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General approach to diffusion of periodically kicked charges in a magnetic field

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The chaotic diffusion of periodically kicked charges in a uniform magnetic field is systematically approached by considering general values of a conserved quantity for the problem, namely the coordinate x_c of the orbit center. The dependence of the diffusion coefficient D on the correlation function C is explicitly given. Assuming "crystalline" resonance conditions, exact closed expressions are derived for C, both at fixed x_c and averaged over x_c. This averaging removes much of the rich structure of $D(K)$ (K is the kicking parameter), found at fixed x_c.

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The interaction of charged particles with an electrostatic wave packet in a magnetic field is a problem of central importance in plasma physics [1]. A case of major interest is that of a uniform magnetic field perpendicular to the direction of propagation of the wave packet. The presence of chaos in this case leads to "stochastic-resonance" heating of the plasma particles and to the appearance of a nonlinear mechanism of damping of the wave packet [1,2]. A basic investigation of this chaotic motion was initiated in the last decade [2–4]. The model system used was that corresponding to a wave packet with a well-defined wave vector $\mathbf{k}$ and time periodic with a very broad Fourier spectrum. The time dependence can then be approximated by a periodic δ function, and the general Hamiltonian describing the system is

$$H=\Pi^2/(2M)+KV(kx)\sum_{s=-\infty}^{\infty}\delta(t-sT). \qquad (1)$$

Here $\mathbf{\Pi}=\mathbf{p}-q\mathbf{A}/c$ is the kinetic momentum of a particle with charge q and mass M in a uniform magnetic field $\mathbf{B}$ (along the z axis), K is a parameter, $\mathbf{k}$ is taken in the x direction, V is a general periodic function with period 2π, and T is the time period. Without loss of generality, the values of M and k will both be set to 1 from now on.

Before discussing the known properties of the system (1) and stating the aims of this paper, let us first represent (1) in a different form. The natural degrees of freedom in a magnetic field are given by [5] the conjugate pairs $(x_c,\ y_c)$ (coordinates of the center of a cyclotron orbit) and $(\Pi_x,\ \Pi_y)$. Defining [2] $u=\Pi_x/|\omega|$, $v=\Pi_y/\omega$, where $\omega=qB/c$ is the cyclotron frequency, and using the relation $x_c=x+\Pi_y/\omega=x+v$ (easily derivable from simple geometry), (1) can be rewritten as follows:

$$H=\omega^2(u^2+v^2)/2+KV(x_c-v)\sum_{s=-\infty}^{\infty}\delta(t-sT). \qquad (2)$$

Since H does not depend on y_c, x_c is *conserved*. One may therefore treat x_c in (2) as a parameter, thus reducing the system to a periodically kicked harmonic oscillator.

As far as we are aware, all the investigations of the system (1) have assumed the very specific value $x_c=0$ in (2) [2] and an *even* function $V(x)$. In most cases, the standard choice $V(x)=-\cos x$ was made [2], and other choices of even $V(x)$ were studied in Ref. [6]. These investigations have led to the discovery of the well-known properties of this system. Since the harmonic oscillator is degenerate (linear in the action), the nonlinear perturbation in (2) is strong [in the sense of Kol'mogorov-Arnol'd-Moser (KAM) theory] for all values of K, especially under resonance conditions, $\alpha\equiv\omega T=2\pi m/n$ (m and n are coprime integers). One then expects, on the basis of general arguments [2], that unbounded chaotic motion of $(u,\ v)$ should exist for arbitrarily small values of K in the resonance case. This motion is observed to take place diffusively [2–4] on a "stochastic web" [see Fig. 1(a)], analogous in some aspects to the Arnol'd web [7]. For $n=3$, 4, 6, the web has crystalline symmetry (triangular, square, hexagonal), while for all other values of $n>4$ it has quasicrystalline symmetry.

However, assuming a specific value for x_c is quite restrictive for several reasons. First, it is obviously impossible to prepare an ensemble of particles all having the same value of x_c. In general, such an ensemble will exhibit all values of $x_c \bmod 2\pi$ in the interval of periodicity $[0,\ 2\pi)$ of $V(x)$. Second, while the basic symmetry of the stochastic web is completely determined by α, its actual structure and dynamics (at fixed K) may drastically change as x_c is varied. This is clearly illustrated in Fig. 1. Notice that if $V(x)$ is even (as, apparently, in all cases considered in the literature), $V(x_c-v)$ in (2) is generally not even. In particular, for $V(x)=-\cos x$ and $x_c=\pi/2$ [see Fig. 1(c)], $V(x_c-v)=-\sin v$, an odd function. As pointed out recently [8], both the classical and quantum dynamics of (2) for odd $V(x_c-v)$ are significantly different than for even $V(x_c-v)$. Finally, it is important to realize that any small y-dependent perturbation in (1) (corresponding, e.g., to a second drift wave [1]) will lead to the destruction of the conserved quantity x_c, and thus to a full four dimensional (4D) phase-space diffusion. If the perturbation is sufficiently small, this diffusion may be approximately decomposed into two components: (i) a slow diffusion in x_c; (ii) a fast diffusion within the web associated with an "instantaneous" value of x_c.

