23 March 1995)

An analytic expression for characteristic function defining a truncated Lévy flight is derived. It is shown that the characteristic function yields results in agreement with recent simulations of truncated Lévy flights by Mantegna and Stanley [Phys. Rev. Lett. **73**, 2946 (1994)]. With the analytic expression for the characteristic function, the convergence of the Lévy process towards the Gaussian is demonstrated without simulations. In the calculation of first return probability the simulations are replaced by numerical integration using simple quadratures.

PACS number(s): 05.40.+j, 02.50.−r

Lévy flights [1,2] and Lévy walks [3,4] are applied in modeling physical systems with spatiotemporal fractality (for review, see Ref. [2]). The characteristic feature of Lévy flight is that it does not converge to the Gaussian stochastic process, instead it is "attracted" towards the Lévy stable process with infinite variance [2]. In Lévy walks the requirement of finite variance of the process is met by introducing spatiotemporal coupling [4]. However, a more direct way to retain the finite variance is by means of truncated Lévy flight, where an upper cutoff to the values of random variables is introduced [5].

The general statistical properties of truncated Lévy flights have been studied recently by Mantegna and Stanley [5]. In their simulations they observed a clear but slow transition from the Lévy to the Gaussian process. The convergence was so slow that a huge number of independent events (a long time) was needed to observe deviations from Lévy stable process, first apparent only in the very far wings of the probability density distribution [5]. In this Brief Report a treatment based on an analytical approach is given, paralleling the simulation study of Mantegna and Stanley. Instead of the abrupt truncation employed in Ref. [5], a smooth exponential regression towards zero is introduced. This makes it possible to derive an analytic expression for characteristic function and enables one to replace simulations by more straightforward calculations. Because the present report supplements the more thorough work of Mantegna and Stanley the exposition is brief and concise.

The truncated Lévy flight with smooth cutoff is generated by random variables $\{z\}$ with distribution

$$f(z)=\begin{cases} A_{-}e^{-\lambda|z|}|z|^{-1-\nu}\ , & z<0 \\ A_{+}e^{-\lambda z}z^{-1-\nu}\ , & z\geq 0\ , \end{cases} \tag{1}$$

where the characteristic exponent is in the range $0<\nu<2$. For values $\nu\geq 2$ one obtains the Gaussian process. The limit distribution $P(x)$ [6] of the sums of random variables $\{z\}$ in the Poissonian stochastic process is defined by the characteristic function [7]

$$\ln\hat{P}(k)=-t\int_{-\infty}^{\infty}(1-e^{-ikz})f(z)dz\ . \tag{2}$$

The characteristic function given by Eqs. (1) and (2) can be expressed in a compact form if $0<\nu<2$, but $\nu\neq 1$ [8]. The lengthy but straightforward calculation yields

$$\ln\hat{P}(k)=c_0-c\frac{(k^2+\lambda^2)^{\nu/2}}{\cos(\pi\nu/2)}\cos\left[\nu\arctan\frac{|k|}{\lambda}\right]\left[1+i\frac{k}{|k|}\beta\tan\left[\nu\arctan\frac{|k|}{\lambda}\right]\right]\ , \tag{3}$$

where $c=t(A_{+}+A_{-})\pi\cos(\pi\nu/2)/[\nu\Gamma(\nu)\sin(\pi\nu)]$ is a scaling factor and $c_0=\lambda^{\nu}/\cos(\pi\nu/2)$ takes care of the normalization of $P(x)$. The asymmetry of the process is defined by the parameter $\beta=(A_{+}-A_{-})/(A_{+}+A_{-})$. The distribution $P(x)$ is now uniquely determined by the analytic expression in Eq. (3), and accurate numerical values for $P(x)$ can be calculated Fourier-transforming $\hat{P}(k)$ numerically, using adaptive quadratures designed for oscillating integrals (e.g., routine D01AKF in the NAG library). An example of the calculated symmetric distribution for $\nu=0.5$ is given in Fig. 1, demonstrating clearly the convergence towards the Gaussian process.

The transition from the Lévy stable process to the Gaussian one becomes evident from the closer inspection of characteristic function. For simplicity the study is restricted to the symmetric cases $\beta=0$ only. The probabili-

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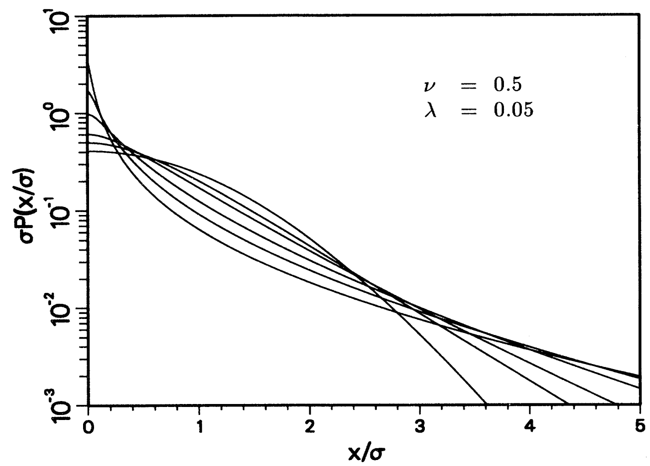

FIG. 1. Probability density distribution $P(x)$ obtained for characteristic exponent $\nu=0.5$ and $\lambda=0.05$. Variable x is scaled by variance σ given in Eq. (5) in order to enhance the convergence towards the Gaussian. Profiles are shown for scaled times $2At=0.5$, 1, 2, 5, 10, and 100 (from top to bottom at origin). Sharp profiles at short times are close to the Lévy stable distribution, and at the longest time shown the profile is nearly identical to the Gaussian.

