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Generating partition for the standard map

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A procedure to obtain the symbolic dynamics for conservative dynamical systems is introduced with reference to the standard map in a strongly chaotic regime. The method extends an approach previously developed for highly dissipative systems. It is based on the construction of a generating partition from homoclinic tangencies and fibers of invariant manifolds. It is found that some arbitrariness in the construction of the partition is unavoidable.

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An effective representation of chaotic dynamics can be achieved by encoding any trajectory as an infinite sequence of symbols. This enables a fruitful mapping: orbits can formally be seen as microstates of some spin chain (the symbols corresponding to spin values). Accordingly, a thermodynamical formalism can be developed [1] to compute relevant statistical averages such as Lyapunov exponents, dynamical entropies, and fractal dimensions [2].

Various approaches have been introduced to encode a given trajectory in phase space. One method relies on the assumption that the code assigned to each periodic orbit remains unchanged when the dynamical system is smoothly modified [3]. The key aspect of this is the identification of some parameter value k_h such that the resulting dynamics is characterized by a complete horseshoe. The encoding of each periodic orbit for the desired parameter value k_0 is obtained by smoothly deforming the orbit from k_h to k_0. Unfortunately, it has been discovered that there exist periodic orbits which, followed along closed paths in parameter space, are transformed into different orbits, thus showing unavoidable ambiguities [4]. A second method is based on the simultaneous introduction of a pseudodynamics along a formal time axis and on the interpretation of the true time variable n as a spatial index [5]. The applicability of this approach is limited to strongly dissipative models.

A last powerful method, which works whenever a horseshoe type mechanism is present in the dynamics, is based on the direct construction of a generating partition (GP) by connecting together the relevant (primary) homoclinic tangencies (HTs) to eventually split the phase space into disjoint atoms [6]. Such a strategy has been successfully applied to both maps and flows [7] and it appears to be of general validity, although there is no rigorous proof that it is always applicable. However, this method too has been implemented only in dissipative systems. In fact, for the Hamiltonian case serious difficulties arise in connecting the primary HTs to form continuous partition lines.

A complete encoding of the dynamics in a conservative system requires taking into account stability islands as well as the chaotic component in which they are embedded. The former problem can in principle be solved by encoding the rotation angles with respect to suitable reference points. An approach in this direction has been developed by Russberg [8] for the piecewise linear standard map in a regime where the phase space is essentially filled by islands. Here, we focus our attention on the complementary problem of a correct description of the evolution in the chaotic component. To this aim we have studied the standard map for a large nonlinearity, such that the stability islands cover a tiny portion of the phase space. In particular, we describe a procedure to identify and connect primary HTs in such a way that the resulting line represents the border of a generating partition.

Let us first briefly recall the main ideas behind the method originally proposed in Ref. [6]. Because of the folding process associated with a horseshoe, if fibers of the unstable (W_u) and stable (W_s) manifolds intersect each other, they must do so twice except for points of tangency. The trajectories stemming from any pair of intersections approach each other both in the past and in the future, as they belong to the same branch of both W_s and W_u. For a partition to be generating, it is sufficient that its border separates the two trajectories at least once. Since the same reasoning applies to any pair of intersections, no matter how close they are, it follows that the only way to distinguish the corresponding symbolic sequences is to set the border of the partition exactly on the tangency point, or on some backward (forward) image of it. As long as one limits the analysis to just one fiber, all choices are equivalent. However, the partition of phase space into distinct atoms requires taking all fibers simultaneously into account. As a consequence, one is faced with the problem of identifying the "primary" tangencies as those effectively used to construct the GP. In practice, one starts with an Ansatz about the region which is expected to contain the primary tangencies (typically the folding region of the horseshoe). Then, different tangencies are connected by following a sort of trial and error approach.

