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Exact exponent λ of the autocorrelation function for a soluble model of coarsening

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The exponent λ that describes the decay of the autocorrelation function $A(t)$ in a phase ordering system, $A(t)\sim L^{-(d-\lambda)}$, where d is the dimension and L the characteristic length scale at time t, is calculated exactly for the time-dependent Ginzburg-Landau equation in $d=1$. We find $\lambda=0.399\,383\,5\ldots$. We also show explicitly that a small bias of positive domains over negative gives a magnetization which grows in time as $M(t)\sim L^{\mu}$ and prove that for the one-dimensional Ginzburg-Landau equation, $\mu=\lambda$, exemplifying a general result.

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The field of phase ordering kinetics has seen a number of new developments in recent years [1]. In particular, the values of the growth exponents z, which describe the time-dependence of the characteristic scale $L(t)$ via $L\sim t^{1/z}$, are known exactly for most models with purely dissipative dynamics [1,2]. For systems with short-range interactions and dynamics which are either nonconserved or obey a local conservation law, the exponent z is usually a dimension-independent integer [2]. Recently, however, it has been realized that for nonconserved dynamics the description of two-time correlations requires a new exponent, whose dependence on the spatial dimension d and on the symmetry of the order parameter is nontrivial [3,4].

The exponent λ can be defined in terms of the general two-point correlation function $C(r;t_1,t_2)=\langle\phi(\mathbf{x},t_1)\,\phi(\mathbf{x}+\mathbf{r},t_2)\rangle$, where ϕ is the order parameter field. In the scaling regime, this is expected to have the scaling form $C(r;t_1,t_2)=f(r/L_1,r/L_2)$, where L_1, L_2 are the characteristic length scales at times t_1 and t_2 [5,6]. In the limit of well-separated times, $L_2\gg L_1$, one anticipates [5] the power-law form $C(r;t_1,t_2)\sim(L_1/L_2)^{d-\lambda}f(r/L_2)$, defining the exponent λ. An especially simple case is where we take $r=0$, and the initial time $t_1=0$. Then the general form reduces to

$$A(t)\equiv C(0;0,t)\sim[\xi_0/L(t)]^{d-\lambda}\ , \tag{1}$$

where ξ_0 is some fixed length related to the initial conditions. The "autocorrelation function" $A(t)$ has been measured in simulations of O(n) models for various spatial dimensions d, [3,7], and in experiments on twisted nematic liquid crystals films [8], and the exponent λ deduced. It generally has a nontrivial value.

There are a few analytical results for λ– the nonconserved O(n) model for $n=\infty$ ($\lambda=d/2$ [4]), and the $d=1$ Glauber model ($\lambda=0$ [9]), while for nonconserved scalar fields in $d=2$ Fisher and Huse [3] have conjectured that $\lambda=3/4$ exactly. In general, however, λ appears to be a nontrivial exponent associated with ordering dynamics, although it is known to satisfy the bound (in our notation) $\lambda\leq d/2$ for nonconserved dynamics [3,10].

In this paper we calculate λ exactly for a soluble model corresponding to the late-time, zero-temperature coarsening dynamics of the time-dependent Ginzburg-Landau (TDGL) equation for a scalar field in $d=1$. The equation of motion is $\partial_t\phi=\partial_x^2\phi-dV/d\phi$, where $V(\phi)$ is a symmetric, double-well potential with minima at $\phi=\pm1$ [e.g., $V(\phi)=(1-\phi^2)^2$]. At late times, when the mean separation L of domain walls is large compared to their intrinsic width $\xi(=[V''(1)]^{-1/2})$, the walls only interact weakly, through the exponential tails of the wall profile function. Then the dynamics is very simple [11–13]. The closest pair of walls move together and annihilate, while the other walls hardly move at all, and the system coarsens by successively eliminating the smallest domains. It is found that the distribution of domain sizes l approaches a scaling form, $P(l)=L^{-1}f(l/L)$. The scaling function $f(x)$ can be exactly calculated [11–13].

In an earlier work [13], we have shown that there is a nontrivial exponent associated with the fraction of the line that has never been traversed by a domain wall (i.e., the fraction of the line where the order parameter ϕ has never changed its sign [14]). This fraction decays as $L^{-(1-\beta)}$, with $\beta=0.824\,924\,12\ldots$. Here we show that the approach developed in [13] can be generalized to calculate λ for this model. The result is $\lambda=0.399\,383\,5\ldots$. A recent simulation of the same model [15] gave the estimate $\lambda=0.43\pm0.01$, which, we think, is in reasonable agreement with our exact result, given that the extrapolation to large L was not straightforward.

