Controlling of chaotic motion by chaos and noise signals in a logistic map and a Bonhoeffer–van der Pol oscillator

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The possibility of the conversion of a chaotic attractor to a strange but nonchaotic attractor is investigated numerically in both a discrete system, the logistic map, and in a continuous dynamical system, the Bonhoeffer–van der Pol oscillator. A suppression of the chaotic property, namely, the sensitive dependence on initial states, is found when an appropriate (i) chaotic signal and (ii) Gaussian white noise are added. A strange but nonchaotic attractor is shown to occur for some ranges of amplitude of the external perturbation. The controlled orbit is characterized by the Lyapunov exponent, correlation dimension, power spectrum, and return map.

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The presence of chaos in many nonlinear dynamical systems has been extensively studied. In recent years a great deal of interest has been focused on controlling chaos [1–6]. There also have been recent attempts to use chaos profitably by synchronizing chaotic orbits [7–10]. The existing control methods are capable of converting a chaotic motion into a regular dynamics, either by stabilizing an unstable periodic orbit embedded in the chaotic attractor [2–5] or by creating a new periodic orbit [1,5,6]. One of the prime motivations to control chaotic motion is its extreme sensitive dependence on initial conditions. The sensitive dependence on initial conditions is characterized by a positive maximal Lyapunov exponent. In the chaotic regime two nearby trajectories diverge exponentially until they become completely uncorrelated and hence future prediction becomes inaccurate. It is thus important to investigate the possibility of the suppression of sensitive dependence on initial conditions and maintaining the strangeness of the attractor, instead of stabilizing a periodic orbit.

Motivated by the above, in this paper we study the conversion of a chaotic attractor into a strange nonchaotic attractor by adding an appropriate (i) chaotic signal and (ii) Gaussian white noise. We carry out our investigation both in a discrete system, the logistic map [11,12]

$$x_{n+1}=x_n\exp[A(1-x_n)],\quad x_n>0\ , \tag{1}$$

and in a continuous dynamical system, the Bonhoeffer–van der Pol (BVP) oscillator [5],

$$\dot{x}=x-x^3/3-y+f\cos t\ , \tag{2a}$$

$$\dot{y}=c(x+a-by)\ . \tag{2b}$$

In Eq. (2) a, b, and c are constant parameters and f is the amplitude of the external periodic force. Recently, Carroll and Pecora [13] studied the effect of adding chaotic and noise signals separately in a Duffing oscillator circuit in a different context. They studied the effect of the added signals on the flipping of the state of the system between two coexisting periodic attractors and observed stochastic resonance. In our present study we use chaotic and noise signals in the context of controlling chaos.

Figure 1 shows the chaotic attractor of the logistic map (1) for $A=3$ in the x_n versus x_{n+1} plane. The Lyapunov exponent of this attractor is ≈ 0.39. Now, we study the effect of the addition of a chaotic signal to Eq. (1) with $A=3$. The logistic map with the addition of a chaotic solution can be written as

$$x_{n+1}=x_n\exp[A(1-x_n)]+Cy_n\ , \tag{3}$$

where y_n is the chaotic solution generated from the logistic map (1). First we study the influence of the chaotic solution generated with $A=3$ from (1). Later, we show the effect of chaotic solution of (1) generated with different values of A.

Figure 2 shows the estimated Lyapunov exponent as a function of the parameter C. From this figure we note that λ is positive for $C<C^*\approx 0.075$ while it takes negative values for $C>C^*$. The Lyapunov exponent with a

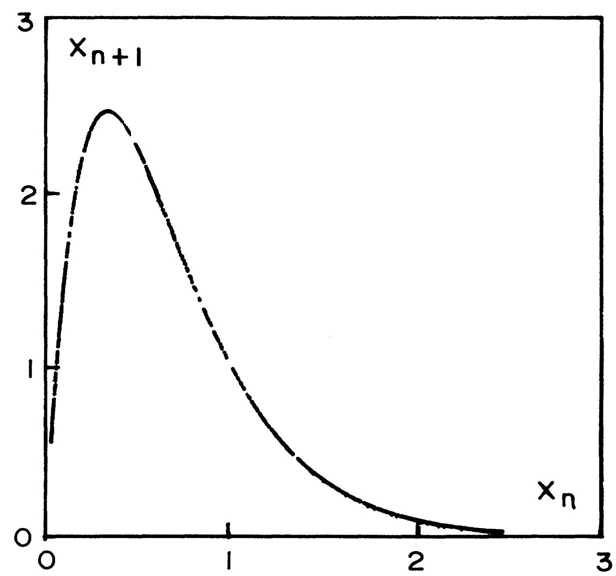

FIG. 1. Chaotic attractor of the logistic mapping (1) for $A=3$.

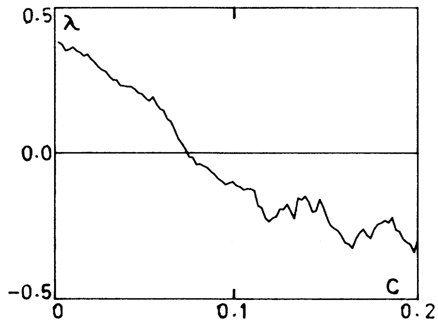

FIG. 2. Estimated Lyapunov exponent versus the parameter C.

