ARTICLES

Reshaping-induced chaos suppression

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We discuss the route for eliminating chaos in nonlinear oscillations by changing only *the shape* of a periodic force. We consider the Duffing oscillator forced with the Jacobi elliptic function sn and, applying a simple averaging technique, show that the phenomenon of chaos suppression due to reshaping of the driving force may be easily explained as an effect solely arising from the change of the effective amplitude of the first harmonic in the Fourier series expansion of the elliptic function. This conclusion is clearly confirmed by direct numerical simulations.

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I. INTRODUCTION

As is well known, the main methods to control chaotic dynamics are based on the idea of the stabilization of unstable periodic orbits by feedback or external signals, and this may be achieved in several different ways (see, e.g., [1–6] to cite a few). One of the methods is to use a small parametric [2,3] or direct [4] force applied at some resonant frequency, and the chaos suppression is then observed as regularization of the motion at exact resonance. Recently, Chacón and Díaz Bejarano [5] have proposed a route for eliminating chaos in nonlinear oscillation by changing only the shape of a weak periodic perturbation. In particular, they used the modified Duffing oscillator,

$$\frac{d^2x}{dt^2} - \alpha x + \beta x^3 = -\gamma \frac{dx}{dt} + F\mathrm{sn}\,(\omega t; m), \tag{1}$$

where $\mathrm{sn}\,(\omega t; m)$ is the Jacobi elliptic function with the modulus m $(0 \leq m \leq 1)$ and, applying the Melnikov-Holmes analysis (see, e.g., [7]), demonstrated that chaos may be suppressed by changing m only, provided the forcing period is fixed. When $m = 0$ we have $\mathrm{sn}(\omega t; 0) = \sin(\omega t)$, i.e., the standard Duffing oscillator, and the idea of Ref. [5] to change the pulse shape selecting $m \neq 0$ seems to originate from the earlier analyzed effect of small-amplitude parametric [2,3] or direct [4] perturbations on chaotic dynamics. However, our recent results have shown that for *nonresonant* parametric perturbations chaos suppression may be effective only for the case of large amplitudes [6]. This immediately implies that higher-order harmonics, which usually appear due to such a reshaping cannot be effective enough to be important in the chaos suppression. The purpose of this paper is to show that the chaos suppression discussed in Ref. [5] is basically produced by the first harmonic of the anharmonic force and the effective contribution of higher-order harmonics may be taken as a simple (small) renormalization of a coefficient in the Duffing oscillator equation. Thus, our results suggest that the phenomenon of the reshaping-induced chaos suppression is nothing but a variation of the amplitude of an effective harmonic driving force.

The paper is organized as follows. In Sec. II we briefly discuss, using the ideas and method of Ref. [6], the effect of a high-frequency *direct* force on the Duffing oscillator. In Sec. III the more general theory developed in Sec. II is applied to Eq. (1) to demonstrate that in such a case the contribution of higher-order harmonics is small. In Sec. IV we confirm the results of our analysis by direct numerical simulations, and last, Sec. V concludes the paper.

II. ANALYTICAL APPROACH

First, we consider the more general problem of the forced dynamics in the presence of a high-frequency periodic perturbation. If such a periodic perturbation is applied to a system driven by a low-frequency force, it may be effective to suppress chaotic dynamics provided the perturbation amplitude is big enough. The effect of a parametric perturbation was considered in [6], but here we are interested in a similar kind of phenomenon but caused by a direct driving force.

