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Traveling salesman problem and Tsallis statistics

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A generalization of the stochastic method of simulated annealing algorithm based on Tsallis statistics is proposed. This algorithm is considerably faster than the traditional ones in solving the traveling salesman problem. Acceptable solutions are found in fewer steps and higher temperatures than both the classical and the fast simulated annealings. Recent developments in solving *NP*-complete problems can be incorporated and improve the performance even more.

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A class of difficult optimization problems is the so-called *NP* (nonpolynomial) complete. For this class of problems, the computational effort for an exact solution grows more rapidly than a polynomial function of size. A well-known example is "the traveling salesman problem" (TSP) in which one must find the shortest path through N given cities only once. There are $(N-1)!/2$ possible paths to be examined in order to find the global minimum. This large number of candidate solutions is not the only difficulty in this problem. The TSP presents a highly rugged landscape, i.e., there are a great number of local minima of a cost function (the length of the path), hence a gradient descent method will fail by reaching local minima instead of a global one. A strategy widely used consists in perturbing a local minimum following a well-defined strategy, by taking a step away from it and checking whether a new minimum is reached or the system returns to the same minimum.

Methods from statistical physics, such as neural network dynamics [1] and simulated annealing [2], besides others (as genetic algorithm [3]), have been applied to find global minima of nonconvex functions and specifically to the TSP. Simulated annealing, on which this work is based, is one of the most celebrated and efficient methods in the task of finding global minima (or an approximate solution). In physical annealing, a metal, for example, is heated to a high temperature and slowly cooled, allowing the atoms to search for a configuration with lowest energy (crystalline state). However, if the metal is quickly cooled (or "quenched") it will eventually be trapped in a metastable configuration (amorphous or polycrystalline state). Classical simulated annealing (CSA), as proposed by Kirkpatrick *et al.* [2], extended the well-known procedure by Metropolis *et al.* [4] for equilibrium Boltzmann-Gibbs statistics: the system is assumed to change from an energy E_1 to energy E_2 with probability $p=\exp[-(E_2-E_1)/kT]$. It is clear that if $E_2<E_1$ the system will *always* accept the change. The annealing consists in decreasing the temperature gradually. Geman and Geman [5] showed that if the temperature decreases as the inverse logarithm of time, the system will end in a *global* minimum. In the TSP the energy function will be the length of the path.

Recently, Szu and Hartley [6] proposed a more efficient procedure, called fast simulated annealing (FSA), where the cooling can be faster (the temperature decreases as the inverse of time). In the Szu and Hartley recipe, the system can jump around the energy landscape according to a Cauchy-Lorentz visiting distribution, instead of the Gaussian used in the CSA. Thereby, longer jumps are allowed describing a Lévy-flight distribution. Both procedures (FSA and CSA) were generalized by Tsallis and Stariolo [7] (GSA), with the

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additional bonus of providing a quicker algorithm than the FSA. The GSA is based on the Tsallis entropy [8]

$$S_q = k \frac{1 - \sum_i p_i^q}{q-1}, \quad (1)$$

where the $\{p_i\}$ are the probabilities of the microscopic configurations. The Shannon entropy

$$S_1 = -k_B \sum_i p_i \ln p_i \quad (2)$$

is recovered in the $q \to 1$ limit. The relation between Tsallis entropy and Lévy flights was recently established [9], by applying the maximum entropy formalism. Lévy walks with fractal dimension γ are generated using suitable constraints plus the variational principle. The value of q determines the fractal dimension of the Lévy walk. From the Tsallis entropy it also is possible to derive a generalized thermodynamics [10] which elegantly preserves the Legendre-transformation structure. It also claimed to provide the solution for the divergent mass problem in the polytropic model of stellar systems [11]. In view of these results, it is worthwhile to test (1) in optimization problems.

In this paper we present an application of the generalized simulated annealing to a combinatorial optimization problem: the TSP. Since we are interested in studying the enhancements specifically due to Tsallis statistics, we chose to work with the traditional techniques for the TSP such as the Lin-Kernighan method [12,13]. Below, we describe the computational implementation of this problem.

(i) The cities are numbered $i = 1, \ldots, N$. A configuration is described by the order in which the cities are visited. Starting from a given city we have $(N-1)!$ permutations. The number of configurations is half of that number because a configuration and its reverse (the cities in the opposite order) have the same length.

(ii) The rearrangements are performed following an efficient strategy [12,13]. Basically there are two types of trial moves: path reversal, where a section of the path is removed and replaced by the cities in the opposite order, and path segment transport, where a section of the path is removed and replaced at a different location randomly chosen.

(iii) The new length is evaluated. We define ΔE as the difference between the length of the configuration before the trial described above and the length of the rearranged configuration. In the CSA by Kirkpatrick *et al.*, the rearranged configuration is accepted with probability

$$p = \min(1, \exp(-\Delta E/kT)). \quad (3)$$

At each temperature, this is the Metropolis algorithm [4] for equilibrium thermodynamics. Here, we generalize the algorithm, introducing the probability

$$p = \min(1, [1-(1-q)\Delta E/kT]^{1/(1-q)}). \quad (4)$$

We recover (3) in the $q \to 1$ limit. Let us stress here that (4) is different from the one presented by Tsallis and Stariolo [7], which is a heat-bath generalization, and also slightly different from the one used in Monte Carlo simulations of the ferromagnetic two-dimensional Ising model in the Tsallis statistics [14]. In the latter the probability (4) must be raised to the q power, as demanded by the generalized thermodynamics [10].