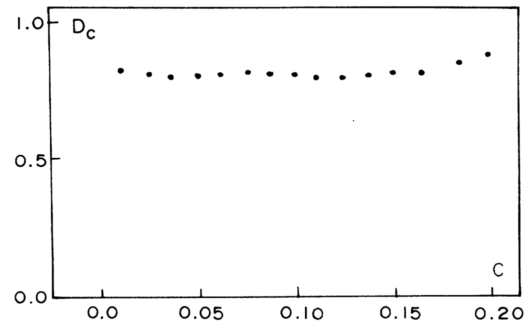

FIG. 4. Variation of correlation dimension as a function of C.

negative value implies that the solution is insensitive to small disturbances in the initial conditions. In other words, chaotic property is suppressed for $C > C^*$ and hence the attractor is nonchaotic. Figure 3 shows the attractor in the x_n versus x_{n+1} plane for $C = 0.1$. The Lyapunov exponent of the attractor is -0.12. Further, we have estimated the correlation dimension D_c of the controlled attractor. To calculate D_c we have used the algorithm proposed by Grassberger and Proccacia [14]. The correlation dimension of the attractor shown in Fig. 3 is estimated as 0.82. The attractor is thus strange. The negative Lyapunov exponent, noninteger dimension, and return map (Fig. 3) imply that the controlled attractor is strange but nonchaotic. The strange nonchaotic attractor has been previously found to occur in numerical studies of many nonlinear systems and also in actual experiments [15–20].

The occurrence of a strange nonchaotic attractor is further confirmed by a power spectrum analysis. For a strange nonchaotic attractor the number of peaks $N(\sigma)$ in a power spectrum exceeding a threshold amplitude σ is scaled as [20]

$$N(\sigma) \propto \sigma^{-\alpha}, \quad 1 < \alpha < 2 . \tag{4}$$

The power spectrum of the attractor (Fig. 3) is obtained using the fast Fourier transform with 4096 data points. From the $\log_{10} N(\sigma)$ versus $\log_{10}\sigma$ plot the value of the scaling exponent α is estimated as 1.45. This is in agreement with the power-law scaling relation (4).

A strange nonchaotic attractor is found for C values for which $\lambda \leq 0$. Figure 4 shows the variation of correlation dimension as a function of the parameter C. Figures 2 and 4 clearly support the occurrence of a strange nonchaotic attractor for $C > C^*$. In the logistic map the interaction between dynamic and stochastic (coupling term) forces gives rise to the strange nonchaotic attractor. The effect of various nonlinear coupling terms such as y^3, $\sin y$, and $\exp(y)$ has also been studied. The critical values of C^* above which the strange nonchaotic attractor occurs corresponding to the coupling terms y^3, $\sin y$, and $\exp(y)$ are found to be 0.0425, 0.168, and 0.0225, respectively. Further we have carried out our analysis using the

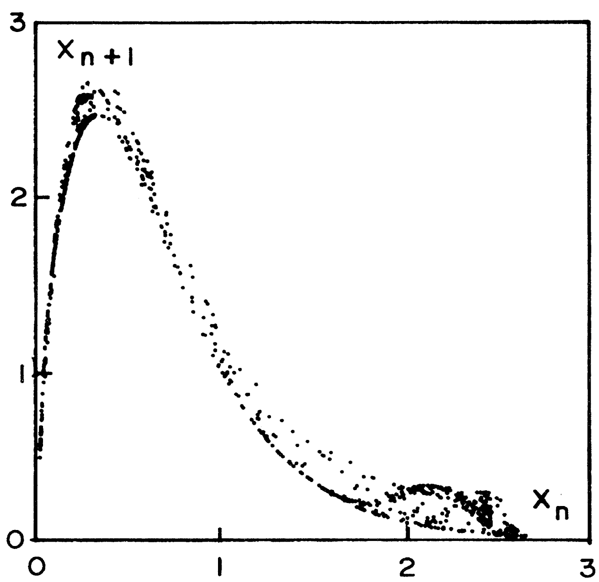

FIG. 3. Strange nonchaotic attractor of Eq. (3) for $C = 0.1$.

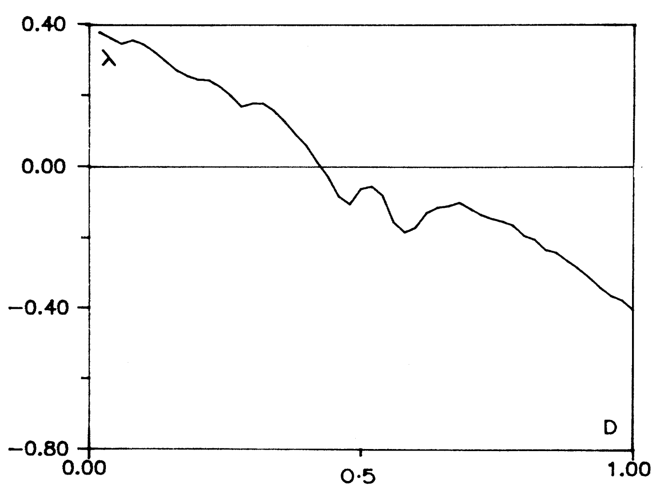

FIG. 5. Estimated Lyapunov exponent versus the amplitude D.

