Population dependence of early relaxation

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The general view of galactic evolution suggests an early period of "violent" relaxation followed by the establishment of a long-lived quasiequilibrium state which is associated with a stationary or steady-state solution of the Vlasov equation. The predicted time scale for the initial violent period, which has been supported by a number of simulations, is of the order of a few galactic crossing times. However, the formation of stellar clusters may constrain the early mixing to a slower diffusion which precedes the violent phase. To explore this possibility, we report on a recent study of the relaxation of a highly virialized (large ratio of kinetic to potential energy) model of a one-dimensional "galaxy" consisting of N parallel mass sheets interacting solely through their mutual gravitational attraction. We show that (1) relaxation consists of a long diffusive phase which can be of the order of a thousand crossing times and depends sensitively on the system population, followed by a short-lived "violent" period lasting less than 100 crossing times, and (2) the overall relaxation time has a minimum for a system consisting of about 30 sheets, suggesting that dynamics is most "chaotic" for this population. However, regardless of the population, the ultimate stationary state exhibits the expected core-halo structure in μ (position, velocity) space. Possible implications of the study for both nonlinear dynamics and astrophysics are considered.

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I. INTRODUCTION

A central question for astrophysics and, more specifically, stellar dynamics, is the prediction of the relaxation time for autonomous gravitating systems. A quick look at galaxies and globular clusters reveals that many elliptical galaxies and globular clusters have little internal structure and are devoid of gas and dust, indicating that they have relaxed to some type of "equilibrium," whereas spiral and irregular galaxies are still evolving [1]. Finer details show that, in fact, the velocity distribution of most ellipticals is not isotropic, indicating that although they are highly relaxed they are not in complete statistical equilibrium. In contrast, many globular clusters are fully isotropic and may be completely relaxed [1,2].

The usual picture of galactic evolution partitions the process into two different time scales, an initial period of "violent relaxation" in which the system rapidly approaches a nearly stationary "metastable" state [1,2] which is a discrete representation of a stationary Vlasov distribution. This state, in turn, gradually approaches complete thermal equilibrium. The time scale for the initial process results from "collisionless" Vlasov evolution while the last, slow stage is a result of "collisional" relaxation. Some authors have offered a refinement of this picture by introducing a third, intermediate, time scale arising from Landau damping [1] which results from the interaction of individual system elements with collective fluctuations. This scenario is predicated on the assumption that the initial distribution of stars is "well mixed" [3] so that the violent phase begins almost immediately. However, observation suggests that many stars form in isolated clusters which, in turn, require an additional period of diffusion prior to realizing the well-mixed condition.

It was shown long ago by Chandrasekhar [4] and others that, due to the low stellar density, the time scale for collisional relaxation of galaxies is greater than the predicted age of the universe, while the more dense globulars are nearly thermalized. Because the early stage is based on initial Vlasov evolution, it was anticipated that its duration is on the order of a typical crossing time and, hence, independent of population (N). In contrast, the final stage encompasses the evolution through a sequence of approximately stationary Vlasov states to complete thermal equilibrium and is population dependent. In a seminal paper Lynden-Bell uniquely approached the question of the final result of the violent phase using methods from quantum statistical physics [5]. He assumed that, because the Vlasov flow is incompressible, the individual mass elements were analogous to fermions, and hence their final distribution in μ space could be represented as a sum of Fermi distributions [6]. An alternative proposal was later brought forth by Shu [3].

In order to test these ideas, a number of investigators have studied the early stages of evolution of model systems. Because the numerical simulation of three-dimensional systems is slow and inexact [7], typically one-dimensional systems were chosen which could be evolved with greater numerical accuracy for a large number of crossing times. The most popular model chosen consists of a system of parallel mass sheets which are constrained to move in a direction perpendicular to their surface. This system, mentioned in the literature by Oort in 1932 [8], has been studied intensively, both numerically and analytically, since about 1967 and has been used to investigate both evolutionary stages. While the Lynden-Bell or Shu conjectures have not been confirmed [9], most simulations do show a rapid progression to a nearly stationary "metastable" state [9–13].