The exponent λ can also be obtained from the rate at which a small initial bias in the order parameter grows with time [16], $\langle\phi\rangle\sim L^{\lambda}$. We demonstrate this explicitly within the present model in the second part of this work.

The calculation of the autocorrelation exponent λ follows closely that presented in Ref. [13]. One starts with random intervals on the line. Each interval I is characterized by its length $l(I)$ and by its overlap $q(I)$ with its initial condition [initially $q(I)=l(I)$ for all I]. At each iteration step, the smallest interval $I_{\min}$ is removed (i.e., the field ϕ is replaced by $-\phi$ in this interval). So three intervals (the smallest interval $I_{\min}$ and its two neighbors I_1 and I_2) are replaced by a single interval I. The length and the overlap of the new interval I are given by

$$l(I)=l(I_1)+l(I_{\min})+l(I_2)\ , \tag{2}$$

$$q(I)=q(I_1)+q(I_2)-q(I_{\min}) \ . \tag{3}$$

Then the average length L of domains and the autocorrelation function A are given by

$$L=\sum_I l(I) \Big/ \sum_I 1 \ , \quad A=\sum_I q(I) \Big/ \sum_I l(I) \ , \tag{4}$$

where the sums are over all the intervals I present in the system.

The argument showing that no correlations develop if none is present initially was given earlier [13] and the calculation is then very similar to that for the evaluation of the exponent β. The only difference is that in the previous work Eq. (3) for the overlap $q(I)$ was replaced by the equation $d(I)=d(I_1)+d(I_2)$ for the length $d(I)$ of interval I that has never been traversed by a domain wall.

We take, for simplicity, the lengths of the intervals to be integers and i_0 to be the minimal length in the system. We also assume that the total number N of intervals is very large. We call n_i the number of intervals of length i and q_i the average overlap of the intervals of length i. At the beginning, $q_i=i$. We denote with a prime the values of these quantities after all the n_{i_0} intervals of length i_0 have been eliminated, so that the minimal length has become i_0+1. Then the time evolution is given by [compare Eq. (2) of [13]]

$$N'=N-2n_{i_0},$$

$$n_i'=n_i\left(1-\frac{2n_{i_0}}{N}\right)+n_{i_0}\sum_{j=i_0}^{i-2i_0}\frac{n_j}{N}\,\frac{n_{i-j-i_0}}{N},$$

$$n_i'q_i'=n_iq_i\left(1-\frac{2n_{i_0}}{N}\right)+n_{i_0}\sum_{j=i_0}^{i-2i_0}\frac{n_j}{N}\,\frac{n_{i-j-i_0}}{N}\,(q_j+q_{i-j-i_0}-q_{i_0}). \tag{5}$$

This is only valid under the condition that $n_{i_0}\ll N$ which is indeed valid when i_0 becomes large and as long as the system consists of a large number of intervals.

We assume that after many iterations, i.e., when i_0 becomes large, a scaling limit is reached where

$$n_i=(N/i_0)f(i/i_0), \quad n_iq_i=N(i_0)^{\lambda-1}g(i/i_0), \tag{6}$$

where λ is the exponent we want to calculate. Because i_0 is so large, we can consider $x=i/i_0$ as a continuous variable. This gives

$$n_i'=[N'/(i_0+1)]f(i/(i_0+1))=\frac{N}{i_0}\left[f(x)-\frac{2}{i_0}f(1)f(x)-\frac{1}{i_0}f(x)-\frac{1}{i_0}xf'(x)\right], \tag{7}$$

$$n_i'q_i'=N'(i_0+1)^{\lambda-1}g\left(\frac{i}{i_0+1}\right)=Ni_0^{\lambda-1}\left[g(x)-\frac{2}{i_0}f(1)g(x)+\frac{\lambda-1}{i_0}g(x)-\frac{1}{i_0}xg'(x)\right] \ . \tag{8}$$

Inserting these expressions in the time evolution equations (5) gives

$$i_0\frac{\partial f}{\partial i_0}=f(x)+xf'(x)+\theta(x-3)f(1)\times\int_1^{x-2}dy\ f(y)f(x-y-1),$$

$$i_0\frac{\partial g}{\partial i_0}=(1-\lambda)g(x)+xg'(x)+2\theta(x-3)f(1)\times\int_1^{x-2}dy\ g(y)f(x-y-1)-g(1)\theta(x-3)\times\int_1^{x-2}dy\ f(y)f(x-y-1). \tag{9}$$

