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Self-similar decay of three-dimensional homogeneous turbulence with hyperviscosity

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Numerical simulations of the Navier-Stokes equations with hyperviscosity $(-1)^{h+1}\Delta^h$ $(h=8)$ show that periodic-box turbulence exhibits self-similar decay. The inertial-range energy spectrum has the scaling law $\mathcal{E}(t)^{2/3}/k^{5/3}$, where $\mathcal{E}(t)$ is the energy dissipation rate at time t. The total energy of the system decreases at $1/t^2$. The concept of constant Reynolds number decay is introduced, enabling us to perform long time averages and reliably measure higher-order correlation functions. Comparisons are made with the case of forced turbulence reported earlier.

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Large-Reynolds-number turbulence is closely described by the inviscid Euler equations at large scales; at very small scales, dissipation provides an energy sink (ultraviolet cutoff). In this case, it is natural to suppose that the physics of turbulence should be insensitive to the specific form of dissipation. In our previous work [1] we have already demonstrated that for the same numerical resolution, we can effectively increase the extent of the inertial range of three-dimensional turbulence by an order of magnitude by using alternative forms of dissipation. We considered replacing the normal Laplacian dissipation by a higher power of the Laplacian, i.e., a hyperviscosity. It was shown that three-dimensional inertial-range dynamics seems to be relatively independent of the form of the hyperviscosity and that modest resolution simulations with high-order hyperviscosity lead to sufficiently extensive inertial ranges that measurement of a broad variety of otherwise intractable quantities can be made. In [1] we observed deviations of the isotropic energy spectrum from the 5/3 Kolmogorov law [2]. It was suggested that this deviation is likely to be related to the large-scale behavior of the system, in particular to the mechanism of energy pumping.

In this Rapid Communication we consider the unforced case, i.e., decaying turbulence. It will be shown that for decaying turbulence in a periodic box, the 5/3 Kolmogorov law is recovered and we observe self-similar decay with the isotropic energy spectrum defined as $E(k) = 4\pi k^2\langle v_i(\mathbf{k})v_i(-\mathbf{k})\rangle$ changing as

$$E(k,t)=C_K\frac{\mathcal{E}(t)^{2/3}}{k^{5/3}}\,G\left(\frac{k}{k_d(t)}\right). \tag{1}$$

Here $\mathcal{E}(t)$ is the energy dissipation rate and $k_d(t)$ is the dissipation cutoff wave number at time t. The Kolmogorov constant C_K measured from our simulations is $C_K\approx 1.2$; the function $G(k/k_d)$ is nearly unity at small k and parametrizes the influence of dissipation for k near k_d. We define k_d as the wave number at which the maximum of the vorticity spectrum $k^2E(k)$ is attained.

The hyperviscosity-modified Navier-Stokes equations are

$$\partial_t v_i+v_j\partial_j v_i=-\partial_i p+(-1)^{h+1}\nu_h\Delta^h v_i, \tag{2}$$

where the pressure p is calculated from the incompressibility condition $\partial_i v_i=0$. As in [1] we use hyperviscosity with $h=8$. The hyperviscosity coefficient ν_h with $h=8$ is chosen so that the hyperviscosity is significant only for $k\geqslant k_d\approx N/3$ (where N^3 is the numerical resolution). The energy balance equation then takes the form

$$\partial_t E(k,t)=T(k,t)-2\nu_h k^{2h}E(k,t). \tag{3}$$

Here $T(k,t)$ is the energy transfer function and the energy flux is defined as

$$J_E(k)=\int_{k'>k}^{\infty}T(k')dk'. \tag{4}$$

It is clear that if (1) holds

$$k_d(t)\propto\left(\frac{\mathcal{E}(t)}{\nu_h{}^3}\right)^{1/(6h-2)}; \tag{5}$$

for $h=8$, $k_d(t)$ is practically independent of t. Neglecting the time derivative of k_d the only energy transfer function compatible with (1) and (3) has the form

$$T(k,t)=\left(\frac{2}{3}\frac{\partial_t\mathcal{E}}{\mathcal{E}}+2\nu_h k^{2h}\right)E(k,t). \tag{6}$$