Until 1982, it was generally accepted that the final evo-

lutionary stage for this one-dimensional system took place in N^2T_c, where N is the system population and T_c is approximately the time required for a mass sheet to traverse the system [14]. By applying nonparametric statistics to the distribution of sheets in position and velocity, Wright, Miller, and Stein showed that, in fact, the system was not relaxed after $2N^2T_c$ [11]. For a reference distribution, they used the exact results derived by Rybicki for the microcanonical ensemble [15]. Shortly thereafter, Luwel, Severne, and Rousseeuw found that complete thermalization seemed to occur in a time scale of only NT_c for a specially chosen set of initial conditions. They conjectured that this would occur for all systems with initial states selected sufficiently close to equilibrium [10]. Further work we performed demonstrated that this was not the case: Rather we found a remarkable sensitivity of the stationary metastable state to the initial condition selected in a given simulation [12,16,17].

In other work, new insights have been obtained concerning the ergodic properties and the final relaxation time scale. By computating Lyapunov exponents associated with different periodic trajectors, we demonstrated the coexistence of both a stable and an unstable region of the phase space for each $N<11$. For $N=11$, the formerly stable region becomes unstable, suggesting that this is a critical population at which the system becomes both ergodic, and mixing [18]. By extrapolating the time required for the Lyapunov exponents generated by trajectories originating near each unstable fixed point to converge to a common value, we predicted that complete thermalization would require on the order of 10^7T_c [19]. Simulations carried out by Tsuchiya, Konishi, and Gouda have just confirmed our prediction. They report that the minimum thermalization time is about $(10^6-10^7)T_c$, and increases with system population [20].

This leaves unanswered the significance of the earlier research by Luwel, Severne, and Rousseeuw. What kind of relaxation did they observe, and what is the significance of the fact that it seemed to occur on the time scale NT_c? It seems clear that this process does not represent collisional relaxation. While it could arise from the violent phase, although this would be counter to current ideas, or represent the intermediate particle-fluctuation time scale (Landau damping) considered by Kandrup and Severne for this system [21], there is a third possibility. The counterstreaming initial conditions selected by Luwel, Severne, and Rousseeuw place the system members in separate clusters from which they must diffuse before the system an be considered "well mixed."

To gain further insight into the early stages of the relaxation process, we prepared the system in a highly virialized counterstreaming initial state (dumbbell, see Fig. 1) in the μ (position, velocity) space and followed its evolution to metastable equilibrium. The choice of tight clustering in each dumbbell lobe is analogous to the relatively small size of typical stellar clusters on a galactic scale. The evolution of these initial dumbbell states is characterized by a period during which each lobe of the dumbbell grows in size until the elements of each region finally commingle. Once the lobes overlap, there is a transition to a metastable core-halo structure. These initial states have the advantage that the duration of the period of *transition* from a dumbbell to the metastable core-halo structure is relatively short and thus easily quantified.

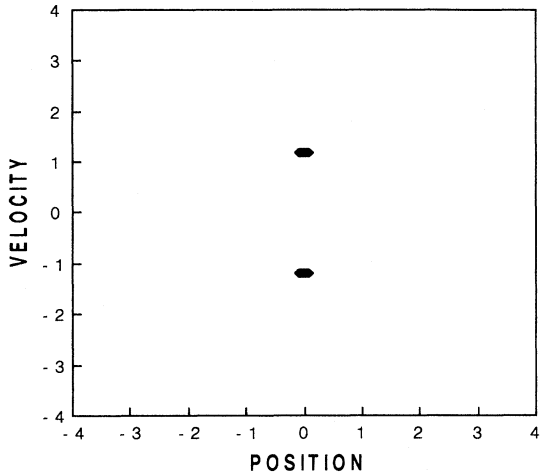

FIG. 1. Initial counterstreaming μ space distribution for $N=200$ sheets. Dimensionless units are defined by Eqs. (5) and (6).

We find that the lifetime of the initial "diffusive" phase, defined as the time preceding the dumbbell to core-halo transition during which two distinct lobes in the μ space are easily distinguished depends strongly on system population and, after passing a minimum, appears to increase linearly with N over a wide range. In this context, it would seem that Luwel, Severne, and Rousseeuw were actually observing the initial "diffusive" phase, rather than thermalization. Moreover, the fact that the transition from "dumbbell" to core-halo structure is rapid suggests that the separation of the latter stages of evolution into both an intermediate and a collisional time scale is unlikely. However, additional work will be required to answer this question fully. A bonus of the study is the unavoidable conclusion that the most rapid exploration of the phase space (most "chaotic" behavior) occurs for N about 30, where the relaxation time has its minimum.