In (6), both n_i and n_iq_i are functions of $x=i/i_0$ and of i_0, and the partial derivatives in (9) mean the derivative with respect to i_0, keeping x fixed. Demanding that the system is self-similar, i.e., that the functions $f(x)$ and $g(x)$ do not change with time [i.e., replacing the left-hand sides of (9) by zero], one finds that the Laplace transforms

$$\phi(p)=\int_1^\infty e^{-px}f(x)dx, \quad \psi(p)=\int_1^\infty e^{-px}g(x)dx, \tag{10}$$

satisfy the following equations (where primes now indicate derivatives):

$$-f(1)e^{-p}-p\phi'(p)+f(1)e^{-p}\phi^2(p)=0 \ , \tag{11}$$

$$-\lambda\psi(p)-g(1)e^{-p}-p\psi'(p)+2f(1)e^{-p}\phi(p)\psi(p)-g(1)e^{-p}\phi^2(p)=0 \ . \tag{12}$$

Defining the function $h(p)$ by

$$h(p)=2f(1)\int_p^\infty\frac{e^{-t}}{t}\,dt \ , \tag{13}$$

the solutions of the above equations are

$$\phi(p)=\tanh[h(p)/2], \tag{14}$$

$$\psi(p)=g(1)\int_p^\infty[1+\phi^2(q)]\frac{1-\phi^2(p)}{1-\phi^2(q)}\,\frac{q^{\lambda-1}}{p^\lambda}\,e^{-q}dq \ . \tag{15}$$

The constants of integration implied by these forms were fixed by the requirement that both ϕ and ψ decay fast enough for large p, as is clear from the definitions (10). So far the parameters $f(1)$, $g(1)$, and λ are arbitrary. We shall see that they are fixed by physical considerations.

Equation (14) for ϕ, which determines the domain size distribution, is of course identical to that obtained in previous work [11–13]. Equation (15) for ψ can be rewritten in the more convenient form

$$\psi(p)=2g(1)\int_p^\infty\frac{e^{h(q)}+e^{-h(q)}}{e^{h(p)}+2+e^{-h(p)}}\,\frac{q^{\lambda-1}}{p^\lambda}\,e^{-q}dq \ . \tag{16}$$

It is helpful to introduce the expansion

$$\int_p^\infty\frac{e^{-q}}{q}\,dq=-\ln p-\gamma-\sum_{n=1}^\infty\frac{(-p)^n}{n\ n!} \ , \tag{17}$$

where $\gamma=-\int_0^\infty dt\ e^{-t}\ \ln t=0.577\,215\,6\ldots$ is Euler's constant.

From the small-p expansion of (14), it is easy to show that, provided the first moment of the domain size distribution exists, one must have $f(1)=1/2$ [11–13]. From now on,

we will consider only this case (see [17] for the discussion of cases where the stationary distribution has long tails). Defining the function $r(p)$ by

$$r(p)=h(p)+\ln p=\int_p^\infty \frac{e^{-q}}{q}\,dq+\ln p \ , \tag{18}$$

one obtains, using (16),

$$\psi(p)=2g(1)\int_p^\infty \frac{e^{r(q)}+q^2e^{-r(q)}}{e^{r(p)}+2p+p^2e^{-r(p)}}\,\frac{q^{\lambda-2}}{p^{\lambda-1}}\,e^{-q}dq \ . \tag{19}$$

Now $r(p)$ can be expanded in powers of p, using (17), and so this last form makes it easier to analyze the singular behavior of $\psi(p)$ at $p=0$. One finds that, for small p,

$$\psi(p)=A+Bp^{1-\lambda}+O(p) \ , \tag{20}$$

where $A=2g(1)/(1-\lambda)$ and

$$\begin{aligned}B&=2g(1)e^{-r(0)}\left[\int_0^\infty \frac{q^{\lambda-1}e^{-q}}{1-\lambda}[r'(q)-1]e^{r(q)}dq\right.\\&\quad\left.+\int_0^\infty q^\lambda e^{-q}e^{-r(q)}dq\right]\\&=2g(1)e^{\gamma}(1-\lambda)^{-1}\int_0^\infty q^{\lambda-2}e^{-q}[(1-q-e^{-q})e^{r(q)}\\&\quad+q^2(1-\lambda)e^{-r(q)}]dq \ .\end{aligned} \tag{21}$$

Now compare (20) with a direct expansion of (10), namely, $\psi(p)=\int_1^\infty dx g(x)[1-px+O(p^2)]$. If the function $g(x)$ is to have a finite first moment then we must have $B=0$ in (20). This condition determines λ as

$$\lambda=0.399\,383\,5\ldots . \tag{22}$$

From numerical simulations of the same model, Majumdar and Huse [15] found the power-law decay $A(t)\sim L^{-\tilde{\lambda}}$, with $\tilde{\lambda}=0.57\pm0.01$, corresponding to $\lambda\equiv d-\tilde{\lambda}=0.43\pm0.01$. There were, however, large corrections to scaling in their numerical data, which we think are the origin of the disagreement between their numerical estimate and our exact result.