Let us consider the driven damped Duffing oscillator

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with two driving forces,

$$\frac{d^2x}{dt^2} - \alpha x + \beta x^3 = -\gamma\frac{dx}{dt} + F\sin(\omega t) + \epsilon\sin(\Omega t), \quad (2)$$

where the frequency Ω is assumed to be *much bigger* than the other frequency, ω. As was shown in Ref. [6], the effect of such rapidly varying oscillations may be analyzed by the method based on the separation of different time scales. Using the ideas of Ref. [6], we look for the solution of Eq. (2) in the form,

$$x = X + A_1\cos(\Omega t) + B_1\sin(\Omega t) + A_2\cos(2\Omega t) + B_2\sin(2\Omega t) + \cdots, \quad (3)$$

where the coefficients $X, A_1, B_1, A_2, B_2, \ldots$ are assumed to be slowly varying functions on the scale $\sim \Omega^{-1}$. Substituting Eq. (3) into Eq. (2) and equating the terms in front of different harmonics, we obtain an infinite system of coupled nonlinear equations for $X, A_1, B_1, \ldots$ which can be effectively solved by applying asymptotic expansions (cf. Refs. [6,8]),

$$A_1 = \frac{a_{11}}{\Omega^3} + \frac{a_{12}}{\Omega^5} + \cdots, \quad B_1 = \frac{b_{11}}{\Omega^2} + \frac{b_{12}}{\Omega^4} + \cdots,$$
$$A_2 = \frac{a_{21}}{\Omega^5} + \cdots, \quad B_2 = \frac{b_{21}}{\Omega^4} + \cdots, \quad (4)$$

and so on. The first terms of the expansions (4) have the simple form,

$$a_{11} = \gamma b_{11}, \quad b_{11} = -\epsilon, \quad (5)$$

and other ones are functions of a_{11} and b_{11} or their derivatives (see also Refs. [6,8]). Substituting now Eq. (5) into the equation for the "averaged" oscillation amplitude X,

$$\frac{d^2X}{dt^2} - \alpha X + \beta X^3 + \tfrac{3}{2}\beta X(A^2 + B^2 + \cdots) = -\gamma\frac{dX}{dt} + F\sin(\omega t), \quad (6)$$

we find the simple result that the slowly varying dynamics of the system is described (up to the terms of order of Ω^{-4}) by the standard Duffing equation with the *renormalized* coefficient

$$\alpha \to \tilde{\alpha} \equiv \left(\alpha - \frac{3}{2}\beta\frac{\epsilon^2}{\Omega^4}\right). \quad (7)$$

It is easy to extend the analysis presented above to cover a more general case when the high-frequency driving force has an arbitrary shape that allows an expansion in a Fourier series, so that in Eq. (1) we should make the change,

$$\epsilon\sin(\Omega t) \to \sum_{n=1}^{\infty}\epsilon_n\sin(n\Omega t). \quad (8)$$

For this case the expansion (3) involves a double sum, it may be found that the effective Duffing equation has the renormalized coefficient,

$$\tilde{\alpha} = \alpha - \frac{3\beta}{2\Omega^4}\sum_{n=1}^{\infty}\frac{\epsilon_n^2}{n^4}. \quad (9)$$

III. CHAOS SUPPRESSION BY AN ELLIPTIC DRIVING FORCE

We consider now the Duffing oscillator driven by an elliptic driving force as in Eq. (1). First, the elliptic function in Eq. (1) may be expanded into the Fourier series,

$$\mathrm{sn}(\omega t; m) = \sum_{n=0}^{\infty}G_n(m)\sin\left[(2n+1)\frac{\pi\omega t}{2K}\right], \quad (10)$$

where

$$G_n(m) = \frac{\pi}{K\sqrt{m}\sinh\left[\frac{\pi K'}{K}\left(n+\frac{1}{2}\right)\right]}, \quad (11)$$

$K \equiv K(m)$ is the elliptic integral of the first kind, and $K'(m) \equiv K(1-m)$. Then, we note that from the viewpoint of the main harmonic (at $n = 0$) the higher-order ($n \geq 1$) harmonics in Eq. (10) may be simply viewed as those produced by (additional) rapidly varying perturbations, provided the oscillation frequency is large enough. Using the approach developed in Sec. II, we derive an effective nonlinear equation for the slowly varying component $X = \langle x\rangle$ of the oscillation amplitude x, where $\langle\cdots\rangle$ stands for averaging over the time scales smaller than the oscillation period $T = 4K(m)/\omega$, e.g., over $T/3$. To do this, we look for a solution of Eqs. (1) and (10) in the form of the asymptotic expansion [cf. Eq. (3)]

$$x = X + \sum_{i,j}[A_{ij}\sin(\Omega_j t) + B_{ij}\cos(\Omega_j t)], \quad (12)$$

where

$$\Omega_j = (2j+1)\pi\omega/2K(m) \quad \text{for} \quad j = 0, 1, 2, \ldots. \quad (13)$$