The details of the study are presented below. We first describe the initialization of the system into the dumbbell configuration. We then describe the criteria for estimating the time of transition, and its dependence on population. In the conclusions we discuss both the dynamical and astrophysical implications of the results.

II. DESCRIPTION OF THE ONE-DIMENSIONAL SYSTEM

The one-dimensional self-gravitating system is composed of N identical mass sheets, each of uniform mass density and infinite in extent in, say, the (y,z) plane. The sheets move freely along the x axis and accelerate as a result of their mutual gravitational attraction. The ith sheet experiences a constant acceleration given by

$$A_i=(2\pi G/N)[N-2i+1]\ , \tag{1}$$

where $1/N$ is the mass of a sheet and G is the universal gravitational constant. At an encounter the sheets pass

freely through each other. The energy of the system is expressed as

$$E=(1/2N)\sum_{1}^{N} v_j^2+(2\pi G/N^2)\sum_{i<j}|x_i-x_j| , \quad (2)$$

where v_i and x_i are the velocity and position of the ith particle, respectively.

The equilibrium velocity and position probability density functions for this system in the microcanonical ensemble were developed by Rybicki [15]. In the limit that $N\rightarrow\infty$ these functions are

$$\theta(\eta)=\pi^{-1/2}\exp(-\eta^2) \quad \text{(velocity)} , \quad (3)$$

$$\rho(\xi)=\tfrac{1}{2}\operatorname{sech}^2\xi \quad \text{(position)} , \quad (4)$$

where

$$\eta=(v/2)(3M/E)^{1/2} , \quad (5)$$

and

$$\xi=(3\pi GM^2/2E)x ; \quad (6)$$

v, x, M, and E represent the velocity, position, total system mass, and total system energy, respectively.

The dynamical time required for a sheet to traverse the system is referred to as the characteristic time and has been expressed by Luwel, Severne, and Rousseeuw (hereafter LSR) [10] in terms of the maximum value of the equilibrium density function $\rho(\eta)$,

$$t_c=(GM\rho_{\max}/\pi)^{-1/2} . \quad (7)$$

III. SIMULATION SETUP

All initial positions and velocities were scaled according to Eqs. (5) and (6) with $M=1$ and $2\pi G=1$. This results in a characteristic time t_c of 2π and forces a total dimensionless energy of $\frac{3}{4}$ for all systems. The center of mass and total momentum were constrained to zero. The evolution of each system was simulated using an exact code with updating occurring at each encounter. All calculations were performed in double precision (16 significant figures) on a VAX 6310, and energy was conserved to better than one part in 10^{10}.

The initial counterstreaming configuration [10,12] in $\mu(x,v)$ space (see Fig. 1) was constructed by the random sampling of uniform distributions for both positions and velocities. First the width of each domain of initial velocities was selected as 0.0001 of the velocity ceiling. The velocities were then chose by uniformly sampling the two intervals in arbitrary units and translating until their sum was zero. Similarly, the positions were chosen next by uniformly sampling a new symmetric interval, translating so that the center of mass is zero, and rescaling the values to obtain the desired virial ratio [twice the (kinetic energy)/(potential energy)] [12]. Finally, Rybicki scaling [see Eqs. (5) and (6)] was applied to obtain an energy of $\frac{3}{4}$ [12].

IV. SIMULATION RESULTS

The virial ratio of the initial configuration was chosen to be 50. Simulations were carried out for eleven

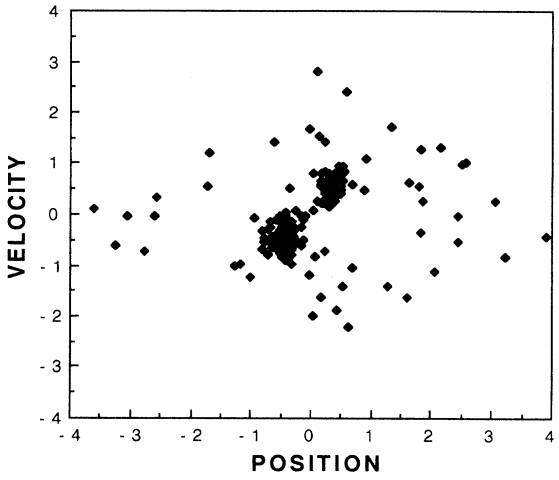

FIG. 2. System distribution after $1700t_c$. Note the appearance of two distinct clusters.