As in [13], one can show that $B\neq0$ would correspond to a power-law decay in $g(x)$ and that such a power law cannot be produced if it is not present in the initial condition. Note that $g(1)$ cannot be determined as one can always multiply all the q_i by a constant without changing our results.

For the remainder of this paper we will look at a related quantity, the growth of an initially small bias in the order parameter, and show that the bias grows as L^μ as the system coarsens (while the bias remains small). Furthermore, we will show explicitly that $\mu=\lambda$ for this model, exemplifying a general result [16].

Consider a sequence of positive and negative domains on a line. We call n_i (m_i) the number of positive (negative) domains of length i. The total number N of positive domains is of course equal to the total number of negative domains, $N=\Sigma_i n_i=\Sigma_i m_i$. When the domains of size i_0 are removed, the new values of n_i, m_i, and N are given by

$$n_i'=\left(1-\frac{2m_{i_0}}{N}\right)+m_{i_0}\sum_{j=i_0}^{i-2i_0}\frac{n_j\ n_{i-j-i_0}}{N^2} \ ,$$

$$m_i'=\left(1-\frac{2n_{i_0}}{N}\right)+n_{i_0}\sum_{j=i_0}^{i-2i_0}\frac{m_j\ m_{i-j-i_0}}{N^2} \ , \tag{23}$$

$$N'=N-n_{i_0}-m_{i_0} \ .$$

Let us write forms for n_i and m_i analogous to the first of equations (6):

$$n_i=(N/i_0)f_1(i/i_0), \quad m_i=(N/i_0)f_2(i/i_0). \tag{24}$$

Then one has

$$n_i'=[(N-n_{i_0}-m_{i_0})/(i_0+1)]f_1(i/(i_0+1)), \tag{25}$$

which gives, for i_0 large (when $x=i/i_0$ can be treated as a continuous variable),

$$\begin{aligned}n_i'=(N/i_0)[&f_1(x)+(1/i_0)\{-f_1(1)f_1(x)-f_2(1)f_1(x)\\&-f_1(x)-xf_1'(x)\}],\end{aligned} \tag{26}$$

and a similar expression for m_i'.

Inserting the forms (24) into (26) gives coupled evolution equations for f_1 and f_2:

$$\begin{aligned}i_0\frac{\partial f_1(x)}{\partial i_0}&=[f_1(1)-f_2(1)]f_1(x)+f_1(x)+xf_1'(x)\\&\quad+\theta(x-3)f_2(1)\int_1^{x-2}dy\ f_1(y)f_1(x-y-1),\end{aligned} \tag{27}$$

and a second equation obtained by interchanging the subscripts 1 and 2. Note that the derivatives on the left-hand sides are with respect to the (implicit) second argument i_0. Introducing the Laplace transforms with respect to the first argument,

$$\psi_n(p)=\int_1^\infty f_n(x)e^{-px}dx \quad (n=1,2) \ , \tag{28}$$

one finds that their evolution is given by

$$\begin{aligned}i_0[\partial\psi_1(p)/\partial i_0]=&[f_1(1)-f_2(1)]\psi_1(p)-p\psi_1'(p)\\&-f_1(1)e^{-p}+f_2(1)e^{-p}\psi_1^2(p) \ ,\end{aligned} \tag{29}$$

and a second equation with subscripts 1 and 2 interchanged.

So far this is completely general. The basic idea is to perform a linear stability analysis around the "symmetric" solution $\psi_1(p)=\psi_2(p)=\phi(p)$, where $\phi(p)$ satisfies (11) with $f(1)=1/2$, in order to determine the rate at which a small perturbation will grow. We therefore take $\psi_1(p)$ and $\psi_2(p)$ to have the forms

$$\psi_n(p)=\phi(p)\pm\epsilon\sigma(p) \ , \tag{30}$$

with

$$f_n(1)=\tfrac{1}{2}\pm\epsilon a \ , \tag{31}$$

with ϵ small and the + (−) sign corresponding to $n=1$ ($n=2$). If the bias represented by the terms in ϵ is a relevant perturbation, $\sigma(p)$ will grow under iteration: $\sigma\sim(i_0)^\mu$ with $\mu>0$ [and similarly, $a\sim(i_0)^\mu$ in (31)]. Subtracting from (29) its counterpart with subscripts 1 and 2 interchanged, and putting $i_0\,\partial\sigma(p)/\partial i_0=\mu\sigma(p)$, yields the eigenvalue equation