The form (6) is self-similar only in the inertial range and is, strictly speaking, incompatible with von Kármán's self-similarity hypothesis [3], although it is compatible with a weaker form of von Kármán's hypothesis valid only at high Reynolds numbers [3]. It is clear that this form of self-similarity necessarily leads to equations for the total energy of the system $K(t)$ of the form

$$\partial_t K(t)=-\mathcal{E}(t),\qquad \partial_t\mathcal{E}(t)=-\frac{3}{2}\frac{\mathcal{E}(t)^2}{K(t)}, \tag{7}$$

from which we infer that $K(t)\sim 1/t^2$ $(t\to\infty)$. This result is in conflict with most theoretical predictions and experimental data. Previous results suggested that $K(t)\sim 1/t^\alpha$ in decaying turbulence with α=1,10/7,6/5 [3,4] (see [5] for a recent review). There are at least two reasons for this discrepancy. First, von Kármán's self-similarity hypothesis, strictly speaking, is only valid asymptotically. The second, and more important reason, is that in practically all laboratory experiments the integral scale of turbulence defined as $L_i=\int k^{-1}E(k)dk/\int E(k)dk$ grows in time, thereby decreas-

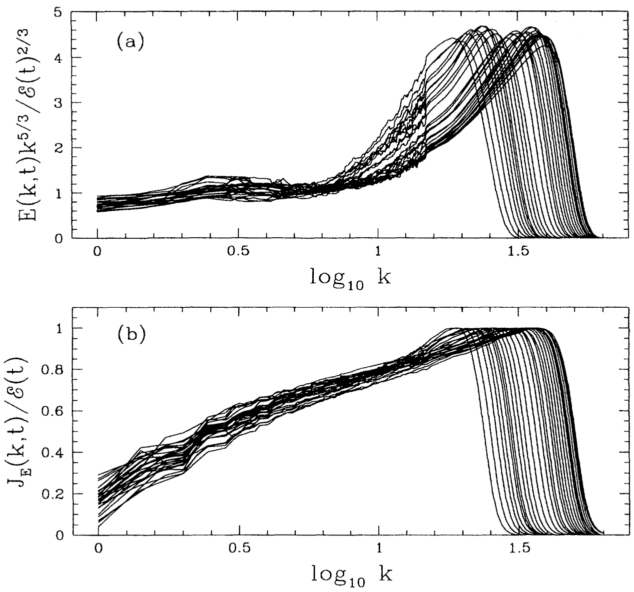

FIG. 1. (a) Scaled energy spectra $E(k,t)k^{5/3}/\mathcal{E}(t)^{2/3}$ and (b) scaled energy fluxes $J_E(k,t)/\mathcal{E}(t)$ as functions of $\log_{10}k$ at different moments of time. The resolution is 128^3.