The standard map represents a simple but general model for testing methods to analyze Hamiltonian systems. We write the transformation F as

$$x_{n+1}=y_n,$$

$$y_{n+1}=-x_n+2y_n-\alpha\cos(y_n)\bmod 2\pi. \qquad (1)$$

We have chosen the value $\alpha=6$ throughout this paper. The variables x and y have been introduced in place of the commonly used $\theta=x$ and $\rho=y-x$, since the resulting representation guarantees that horizontal lines are mapped onto ver-

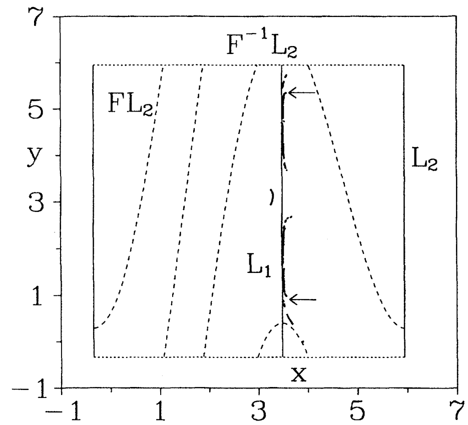

FIG. 1. Approximate generating partition. The primary region of phase space is obtained by using the vertical line L_2 (solid) and its preimage $F^{-1}L_2$ (dotted). This region is then partitioned by L_1 (solid) and the image of L_2 (dashed). Dots denote homoclinic tangencies classified as primary according to their vicinity to L_1. The arrows point to the regions reported in Fig. 2.

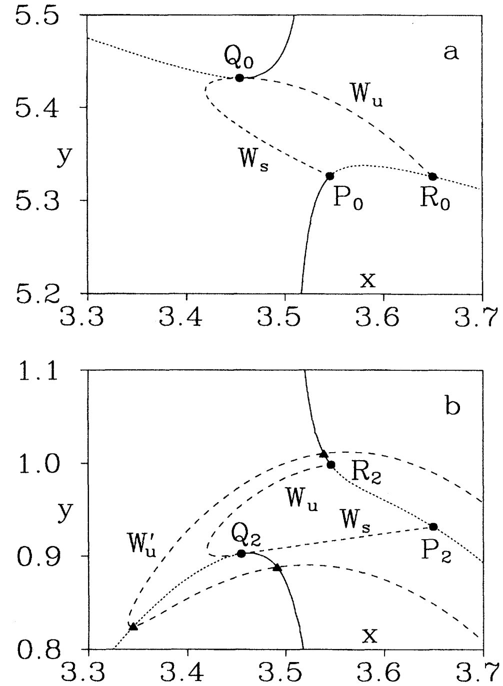

FIG. 2. Enlarged picture of the avoided crossings at the two arrows in Fig. 1; (b) is the second forward image of Fig. 2(a). Solid lines denote those HTs which are necessarily classified as primary. Out of the other tangencies (dotted lines), the points along P_0R_0 may or may not be classified as primary; when not, their second image, lying on R_2P_2, must be taken as primary. Q_0 and its second image Q_2 are identified as the points where two sequences of HTs meet and collapse. The stable and unstable manifolds (dashed lines) departing from Q_0 (Q_2) intersect the strand of HTs in P_0 (P_2) and R_0 (R_2), respectively. A branch of the unstable manifold containing three tangencies (triangles) is also shown in (b).

tical lines, thus making the partition look more natural. Let us note that map (1) is invariant under the composition of a time reversal plus the exchange of x and y variables, and under the transformation $(x,y)\rightarrow(\pi-x,\pi-y)(\mathrm{mod}\ 2\pi)$.

The map exhibits folding regions situated approximately at the vertical lines defined by $x=\sin^{-1}(-2/\alpha)$. We specifically choose the two lines L_1 ($x=3.481\ldots$) and L_2 ($x=5.943\ldots$) to be the basis for the construction of an approximate generating partition.

Since the phase space is a torus, there are no natural boundaries along both the x and y directions. One must, therefore, break the continuity by introducing two sets of transversal lines separated by a distance 2π horizontally and vertically, respectively. This can, for instance, be done by using the vertical line L_2 and its horizontal preimage $F^{-1}L_2$. As a result, the plane is partitioned into infinitely many equivalent squares S. Any other pair of transversal lines is, in principle, equivalent; the idea of using a folding line such as L_2 is inspired by an attempt to minimize the number of partition elements. If the second folding line L_1 is also used, then S is split into two elements. The resulting partition is not sufficiently fine grained to account for the multiplicity of trajectories generated by map (1). In fact, the (pre)images of the two elements intersect different copies of S. One is therefore led to split each of the two elements into as many atoms as the number of copies of S which are visited. This is automatically obtained by using FL_2 as a further dividing line. As a result, one obtains a partition which should be approximately generating (see Fig. 1). In fact, a check done with periodic orbits of increasing period shows that a large fraction of them is correctly discriminated by the above partition. There are, however, a number of orbits described by the same code. This problem is not at all unexpected, since the partition has been constructed starting from rather arbitrary lines identified by just looking at one application of the map; it is well known that a HT involves an infinity of steps.