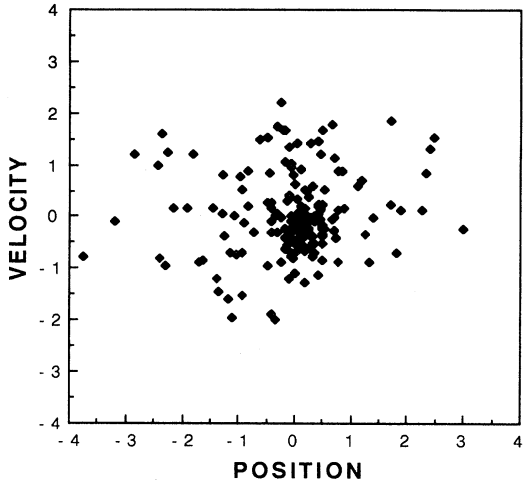

FIG. 3. System after $1800t_c$. Note that the clusters have disappeared and core-halo structure has been established.

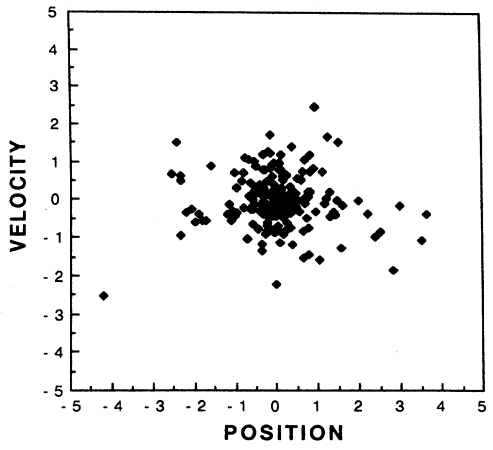

FIG. 4. System after $20\,000t_c$. Note the persistence of core-halo structure.

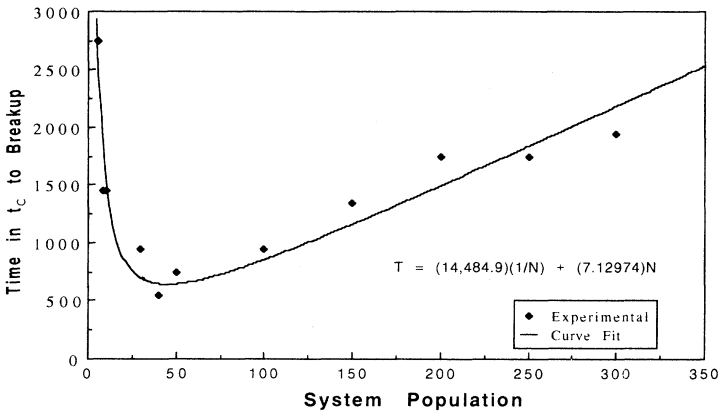

FIG. 5. Time of cluster breakup versus system population. Note the minimum at $N=40$ and the linear dependence on N for asymptotically large population.

different system populations ($N=6$, 8, 10, 30, 40, 50, 100, 150, 200, 250, and 300). In each case the evolution in μ space was carefully observed by plotting the configuration every 100 characteristic times. In all cases, initially the dumbbell configuration is conserved, i.e., the two clusters orbit each other in the μ space consistent with a simple, two particle, system. Gradually, individual masses are seen to diffuse out of the clusters. This results from the random order of crossings which occurs during the short piece of the cycle where the clusters' *positions* overlap. However, the identification of two distinct clusters persists for some time. Finally, within a short period of time, the two cluster structure collapses abruptly leaving a central core and a diffuse halo. The core-halo distribution appears to be stationary and no further systematic evolution is observed on these time scales. The sequence for a 200 sheet system is illustrated in Figs. 1–4.