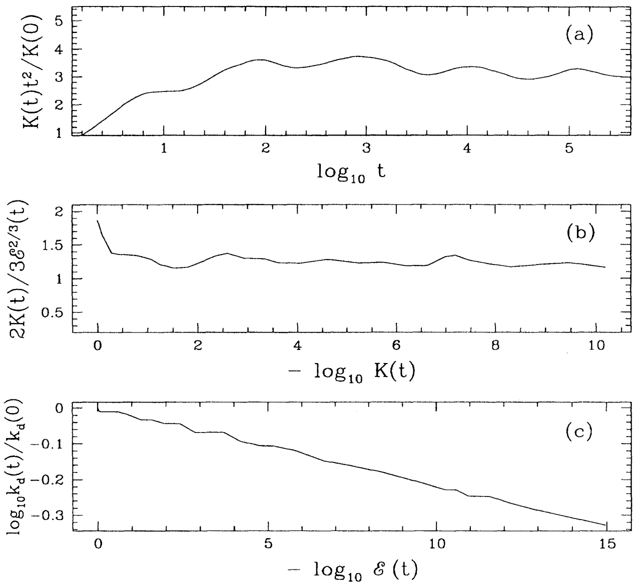

FIG. 2. (a) Total energy scaled by $K(t)t^2/K(0)$ as a function of $\log_{10}(tv_{\rm rms})$; (b) the ratio $2K(t)/3\mathcal{E}^{2/3}(t)$ as a function of $-\log_{10}[K(t)/K(0)]$; (c) the dissipation scale $\log_{10}[k_d(t)/k_d(0)]$ as a function of $\log_{10}[\mathcal{E}(t)/\mathcal{E}(0)]$. The resolution is 128^3.

ing the decay rate of K. In our periodic box, L_i may grow slightly at early times. But when L_i reaches the size of the box, it remains constant thereafter. It is indeed difficult to have an experimental setup in a laboratory experiment that is equivalent to a periodic box so that the $K\sim 1/t^2$ law has never been observed. In numerical simulations with a Laplacian viscosity the Reynolds number is usually too low for inertial-range asymptotic laws to be reliably measured.

We solve (2) using a parallel pseudospectral code [6]. The data for the forced turbulence case [1] are used as the initial conditions for the decay. First we performed one long run with resolution 128^3 in a periodic box of size $L=2\pi$ in each direction. The total run time is 2×10^5 large eddy turnover times $\tau_0\approx 1/v_{\rm rms}(0)$ (where $v_{\rm rms}(t)=[2K(t)/3]^{1/2}$). The scaled isotropic energy spectra $E(k,t)$ and corresponding energy fluxes are shown in Figs. 1(a) and 1(b) at the different moments of time as functions of $\log_{10}(k)$. It is apparent from Fig. 1 that energy spectra and energy fluxes scale with time according to Eqs. (1) and (6). There are obviously some fluctuations of the energy flux and the energy spectrum. Although the Kolmogorov 5/3 law for the energy spectrum is observed, the energy flux is no longer constant. According to (4),(6) in the inertial range, the energy flux has a correction of the form $J_E(k)=\mathcal{E}(1-A/(kL_i)^{2/3})$, where $A\approx 1$. Asymptotically this correction becomes small for large k but the crossover to the asymptotic behavior is very slow. Another interesting feature is the flattening of the energy spectrum near k_d. This "bottleneck" part of the energy spectrum has already been discussed in [1]. In Fig. 2(a) we plot the total energy of the system as a function of time. It turns out that after a few large eddy turnover times, the total energy $K(t)\sim 1/t^2$ in accordance with the self-similar behavior of energy spectra and (7). Moreover, if we plot the ratio of $K(t)/\mathcal{E}^{2/3}(t)$ as the function of $\log_{10}K(t)$[Fig. 2(b)] we see that this ratio is constant across approximately ten decades of variation of $K(t)$, supporting the self-similarity of energy spectra in the form (1). As may be seen from Fig. 2(c) while $\mathcal{E}(t)$ changes 15 orders of magnitude the dissipation scale $k_d(t)$ decreases less then two times in agreement with (5). The fact that the dissipation scale changes in time very slowly allows us to introduce the concept of constant Reynolds number decay.