Before starting the discussion about the refinement of L_1 and L_2, let us notice that the line L_2 can be transformed into L_1 by exploiting the symmetry of map (1). We will, therefore, study only one folding line, namely, L_1. Moreover, since any piece of unstable manifold eventually fills the chaotic component densely, we can restrict ourselves to the unstable manifold W_u of the hyperbolic fixed point $O=(\pi/2,\pi/2)$. The manifold is formally parametrized as $W_u=(x_u(s),y_u(s))$ and the functions $x_u(s)$, $y_u(s)$ are expanded in a power series of s. The coefficients of the series can be determined by demanding that the curve is left invariant by the map, i.e., $F(x_u(s),y_u(s))=(x_u(\lambda s),y_u(\lambda s))$, where λ is the unstable eigenvalue of the stability matrix for the fixed point $O=(x_u(0),y_u(0))$. HTs can be located by iterating a piece of W_u until it shows a notable curvature, which happens in the vicinity of L_1 (L_2). The precise location of a HT is then determined by monitoring the curvature of forward iterates of W_u [7]. In fact, a HT is just a folding point, i.e., it is characterized by a diverging curvature.

The above procedure leads to a tentative set of primary HTs which are seen to align approximately along L_1. In analogy with dissipative maps, it appears natural to connect such points in ascending order, according to their y coordinate. Although the resulting curve is in some places relatively smooth, discontinuities are clearly visible. In dissipative maps, this is not considered to be a serious problem. The fact that the attractor does not fill the whole phase space gives one a large degree of freedom in connecting HTs that are far apart, as long as they are not separated by pieces of the attractor. This is no longer true in a conservative map, where the entire phase space is typically filled by a single ergodic component (with the exception of stability islands which need to be considered separately).

In order to better clarify what happens around each discontinuity, let us look closer at one example, namely, the pair of jumps indicated by arrows in Fig. 1 and enlarged in Fig. 2 [the region in Fig. 2(b) is the second iterate of that depicted in Fig. 2(a)]. We realize that the jump is the consequence of an avoided crossing between two lines of HTs. The discontinuity is in fact caused by the intersection of what will become a border of our generating partition with a forward or backward image of itself. Such a phenomenon is clearly seen in Fig. 2 where forward and backward images (dotted lines) of the "primary" tangencies (solid lines) have been added. Therefore one is faced with the question of which HTs should be used to discriminate different trajectories. Some degree of arbitrariness is apparent for tangencies which return back to the folding region. In principle, discontinuities arising from the intersections of a dividing line with forward and backward images of itself are present everywhere, but the jumps appear to diminish with the respective number of iterates needed to return to the folding region. We can therefore attack this problem starting from the larger gaps.

In Fig. 2(b) it is seen that three distinct tangencies are identified on those fibers of W_u which are not too close to the jump. The first and the last of such points are unambiguously classified as primary points, whereas the middle one corresponds to the second iterate of a tangency classified as primary [in Fig. 2(a)]. Upon shifting the fiber of reference towards the critical region, the two lower HTs meet and eventually disappear, preventing a continuation of the dividing line. This process was already discovered in dissipative maps upon changing a control parameter [9]. In particular, it was shown how it is associated with the difficulty of providing a unique characterization of the symbolic encoding of periodic orbits [4]. In a conservative system, like the standard map under investigation, the same problem occurs for any parameter value, since moving with continuity across the fibers of W_u is like changing a parameter of the dynamics.