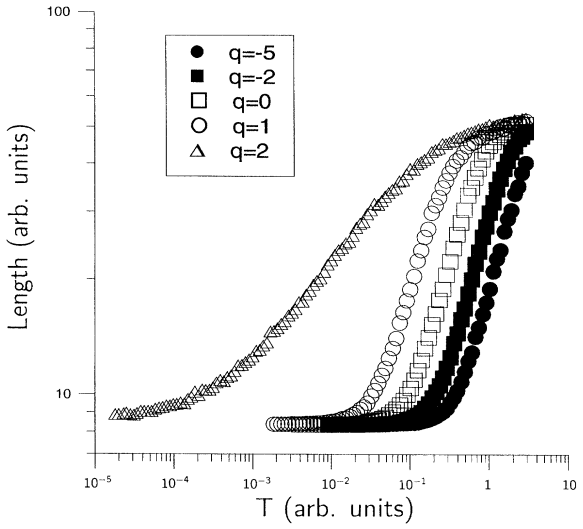

FIG. 1. Average path length versus temperature for 50 different initial configurations of $N = 150$ cities. The $q = 1$ case is CSA, using Boltzmann-Gibbs statistics.

(iv) The annealing schedule starts from a high temperature considerably larger than the largest ΔE. We kept T as constant for $100N$ trial reconfigurations or for $10N$ successful rearrangements, whichever comes first [13]. After this the temperature is decreased by 10% (one annealing step, hereafter). In our computer tests, we stop the process if no trial reconfiguration is accepted after five successive temperature steps. At this point, we expected the solution to be very close to the optimal one.

Let us present some results from numerical simulations. In Fig. 1 we present how the path length decreases as the temperature decreases in the annealing schedule. The results refer to $N = 150$ cities. The average is taken over the same 50 initial configurations *for each value of* q. The random number sequences are initialized with the same seed for each initial configuration. Clearly, good solutions are found faster when $q < 1$. In particular, our numerical simulations suggest that the best results (as far as time is concerned) improve more and more when q decreases towards $-\infty$. It is worth stressing that large absolute values of q lead to numerical roundoff errors, mainly due to the $1/(q-1)$ power in (4). For $q > 1$ more annealing steps are needed to reach a good solution. This fact illustrates the usefulness of considering negative values of q. As can be seen in Fig. 1, the results are more drastically modified by $q > 1$ than the opposite direction. To confirm the good performance of the procedure in Fig. 2 we present the number of steps needed to reach a stationary state versus the number of cities for $q = 1$ and $q = -5$. Although the qualitative behavior is the same for both values, represented by the exponent α, the number of annealing steps is considerably smaller for negative q.

It is well established, for CSA, that a fast cool increases

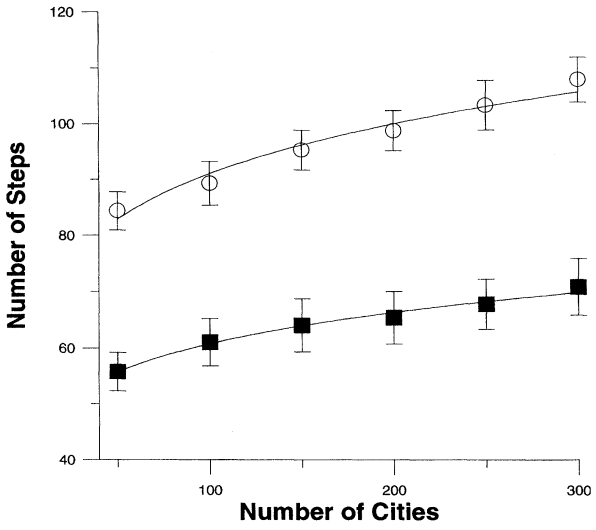

FIG. 2. Number of annealing steps versus number of cities (N). The results from CSA are represented by empty circles, whereas the $q=-5$ case are represented by squares. The lines are least square fits to a power law steps $\sim N^{0.14}$.

the probability of the system to be trapped in a metastable state. Hence we have to test the quality of the minima reached by the GSA. We present in Fig. 3 the average final length and the shortest path for 50 realizations for the same arrangement of $N=150$ cities. As can be seen, the quality of results is the same for any value of q, i.e., the GSA is as efficient in reaching the optimal solution as the CSA but, in addition, is faster in converging for good solutions. A possible extension for searching the optimal solution is also perform the "annealing steps" in the q variable, for a given temperature. It would act as an additional perturbation and some additional minima could be reached.

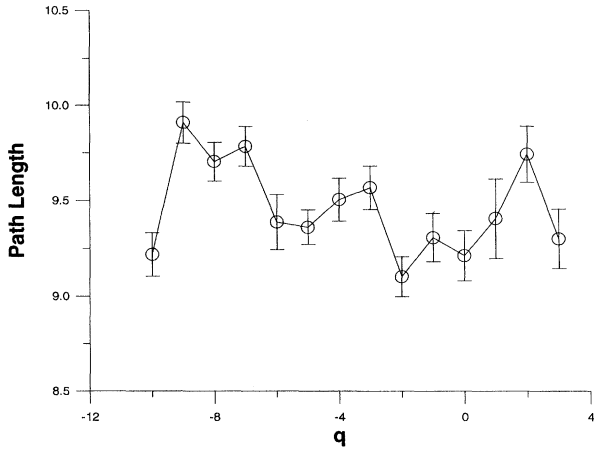

FIG. 3. Final length versus q. We use the same initial configurations for the $N=150$ cities. The error bars are dispersions in the final length.

In summary, a stochastic generalized simulated annealing was presented. We tested it in a *NP*-complete problem: the traveling salesman problem. We found approximate solutions considerably faster than the traditional methods of simulated annealing (CSA). Since the modification is easily implemented, all the recipes [3] applied in the TSP can be implemented in the present generalization, improving the performance even more. This generalization makes possible the solution of larger systems than have been presented.

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