The Reynolds number for the Navier-Stokes equations with hyperviscosity may be defined as some power of k_dL_i [1]. Let us tune the hyperviscosity coefficient $\nu_h(t)$ in order to keep k_d fixed [see (5)]. The only relevant parameter with the dimension of time is the dissipation rate $\mathcal{E}(t)$. We checked that if we rescale any correlation function by the appropriate power of $\mathcal{E}(t)$ the data collapse to one curve. This enables us to conduct long time averages for nearly infinite in time, "stationary," constant Reynolds number decay. We performed two series of stationary decay runs with resolutions 128^3 and 256^3. The total time of integration expressed in units of a local large eddy turnover time $1/v_{\rm rms}(t)$ is approximately 200 and 15, respectively. We measured the same correlation functions as described in [1] and checked that rescaling by an appropriate power of $\mathcal{E}(t)$ leads to the collapse of the correlation functions measured at different moments of time. For example, we plot in Figs. 3(a) and 3(b) the rescaled energy spectra and energy fluxes measured at different moments of time for 128^3 resolution. It has been checked that the observed scatter is due to statistical fluctuations. To obtain statistically reliable data, we performed time averages with rescaling. In what follows, such statistical averaging is always assumed.

The use of hyperviscosity and the concept of a constant Reynolds number decay enables us to address an important question regarding higher-order correlation functions for the case of decaying turbulence. As in [1] we define the isotropic

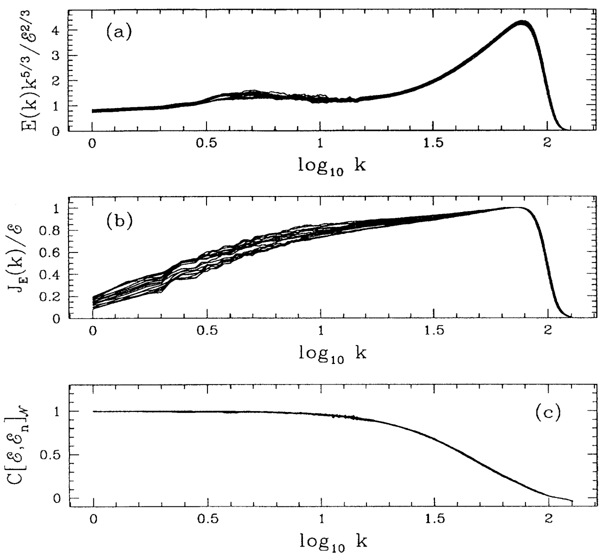

FIG. 3. Decay at constant Reynolds number. The resolution is 256^3. The curves at different moments of time are superimposed on each other. (a) Scaled energy spectra $E(k,t)k^{5/3}/\mathscr{E}(t)^{2/3}$; (b) scaled energy fluxes $J_E(k,t)/\mathscr{E}(t)$; (c) normalized strain-dissipation rate correlation function $\mathscr{C}[\mathscr{E},\mathscr{E}_n](k)/\{\mathscr{C}[\mathscr{E},\mathscr{E}](k)\mathscr{C}[\mathscr{E}_n,\mathscr{E}_n](k)\}^{1/2}$ as functions of $\log_{10}k/k_d$.

correlation function of any two operators in spectral space as $\mathscr{C}[A,B](k)=2\pi k^2\langle A(-\mathbf{k})B(\mathbf{k})+A(\mathbf{k})B(-\mathbf{k})\rangle$. We find that for correlation functions that come predominantly from small-scale fluctuations the results are very close to the ones