The expansion coefficients A_{ij} and B_{ij} are found with the help of a power series in the small parameter Ω_j^{-1}, which allow one to derive an effective equation for the slowly varying component X with any accuracy. The resulting "averaged" equation corresponds, in fact, to an effective harmonically driven damped Duffing oscillator

$$\frac{d^2X}{dt^2} - \tilde{\alpha}X + \beta X^3 = -\gamma\frac{dX}{dt} + FG_0(m)\sin(\Omega_0 t), \quad (14)$$

with the *renormalized* coefficient,

$$\tilde{\alpha} = \alpha - \frac{3\beta F^2}{2\Omega_0^4}\sum_{n=1}^{\infty}\frac{G_n^2(m)}{(2n+1)^4}, \quad (15)$$

where the frequency Ω_0 is assumed to be fixed. The Melnikov-Holmes condition to prevent chaos in Eq. (14) may be written in the following simple form:

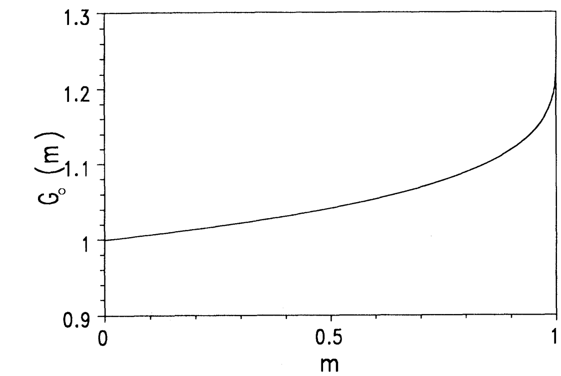

FIG. 1. Dependence of the amplitude $G_0(m)$ of the first harmonic in the expansion (10) on the modulus m of the elliptic force.

$$\gamma > \frac{3\pi F\sqrt{\beta}\Omega_0 G_0(m)}{(2\tilde{\alpha})^{3/2}}\mathrm{sech}\left(\frac{\pi\Omega_0}{2\sqrt{\tilde{\alpha}}}\right). \tag{16}$$

The results (14) to (16) mean that for the case $F^2 \ll \Omega_0^4$ the effective contribution of higher-order harmonics in the phenomenon of eliminating chaotic dynamics on the time scales $\sim T$ is *very small*, and the main effect is expected from the contribution of the first harmonic ($\sim G_0$). The function

$$G_0(m) = \frac{\pi K(m)}{\sqrt{m}\,\sinh[\pi K'(m)/2K(m)]} \tag{17}$$

depends on m and this dependence is shown in Fig. 1. Therefore, we expect that the main features of the reshaping-induced chaos suppression discussed in Ref. [5] may be seen dealing solely with the amplitude $G_0(m)$ of the first harmonic in the Fourier expansion (10), (11), and the phenomenon of the chaos control due to a reshaping of the amplitude of the driving force [5] may be understood as that arising from a change of the am-