For each population we have been able to localize t_b, the time at which breakup occurs, to within $100t_c$. Of course, for small system populations it was more difficult to distinguish the two types of structure in μ space. The central concern of this study is to estimate the dependence of the cluster duration on system size. For large systems it was apparent that t_b increases linearly with N. However, a minimum value is obtained for N about 40. As N decreases below 40, t_b increases once again, but much more rapidly. In Fig. 5 we have plotted t_b versus N with a fitting curve of the form $f(N)=a/N+bN$ which appears to represent the crude data as well as can be expected.

V. DISCUSSION OF RESULTS AND CLOSING COMMENTS

The results of the study clearly show that the duration of the initial "diffusive" phase of relaxation depends on system population. These results were not observed in earlier simulations of this system [9,11,14] or in simulations of three-dimensional systems [1,2,22] because the initial conditions were nearly "well mixed" and did not include the presence of localized clusters. While it is not surprising that collisional relaxation to complete thermal equilibrium depends on population, the N dependence of the early phase is more difficult to understand. One possible explanation is that the initial state selected yields uncharacteristic results. In our case, if the initial dumbbell state were squeezed into two symmetrically placed mass points, then it would define an unstable periodic solution to the Vlasov equation. The fact that the actual initial state is "close" to this periodic solution may inhibit strong mixing in the early phases, leaving phase mixing as the sole initial relaxation mechanism [1]. The investigation of the population dependence of initial relaxation resulting from different initial conditions would be required to fully answer this question. However, it is hard to imagine that population dependence will simply disappear.

It has been demonstrated that the failure of small systems ($N<11$) to rapidly converge to equilibrium is due to the segmentation of the phase space into distinct coexisting ergodic and stable components [18]. The failure to observe thermalization of large systems (say $N>50$) in simulations for times upwards of 10^3t_c is due to the weakening of the pairwise interaction between separated sheets in the Vlasov limit. (The mass vanishes as $1/N$ so the force vanishes as $1/N^2$.) Thus the acceleration experienced by a given sheet becomes dominated by the mean field arising from the long range force. Since nearby sheets experience approximately the same mean field, orbit separation in phase space and, consequently, thermalization becomes a slow crawl. Since complete thermalization has only been achieved in one study [20], the dependence of this relaxation time on population is unknown. With these thoughts in mind, it is not surprising that the most rapid mixing in phase space occurs for N on the order of 30. For smaller values, remnants of segmentation probably limit the rate of spreading of nearby trajectories in phase space, while the system is too close to a Vlasov fluid for larger values.

Although there are important differences between evolution in one and three dimensions, the results of this study may have implications for astrophysics (i.e., three-dimensional self-gravitating systems) as well as nonlinear dynamics. They suggest that, for some classes of initial conditions, the initial relaxation of a galaxy may be less "violent" than anticipated. Thus the observation that many galaxies are "irregular," i.e., cannot be identified

with approximately symmetric, stationary, Vlasov states, *may* be primarily due to a prolonged initial relaxation period resulting from the concentration of young stars in stellar clusters. In preliminary studies we have found that relaxation in systems initiated with three or four distinct clusters follow the sample pattern reported here and reinforce this picture.

A better understanding of initial relaxation would result from an investigation of the connection between Vlasov evolution and discrete particle simulations. Numerical integrations of the Vlasov equation clearly show that, for a typical initial mass distribution, as the fluid evolves structure develops on increasingly finer (smaller) length scales in the μ space [23,24]. Thus relaxation to a metastable equilibrium cannot occur without some type of coarse graining, or other information sink. An N-particle simulation intrinsically provides such a sink in that it can only effectively imitate the Vlasov evolution while the discrete particle masses are smaller than the mass associated with a Vlasov fluid element with dimensions on the order of the shortest prevalent wavelength. However, once equality is reached, say at t_e, the evolutions should separate. The time scale for this separation clearly depends on system population, and has not been investigated. It would be worthwhile to compare the time required following t_e for the coarse-grained Vlasov fluid to achieve a stationary state with that of the discrete system. This is a suggestion for future work which may provide insight into the surprising results of this study.

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