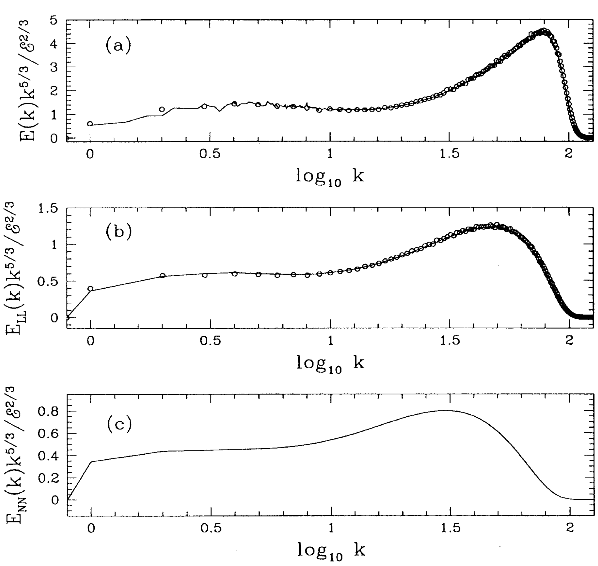

FIG. 4. Scaled by $k^{5/3}/\mathscr{E}^{2/3}$ (a) isotropic energy spectrum $E(k,t)$; (b) transverse energy spectrum $E_{LL}(k,t)$; and (c) longitudinal energy spectrum $E_{NN}(k,t)$ as functions of $\log_{10}k$ for the case of decay with constant Reynolds number. The resolution is 256^3. Points are the isotropic energy spectrum calculated from the transverse and longitudinal energy spectra and the transverse energy spectrum calculated from the longitudinal energy spectrum according to (8).

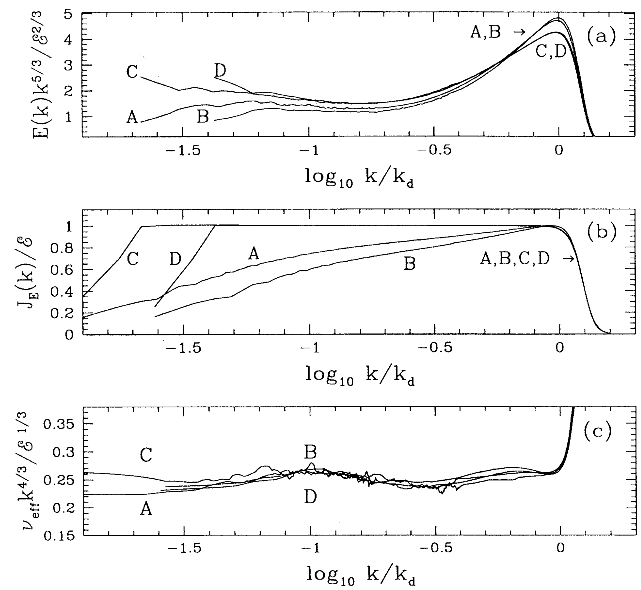

FIG. 5. Comparison between the case of constant Reynolds number decay and white-in-time forced turbulence. (a) Scaled energy spectra $E(k)k^{5/3}/\mathscr{E}^{2/3}$; (b) scaled energy fluxes $J_E(k)/\mathscr{E}$; (c) effective viscosity $\nu_{\rm eff}(k)$ [see (9)] scaled by $k^{4/3}/\mathscr{E}^{1/3}$ plotted as functions of $\log_{10}k/k_d$. Resolutions 256^3 (A), 128^3 (B) for decaying turbulence. Resolution 256^3 (C), 128^3 (D) for white-in-time turbulence.

obtained in [1]. This includes the energy dissipation rate $[\mathscr{E}=\nu_h(\Delta^{h/2}u_i)^2]$ correlation function $\mathscr{C}[\mathscr{E},\mathscr{E}](k)\sim k^{-\gamma}$, strain-vorticity and nonlinear term correlation functions. For example, a run with resolution 256^3 leads to the result $\gamma\approx 0.75$ and to the intermittency exponent $\mu=1-\gamma\approx 0.25$, consistent with [7]. On the contrary, isotropic spectra of the turbulent energy $K=v_iv_i/2$, which come from predominately from large-scale fluctuations, have somewhat flatter spectra $\sim 1/k^{3/2}$ than observed in [1]. As in [1] we can make a statement that our definition of dissipation rate $\mathscr{E}$ is statistically equivalent to that with renormalized normal viscosity. Indeed, we note that the physical dissipation is the product of viscosity and $\mathscr{E}_n=\frac{1}{2}\omega^2+S^2$ [where $\omega_i=\varepsilon^{ijk}\partial_j v_k$ and $S_{ij}=\frac{1}{2}(\partial_i v_j+\partial_j v_i)$ are vorticity and strain, respectively]. Remarkably, the measured correlator $\mathscr{C}[\mathscr{E},\mathscr{E}_n]/(\mathscr{C}[\mathscr{E},\mathscr{E}]\mathscr{C}[\mathscr{E}_n,\mathscr{E}_n])^{1/2}$ nearly equals 1 in the range of small k where asymptotic scaling may be claimed [see Fig. 3(c)].