Because of all these reasons, a systematic approach to the chaotic diffusion in the system (1) should take into consid-

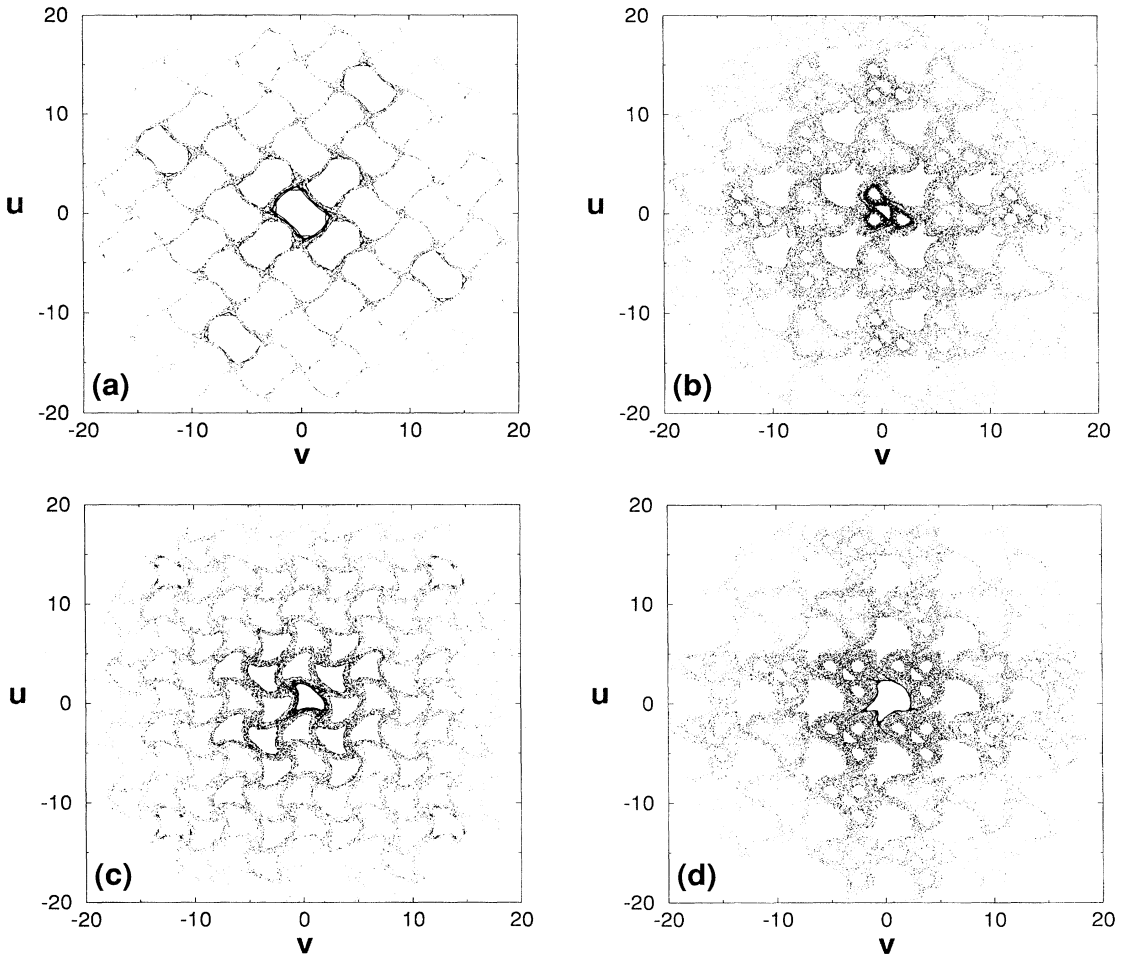

FIG. 1. Portions of the stochastic webs for $m/n=1/4$, $V(x)=-\cos x$, $K=1.4$, and several values of x_c: (a) $x_c=0$, (b) $x_c=2\pi/8$, (c) $x_c=4\pi/8$, (d) $x_c=6\pi/8$. Each of these plots contains 40 000 points of chaotic orbits, generated by iterating an ensemble of initial conditions near the origin with the Poincaré map (3). Notice the diffusive decay away from the origin, and the crystalline square symmetry shared by all the webs.