$$\mu\sigma=2a\phi-p\sigma'-ae^{-p}-ae^{-p}\phi^2+e^{-p}\phi\sigma \ , \tag{32}$$

with solution

$$\sigma(p)=a\int_p^\infty \frac{\phi^2(q)e^{-q}+e^{-q}-2\phi(q)}{q}\left(\frac{q}{p}\right)^\mu \frac{1-\phi^2(p)}{1-\phi^2(q)}\,dq \ . \tag{33}$$

The integration constant was fixed as before by the requirement that $\sigma(p)$ decrease as $\exp(-p)/p$ for large p, which follows from (28), (30), and (31). Demanding once more that $\sigma(p)$ be regular at $p=0$ [so that the first moments of $f_1(x)$ and $f_2(x)$ exist] yields the following equation for μ:

$$\int_0^\infty dq[e^{-q}\phi^2(q)+e^{-q}-2\phi(q)]\frac{q^{\mu-1}}{1-\phi^2(q)}=0. \tag{34}$$

Using $\phi(q)=(e^{r(q)}-q)/(e^{r(q)}+q)$, which follows from (14) and (18), gives the condition

$$\int_0^\infty dq[(e^{-q}-1)e^{r(q)}+q^2(e^{-q}+1)e^{-r(q)}]q^{\mu-2}=0 \tag{35}$$

for μ, with solution $\mu\simeq 0.399\,38\ldots$. Comparison with (22) suggests that $\mu=\lambda$. In fact, using integration by parts one can show that condition (35) for μ is identical to (21) (with $B=0$) for λ, and so $\mu=\lambda$ exactly.

The result $\mu=\lambda$ is, in fact, quite general. For TDGL dynamics, it has been discussed elsewhere [16]. Let us derive it for any kind of dynamics of an Ising model. Consider a system of N Ising spins in dimension d. We call $P(\{S_i(t)\}\,|\,\{S_i(0)\})$ the probability of finding the system in the spin configuration $\{S_i(t)\}$ at time t given that it was in configuration $\{S_i(0)\}$ at time 0. We assume that the system evolves in a zero magnetic field and that the dynamics preserves the $\pm$ symmetry, namely, $P(\{S_i(t)\}|\{S_i(0)\})=P(\{-S_i(t)\}\,|\,\{-S_i(0)\})$.

Suppose that one starts with an initial condition $\{S_i(0)\}$ chosen completely at random, then the correlation $\langle S_i(t)S_j(0)\rangle$ is given by

$$\langle S_i(t)S_j(0)\rangle=(1/2^N)\sum_{\{S(t)\}}\sum_{\{S(0)\}} S_i(t)S_j(0) \times P(\{S_i(t)\}\,|\,\{S_i(0)\}) \ , \tag{36}$$

where $\Sigma_{\{S(t)\}}$ indicates a sum over the 2^N configurations at time t.

Suppose, on the other hand, that one starts with a weakly magnetized initial condition, i.e., the initial configuration $\{S_i(0)\}$ is chosen with probability

$$Q(\{S_i(0)\})=\prod_{i=1}^N \frac{1+m(0)S_i(0)}{2}\simeq\frac{1+m(0)\sum_j S_j(0)}{2^N}$$

when $m(0)$ is infinitesimal. Then the magnetization $m(t)$ per spin at time t is a function of $m(0)$, and to first order in powers of $m(0)$ one has

$$m(t)=\sum_{\{S(t)\}}\sum_{\{S(0)\}} P(\{S_i(t)\}|\{S_i(0)\})Q(\{S_i(0)\})\frac{\sum_j S_j(t)}{N}$$

$$\simeq m(0)\frac{\sum_i\sum_j\langle S_i(t)S_j(0)\rangle}{N} \ . \tag{37}$$

Therefore if one assumes that due to some coarsening phenomenon the two-point function scales as

$$\langle S_i(0)S_j(t)\rangle\simeq L^{-(d-\lambda)}f(R_{ij}/L)$$

where R_{ij} is the distance between sites i and j, one finds that

$$m(t)\simeq L^\lambda m(0)\int d^dR f(R) \ ,$$

which means that the magnetization and the autocorrelation exponents are the same.

To summarize, we have derived a nontrivial value for the exponent λ within an exactly soluble model, and shown explicitly that the growth of an initial bias in the order parameter is controlled by the same exponent.

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