The results plotted in Figs. 4(a) and 4(b) show that our system is locally isotropic. Besides isotropic energy spectra we measure independently the longitudinal E_{NN} and transverse E_{LL} energy spectra. For an isotropic system [3]

$$E_{LL}(k)=\frac{1}{2}E_{NN}(k)\left(1-\frac{d\log E_{NN}(k)}{d\log k}\right),$$

$$E(k)=-\frac{1}{2}\frac{d[E_{NN}(k)+2E_{LL}(k)]}{d\log k}. \qquad (8)$$

In Fig. 4 we plot directly measured longitudinal, transverse, and isotropic spectra for the 256^3 run. We also plot, in the same figures, the transverse spectrum recalculated from the longitudinal spectrum and the isotropic spectrum recalculated from the longitudinal and transverse spectra according to (8). It is clear from this data that the flow is close to isotropic.

In Figs. 5(a) and 5(b) we compare energy spectra and energy fluxes obtained in constant Reynolds number decay with data obtained previously for the case of white-in-time forcing. We still see some small deviations of energy spectrum from the 5/3 law for the decaying run with the resolution 256^3, but these deviations are much smaller than those in the forced case. The run with 256^3 resolution has a larger Reynolds number than the one with 128^3 resolution. The correction to the constant energy flux in this case is smaller. The energy spectra in these two cases do not completely coincide when plotted as a function of k/k_d. This may imply that at larger Reynolds number we may see larger deviations from the 5/3 law and that the present results are not asymptotic.

The most important conclusion from this comparison is that the energy spectrum scaling law may depend on large-scale structure of the flow and this large-scale structure is different in cases of forced and decaying turbulence. Indeed, in the case of decaying turbulence, the total energy is a smooth function of time, while for forced turbulence it is a strongly intermittent function of time. There are moments in time when the total energy changes sharply on the time scale of one large eddy turnover time, although some of the correlation functions seem to be independent of the mechanism of forcing. In our previous work [1] we suggested the definition of an "effective viscosity" in terms of the vortex stretching term $\Gamma_i \equiv S_{ij}\omega_j$ as $\nu_{\text{eff}} k^2 \omega_i = \Gamma_i$. As in [1] we find that

$$\nu_{\text{eff}}(k) = \frac{\mathscr{E}[\omega_i, \Gamma_i](k)}{k^2 \mathscr{E}[\omega_i, \omega_i](k)} \approx 0.25 \frac{\mathscr{E}^{1/3}}{k^{4/3}}, \tag{9}$$

which is remarkably independent of k_d [see Fig. 5(c)]. This scaling law is very robust and seems also to be insensitive to the large-scale forcing.

Our results suggest that inertial-range dynamics may be independent of the particular mechanism of small-scale dissipation but are likely to be strongly dependent on the forcing mechanism. The clear deviation of the inertial-range energy spectrum from the Kolmogorov law observed in [1] is likely to be related to the large-scale behavior of the system. On the contrary, for freely decaying turbulence we seem to regain the 5/3 Kolmogorov law, but the energy flux becomes constant only asymptotically. We emphasize that caution is required in comparing our results with most laboratory experiments. Although most laboratory experiments involve decaying turbulence, even in this case the result might be sensitive to the global geometry of the system. We conclude that further understanding of the sensitivity of turbulent energy cascades to forcing mechanisms is needed.

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