eration *general* values of x_c, as well as the problem of averaging physical quantities, e.g., the diffusion coefficient, over x_c. Such an approach is introduced in this Rapid Communication. We start by writing the general Poincaré map for (2). Assuming, without loss of generality, $\omega=1$ from now on, this map is easily derived as in Ref. [2]:

$$\begin{aligned} u_{s+1}&=[u_s+Kg(v_s)]\cos\alpha+v_s\sin\alpha, \\ v_{s+1}&=-[u_s+Kg(v_s)]\sin\alpha+v_s\cos\alpha, \end{aligned} \tag{3}$$

where $(u_s,\ v_s)$ are the values of $(u,\ v)$ at time $sT-0$, and $g(v)\equiv f(x_c-v)$, with $f(x)=-dV(x)/dx$, is the force function. Defining $z_s=u_s+iv_s$, the map (3) can be written more compactly as

$$\begin{aligned} z_{s+1}&=[z_s+Kg(v_s)]e^{-i\alpha}=z_0e^{-i\alpha(s+1)} \\ &\quad +K\sum_{j=1}^{s+1} g(v_{s+1-j})e^{-ij\alpha}, \end{aligned} \tag{4}$$

after iterating backward in time. Under resonance conditions, $\alpha=2\pi m/n$, the resonance is fully realized every n iterations. The diffusion coefficient is then given by

$$D=\lim_{b\to\infty}\frac{1}{2bn}\langle|z_{bn}-z_0|^2\rangle_E\ , \tag{5}$$

where b is an integer, and $\langle\ \rangle_E$ means averaging over a large ensemble $E=\{(u_0,\ v_0)\}$ of initial conditions. Cases of anomalous diffusion [4,9] are $D=0$ (subdiffusion) and $D=\infty$ (superdiffusion). Using (4) with $s=bn-1$ and $j=rn+l$ $(r=0,\ldots,b-1;\ l=0,\ldots,n-1)$, Eq. (5) can be expressed as follows:

$$D=\lim_{b\to\infty}\frac{K^2}{2bn}\sum_{l,l'=0}^{n-1} e^{i\alpha(l-l')}\sum_{r,r'=0}^{b-1}\langle g(v_{rn+l})g(v_{r'n+l'})\rangle_E. \tag{6}$$

As in the usual approach to the calculation of D [10–12], we shall assume that the ensemble E is (essentially) invariant under the map (3). For example, E may be chosen as a set of infinitely long chaotic orbits [11], or as a set of unstable periodic orbits and accelerator modes in the chaotic region [12] with maximal period bn [giving a good approximation to D in (6) if b is sufficiently large]. For crystalline webs, E may be chosen as one unit cell (see below) [13]. When E is invariant, the average in (6) depends only on the difference $(r'-r)n+l'-l$, and (6) thus reduces to

$$D=K^2\left[C_0/2+\sum_{l=1}^{n-1}\cos(l\alpha)C_l+\sum_{l=0}^{n-1}\cos(l\alpha)\sum_{r=1}^{\infty}C_{rn+l}\right], \tag{7}$$

where $C_j=\langle g(v_0)g(v_j)\rangle_E$ is the force-force correlation function. As in Ref. [10], relation (7) follows exactly from (6) only if C_j falls off with j faster than j^{-1}.

In the case of crystalline webs, one can obtain explicit expressions for C_j. We define, as in the case of Taylor-Chirikov maps [10], the characteristic functions

$$\chi_j(p_0,p_1,\ldots,p_j)=\left\langle \exp\left(i\sum_{l=0}^{j}p_l v_l\right)\right\rangle_E , \tag{8}$$

where p_l, $l=0,\ldots,j$, are integers. The correlation function C_j may then be expressed as follows:

$$C_j=\sum_{r,r'=-\infty}^{\infty} g_r g_{r'}\chi_j(r,0,\ldots,0,r'), \tag{9}$$

where $g_r=(2\pi)^{-1}\int_0^{2\pi}dv\exp(-irv)g(v)$ are the Fourier coefficients of $g(v)$, and in χ_j only $p_0=r$ and $p_j=r'$ are nonzero. To obtain an explicit expression for this χ_j, we start from the "Newton" equation for v_s, which is easily derived from (3):

$$v_{s+1}-\eta v_s+v_{s-1}=\epsilon g(v_s), \tag{10}$$

where $\eta=2\cos\alpha$ and $\epsilon=-K\sin\alpha$. Denoting by $G_r(\epsilon)$ the Fourier coefficients of $\exp[i\epsilon g(v)]$ (defined as g_r above), we get from (8) and (10) the recursion relation

$$\chi_j(p_0,0,\ldots,0,p_{j-1},p_j)=\sum_{r=-\infty}^{\infty}G_r(p_j\epsilon)\chi_{j-1}(p_0,0,\ldots,0,-p_j,p_{j-1}+\eta p_j+r). \tag{11}$$