From the point Q_2, where two strands of HTs collide, one would like to find a way to connect the partition to the nearby sequence of HTs, thus bridging the gap arising from the apparent avoided crossing. Let us focus our attention on the closed region U delimited by the line of HTs between R_2 and P_2 and by the fibers of stable and unstable manifold departing from Q_2. With reference to Fig. 2, it is seen that trajectories visiting U can be discriminated against companion orbits (lying on the opposite side of a dividing line) either when they lie in $F^{-2}U$, or when they are in U itself. We conjecture that any curve C lying in U and connecting Q_2 with a point S on the strand of HTs between P_2 and R_2 is appropriate, provided that $F^{-2}C$ also is used in $F^{-2}U$ in a self-consistent manner. Two of the infinitely many possible choices for C appear to be most natural: W_s and W_u themselves. This same ambiguity arises for any point on the dividing line which returns to the folding region. Thus we have

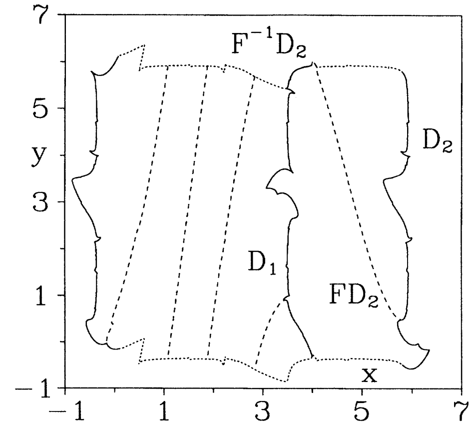

FIG. 3. The generating partition as constructed from primary HTs and suitable pieces of unstable manifolds. In analogy to Fig. 1 we have used the dividing line D_2 (solid) and its preimage (dotted) to define the primary region of phase space. This is then partitioned by D_1 (solid) and the image of D_2 (dashed).

an infinity of bubbles analogous to U. It is therefore convenient to adopt everywhere the same choice. The line D_1 resulting from the application of this procedure to the larger gaps is plotted in Fig. 3, where fibers of the unstable manifold have been used.

A generating partition can be constructed by using D_1 and its symmetric equivalent D_2 analogously to the construction of the preliminary partition from the lines L_1 and L_2. This finally results in a seven letter alphabet as shown in Fig. 3. We have tested the partition on all periodic orbits, both stable and unstable, up to length 9 ($\approx$30 000 orbits), and found that the symbol sequences were unique except for a period-6 orbit and four period-8 orbits around a stable period-2 region, sharing that of the mother orbit. A correct encoding of such orbits requires an *ad hoc* treatment of the corresponding stability island [8].

From the existence of seven different period-2 orbits, it turns out that at least five symbols are needed for a correct encoding of the dynamics. One might try to combine some of the atoms of the seven letter alphabet of Fig. 3 into larger elements. However, from the study of all possible combinations of atoms it is verified that only the triangular region

TABLE I. Block entropies H_k defined in Eq. (2) for lengths $k \leq 11$.

k	H_k
1	1.77292
2	1.69554
3	1.59844
4	1.52601
5	1.47195
6	1.43028
7	1.39680
8	1.36969
9	1.3475
10	1.326
11	1.312

appearing at small y values can be assimilated to another region without loss of information. It is therefore very likely that 6 represents the minimum number of symbols.

A further check of the correctness of the partition has been performed by estimating the Kolmogorov-Sinai entropy from the block entropies

$$H_k = -\sum_i p_i(k) \ln p_i(k), \qquad (2)$$

where $p_i(k)$ is the probability of the ith symbol sequence of length k and the sum is taken over all sequences of length k. The block entropies computed for $k \leqslant 11$ and reported in Table I appear to converge slower than exponentially and faster than algebraically. By taking this into account, the extrapolated value of H_∞ is in good agreement with the numerical estimate of the maximum Lyapunov exponent $\lambda \simeq 1.1365\ldots$, as expected from the Pesin relation.

We can therefore conclude that the partition constructed in this paper is a good generating partition for the main ergodic component. Moreover, since we have not made use of the specific structure of the standard map other than for symmetry reasons, we believe that our approach can be applied to any two-dimensional conservative map and even in principle to Hamiltonian ordinary differential equations. The last problem which still remains to be solved concerns the construction a unique approach encompassing both the method described here and winding-number-type arguments for stable islands.

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