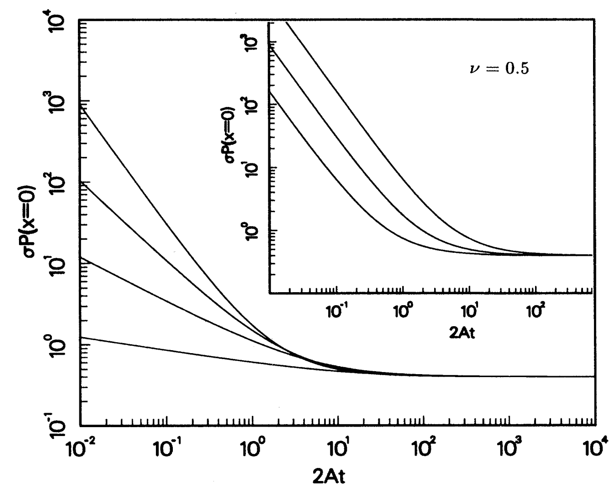

FIG. 2. The probability of first return $P(x=0)$ for values $\nu=0.50$, 0.66, 0.95, and 1.50 (from top to bottom). In all cases $\lambda=0.05$. The probability $P(x)$ is multiplied by variance σ given by Eq. (5), enhancing the convergence towards the Gaussian, signalled by the approach to a constant value $(2\pi)^{-1/2}$. At short times the values $\sigma P(x=0)$ are proportional to $t^{1/2-1/\nu}$, in agreement with Eq. (8). In the inset is shown the effect of parameter λ on the convergence. Results for $\lambda=0.05$, 0.05, and 0.005 are shown (from left to right) for $\nu=0.5$.

ty density $P(x)$ at long times is determined by the asymptotic form of $\hat{P}(k)$ at small values of k,

$$\ln\hat{P}(k)\approx-\tfrac{1}{2}\sigma^2(t)k^2\ , \tag{4}$$

where the variance can be calculated from the characteristic function,

$$\sigma^2(t)=-\left.\frac{\partial^2\hat{P}(k)}{\partial k^2}\right|_{k=0}=t\left[\frac{2A\pi(1-\nu)}{\Gamma(\nu)\sin(\pi\nu)}\right]\lambda^{\nu-2}\ . \tag{5}$$

Equation (4) is the signature of the Gaussian distribution. The convergence to the Gaussian process is caused by the screening factor, which makes $f(z)$ decrease faster than z^{-3} for large values of z. In the Gaussian regime the first return probability $P(x=0)$ is given by

$$P(x=0)=\frac{1}{\sqrt{2\pi}\sigma(t)}\propto t^{-1/2}\ . \tag{6}$$

At short times the screening has no essential effect on $P(x)$ and the distribution $P(x)$ is generated by the formula of Lévy and Khintchine [7], obtained in the limit $\lambda\to 0$,

$$\ln\hat{P}(k)\approx-c|k|^\nu\left[1+i\frac{k}{|k|}\beta\tan(\nu\pi/2)\right]\ . \tag{7}$$

In this regime the probability of first return of the symmetric process ($\beta=0$) is given by

$$P(x=0)=t^{-1/\nu}\frac{\Gamma(1/\nu)}{\nu\pi}\left[\frac{\nu\Gamma(\nu)\sin(\pi\nu)}{2A\pi\cos(\pi\nu/2)}\right]^{1/\nu}\ , \tag{8}$$

signaling now the dominance of Lévy stability. These asymptotic results are identical (apart the constant factor due to parametrization) to results given by Mantegna and Stanley. At intermediate times the probability of first return is calculated by quadratures using the analytic expression in Eq. (3). Results for some representative cases are shown in Fig. 2, where asymptotic regions and the smooth transition between them are clearly exposed.

In conclusion, we have modified the truncated Lévy flight model of Mantegna and Stanley in such a way as to allow an analytical calculation of the characteristic function and determination of the complete probability density distribution by simple quadratures. Using only the characteristic function the existence of a smooth but slow transition of the Lévy stable process towards the Gaussian process can be demonstrated. Furthermore, many properties of interest can be derived directly from the characteristic function, e.g., the variance and probability of first return, in agreement with the simulation results.

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[8] For asymmetric processes $A_+ \neq A_-$ in the range $1<\nu<2$ the integrand is regularized by replacing the term $[1-\exp(-ikz)]$ with $[1-ikz-\exp(-ikz)]$. The effect of this regularization is to remove the singularity in the origin without affecting the values of the probability distribution $P(x)$, as discussed in detail in Ref. [7].