Notice that relation (11) involves characteristic functions with only the first and last two indices p_l nonzero. Since the p_l's are general integers, relation (11) is well defined only if η is an integer, which corresponds precisely to the crystalline webs ($\eta=0$ for $n=4$ and $\eta=\mp1$ for $n=3$, 6). The simplest choice for the invariant ensemble E in the crystalline case is just the unit cell of the crystal [13]. With this choice, which we shall adopt from now on, one can easily verify that $\chi_1(p_0,p_1)=\delta_{p_0,0}\delta_{p_1,0}$ for all the crystalline webs. Using this result, and iterating (11) backward, we get the formula

$$\chi_j(p_0,0,\ldots,0,p_j)=\sum_{r_1=-\infty}^{\infty}\cdots\sum_{r_{j-3}=-\infty}^{\infty}\prod_{l=0}^{j-2}G_{r_{l+1}-\eta r_l+r_{l-1}}(r_l\epsilon), \tag{12}$$

with $r_{-1}=r_{j-1}=0$, $r_0=p_j$, and $r_{j-2}=p_0$. A simple but crucial observation is now that the x_c dependence of the Fourier coefficients g_r and $G_r(\epsilon)$ (see definitions above) can be explicitly written from the definition $g(v)=f(x_c-v)$:

$$g_r=f_r e^{-irx_c},\quad G_r(\epsilon)=F_r(\epsilon)e^{-irx_c}, \tag{13}$$

where f_r and $F_r(\epsilon)$ are, respectively, the Fourier coefficients of $f(-v)$ and $\exp[i\epsilon f(-v)]$. Using (12) and (13) in (9), we obtain, after a straightforward calculation, the following closed expression for C_j:

$$C_j=\sum_{r_0=-\infty}^{\infty}\cdots\sum_{r_{j-2}=-\infty}^{\infty}f_{r_0}f_{r_{j-2}}\exp\left[-i(2-\eta)\sum_{l=0}^{j-2}r_l x_c\right]\times\prod_{l=0}^{j-2}F_{r_{l+1}-\eta r_l+r_{l-1}}(r_l\epsilon). \tag{14}$$

Because of the relatively simple dependence of C_j on x_c in (14), averages of C_j and of D [see (7)] over x_c can easily be performed in several cases. The simplest and most natural case is that of uniform average. This gives our main result

$$\bar{C}_j=\frac{1}{2\pi}\int_0^{2\pi}dx_c C_j=\sum_{r_0=-\infty}^{\infty}\cdots\sum_{r_{j-2}=-\infty}^{\infty}\Delta\left(\sum_{l=0}^{j-2}r_l\right)f_{r_0}f_{r_{j-2}}\times\prod_{l=0}^{j-2}F_{r_{l+1}-\eta r_l+r_{l-1}}(r_l\epsilon), \tag{15}$$

where $\Delta(r)=\delta_{r,0}$ is the Kronecker function. The average diffusion coefficient D_{av} is then obtained from (7) with $\bar{C}_j$ replacing C_j. Because of the restriction $\Sigma_{l=0}^{j-2}r_l=0$, the computation of $\bar{C}_j$ using (15) is usually shorter than the computation of C_j.

As an example, consider the case of $V(x)=-\cos x$ [$f(-v)=\sin v$]. In this case, one immediately obtains from the definition of C_j that $C_0=1/2$ and $C_1=0$ for all x_c, so that also $\bar{C}_0=1/2$ and $\bar{C}_1=0$. The next two values of C_j (for $x_c=0$) and $\bar{C}_j$ are easily calculated from (14) and (15):

$$C_2=-\frac{1}{2}J_{-\eta}(\epsilon),\quad C_3=\frac{1}{2}[J_{1+\eta}^2(\epsilon)-J_{1-\eta}^2(\epsilon)],$$

$$\bar{C}_2=0,\quad \bar{C}_3=\frac{1}{2}J_{1+\eta}^2(\epsilon), \tag{16}$$

where $J_l(\epsilon)$ is a Bessel function. Results for the diffusion coefficient, both at $x_c=0$ and averaged over x_c, are shown in Figs. 2 and 3 for $m/n=1/4$ (square symmetry) [14]. The