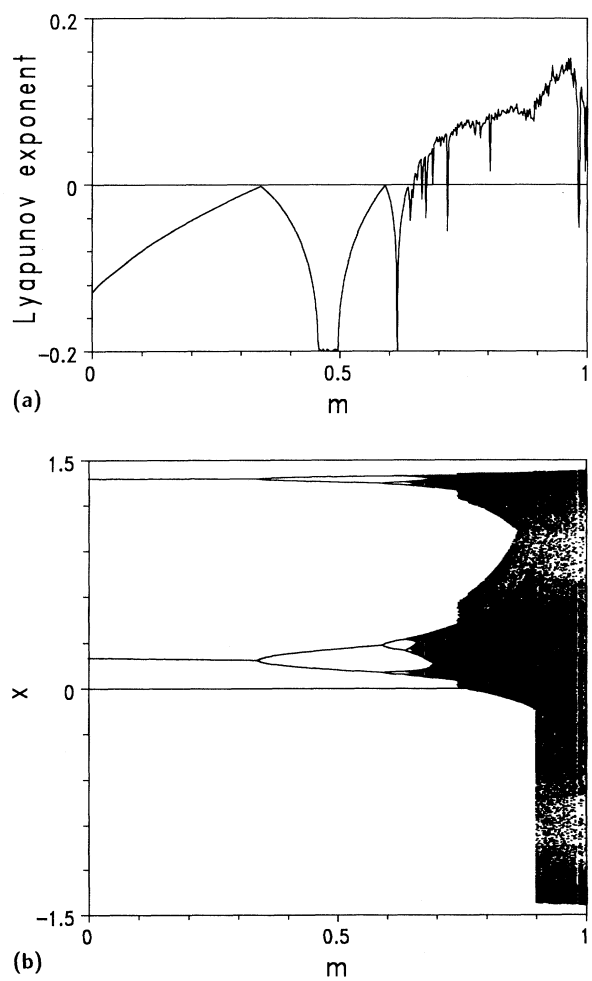

FIG. 2. Lyapunov exponent (a) and the bifurcation diagram (b) vs the modulus m of the elliptic driving force in Eq. (1) .

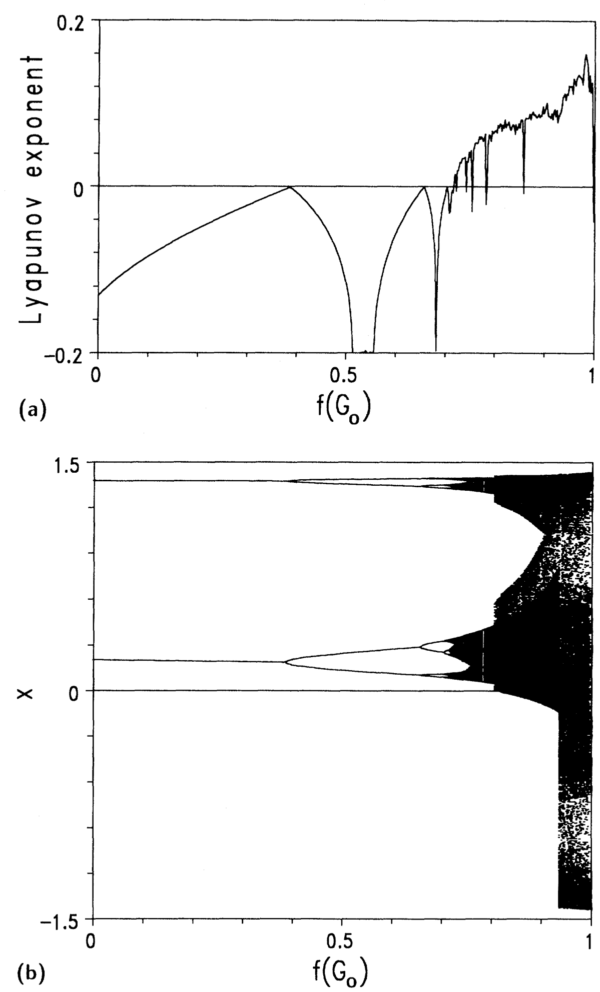

FIG. 3. The same as in Fig. 2 but for the standard Duffing equation (18) with a sinusoidal driving force whose amplitude is taken as the amplitude of the first harmonic in Eq. (10). Note a (small) shift of the plots in comparison with Figs. 2 (a) and 2(b), respectively. However, the global behavior and especially the scenario of the chaos suppression are the very same.

plitude $FG_0(m)$ of the effective *harmonic* force in the "averaged" equation (14).