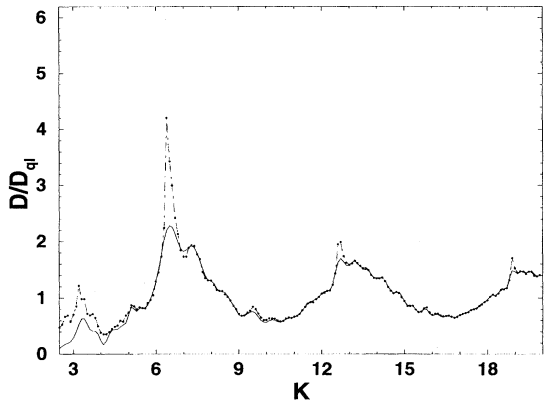

FIG. 2. Solid line: D/D_{ql} as a function of K for $m/n=1/4$, $V(x)=-\cos x$, and $x_c=0$. Here D is calculated using (7) up to, and including $C_{j=6}$, as explained in the text. Dot-dashed line: Numerical results obtained from (5) by iterating 400 times with the map (3) ($x_c=0$) a 200×200 ensemble of chaotic initial conditions around a hyperbolic fixed point.

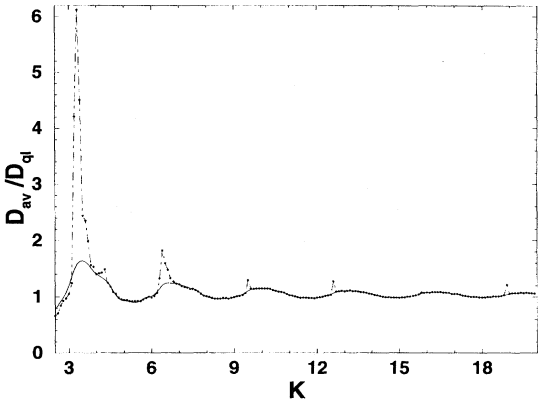

FIG. 3. Solid line: D_{av}/D_{ql} as a function of K for $m/n=1/4$ and $V(x)=-\cos x$. Here D_{av} is calculated using (7) up to, and including $\bar{C}_{j=8}$ (see text). Dot-dashed line: Numerical results obtained from (5) by iterating 1200 times with (3) a 50×50 ensemble E uniformly distributed in a unit cell, and averaging over 50 values of x_c uniformly distributed in $[0,\ 2\pi)$.

values of D and D_{av} are normalized by the "quasilinear" value $D_{ql}=K^2/4$, corresponding to the leading term $C_0/2$ in (7). In Fig. 2, we plot D/D_{ql} as a function of K, where D is calculated using the expansion (7) up to, and including $C_{j=6}$ (i.e., $r=1$ and $l=2$). Notice that for $m/n=1/4$ only correlation functions C_j with even j appear in (7), since $\cos(l\alpha)=0$ for odd l. The values of C_j (and $\bar{C}_j$) for $j>3$, not given in (16), were calculated from (14) [and (15)] by keeping in the sum only terms contributing factors $F_r(r'\epsilon)=J_r(r'\epsilon)$ with $r<20$. The results are compared with numerical results of D, obtained from (5) by using an ensemble E of chaotic orbits (see caption). The agreement is very good, except near values of K where accelerator islands are born and are relatively large. Here the chaotic diffusion is actually anomalous (superdiffusion) [9], and $D=\infty$. In Fig. 3, we plot D_{av}/D_{ql} as a function of K, where D_{av} is calculated using (7) up to, and including $\bar{C}_{j=8}$. Again, except near K values where accelerator islands exert influence, these results are in good agreement with those obtained by averaging (5) over an x_c ensemble (see caption). As one could expect, this averaging removes much of the rich structure of $D(K)$ at fixed x_c (see Fig. 2), and $D_{av}(K)$ approaches the quasilinear value more rapidly than $D(K)$ as K is increased.

In conclusion, we have introduced a systematic approach to the problem of chaotic diffusion of periodically kicked charges in a magnetic field. Unlike previous approaches [2–4], we consider here general values of the conserved quantity of the problem, i.e., the coordinate x_c of the orbit center. The dependence of the diffusion coefficient D on x_c can thus be studied, and the averaging of D over x_c, which is naturally required on physical grounds, can be effectively performed.

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[13] In fact, it is easily shown that the nth power of the map (3) in the crystalline case is invariant under translations with lattice vectors defining the unit cell.
[14] The results for D ($x_c=0$), up to, and including C_2, coincide with those obtained in Ref. [4].