IV. RESULTS OF NUMERICAL SIMULATIONS

To verify the idea presented above we have performed independently the computer simulations of Eq. (1) and the effective Duffing equation (14). First, the problem (1) of the elliptic driving force was considered under the assumption of Ref. [5] that the period is constant, the latter was achieved by dividing the argument of the elliptic function sn by $1/K(m)$. The parameters of the Duffing oscillator were chosen to be $\alpha = \beta = \omega = 1.0$ and $\gamma = 0.27$. The initial condition always was $x(0) = 0$ and $dx(0)/dt = 0.001$. For the sinusoidal driving force (i.e., when $m = 0$) there occurs an asymmetric periodic attractor at positive values of x (see Fig. 2). Increasing m, a period doubling route to chaos is observed ending up in an asymmetric chaotic attractor with a positive Lyapunov exponent. Further increasing yields a heteroclinic symmetry restoring crisis that merges the two attractors occurring at either sides of the potential well for $x > 0$ and $x < 0$. As one can see in Fig. 2(a), the Lyapunov exponent increases after the crisis. Thus, changing m from 0 to 1, one can suppress chaos in the case of the higher-dimensional, symmetric attractor as well as in the lower-dimensional, asymmetric attractor case. Furthermore, one can observe the typical occurrence of a "scenario out of chaos," that always occurs in suppressing chaos by parametric perturbations [2,3,6]. It results from the fact that the perturbations can be reduced to an effective control parameter change.

To identify the effect produced solely by the first harmonic in the chaos suppression presented in Figs. 2(a) and 2(b), we have done the same simulations for the standard Duffing equation,

$$\frac{d^2x}{dt^2} - \alpha x + \beta x^3 = -\gamma\frac{dx}{dt} + f(G_0)\sin(\omega t), \qquad (18)$$

where $f(G_0)$ is proportional to the amplitude of the first harmonic in the expansion (10). The results are presented in Fig. 3 where the amplitude is scaled by the function inverse to $G_0(m)$. In this way the plots in Fig. 3 can directly be compared to those calculated using the elliptic driving force $\mathrm{sn}\,(\omega t/K; m)$ and shown in Fig. 2. The plots in Figs. 3(a) and 3(b) clearly show up a direct equivalence to the former ones, Figs. 2(a) and 2(b), respectively. One can also observe the period doubling cascade first resulting in an asymmetric attractor and then merging into the larger, symmetric attractor via crisis.

However, the scenario shown in Fig. 3(b) and the Lyapunov exponent shown in Fig. 3(a) are clearly seen to be shifted a little bit in the control parameter resulting from the (small) influence of the higher-order harmonics existing in the case of the anharmonic driving force. Nevertheless, the global behavior and especially the suppression of chaos are the very same, showing up that the conclusions made in the previous section are confirmed.

V. CONCLUSION

In conclusion, we have considered both analytically and numerically the chaos suppression by changing only the shape of a periodic force taking the driving force as the Jacobi elliptic function sn. As follows from our analysis and numerical results, the effect predicted in Ref. [5] may be simply explained by a variation of the amplitude of the first harmonic, whereas a contribution of higher-order harmonics is shown to be small. Thus, this confirms that there are only *two independent* (known up until now) ways of controlling chaos — the feedback method (see, e.g., [1]) and the periodic forcing method (see, e.g., [2–4]) whereas the method suggested in Ref. [5] may be effectively reduced to the known one.

[1] E. Ott, C. Grebogi, and Y.A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).

[2] R. Lima and M. Pettini, Phys. Rev. A **41**, 726 (1990).

[3] L. Fronzoni, M. Giocondo, and M. Pettini, Phys. Rev. A **43**, 6483 (1991).

[4] Y. Braiman and I. Goldhirsch, Phys. Rev. Lett. **66**, 2545 (1991).

[5] R. Chacón and J. Díaz Bejarano, Phys. Rev. Lett. **71**, 3103 (1993).

[6] Yu.S. Kivshar, F. Rödelsperger, and H. Benner, Phys. Rev. E **49**, 319 (1994).

[7] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer, New York, 1983).

[8] Yu.S. Kivshar, N. Grønbech-Jensen, and R.D. Parmentier, Phys. Rev. E **49**, 4542 (1994).