TABLE I. Estimated critical value C^* for different A values used to generate the chaotic solution. The generated chaotic solution is added to Eq. (3).

A	C^*
2.8	0.051
2.9	0.067
3.0	0.075
3.1	0.064
3.3	0.101
3.4	0.086
3.5	0.090
3.7	0.187
3.8	0.158

chaotic solution generated for various values of A. For C values in the interval (0,0.2) we have estimated the Lyapunov exponent of attractors of (3) and determined the value of C^* above which λ becomes negative. Table I gives C^* values for different A values.

Next we study the effect of noise added to the chaotic solution of the logistic map. Cy_n in Eq. (3) is replaced by $D\eta(n)$, where $\{\eta(n)\}$ are independent random numbers with mean m and standard deviation σ. We study the influence of noise by fixing A at 3 and by varying the amplitude D of the noise for $m=0.2$ and $\sigma=0.1$. Figure 5 shows the mean Lyapunov exponent obtained by averaging over 500 realizations of $\eta(n)$ as a function of D. For $D<D^*$ the Lyapunov exponent is positive. In this case the long time motion is still chaotic. For $D>D^*$ the Lyapunov exponent is negative and the attractor is nonchaotic. Figure 6 shows the controlled attractor for $D=0.6$. This figure clearly indicates that the attractor is strange.

We have also studied the influence of a chaotic signal in the BVP equation (2). The parameters a, b, c, and f are fixed at 0.7, 0.8, 0.1, and 0.74, respectively, for which chaotic motion is observed. In Eq. (2(a) f cos$t$ is replaced by $f\cos t+Du(t)$, where $u(t)$ is the chaotic solution generated from the logistic map $u_{n+1}=4u_n(1-u_n)$. D is the amplitude of the chaotic solution added to the driving force. The iteration of the logistic map lies in the interval (0,1). The solution is converted into the range $(-1,1)$. The chaotic solution generated from the map is added to the BVP equation after every $(2\pi/50)$ time step. Figure 7 shows the variation of the maximal Lyapunov exponent as a function of D. It is seen that for a range of D values λ is negative, that is, the motion is nonchaotic. Figure 8 shows the Poincaré map of the attractor for $D=0.3$. The λ of the attractor is negative, as seen from Fig. 7. From the negative value of the Lyapunov exponent and Poincaré map (Fig. 7) we conclude that the controlled at-

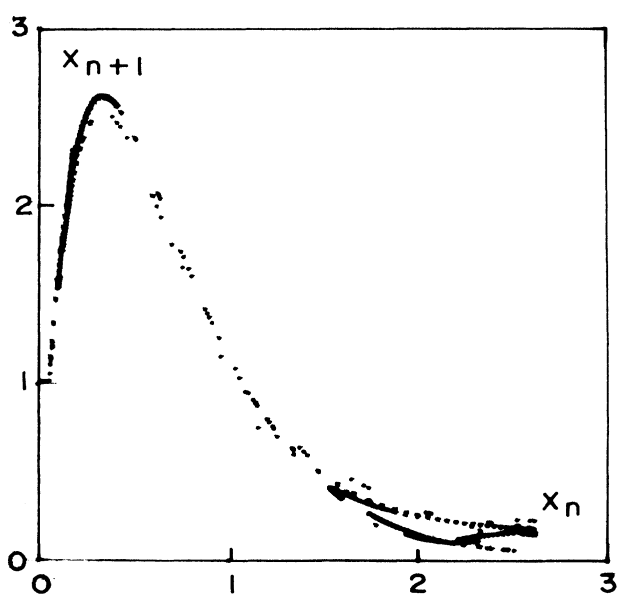

FIG. 6. Strange nonchaotic attractor in the presence of external noise with $D=0.6$.

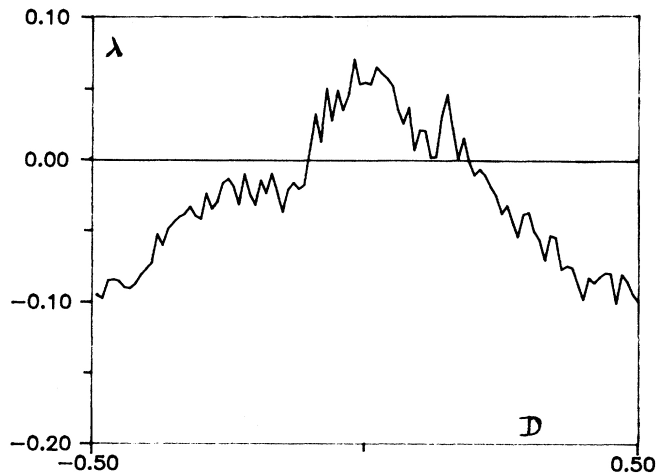

FIG. 7. Maximal Lyapunov exponent as a function of D for the BVP oscillator.

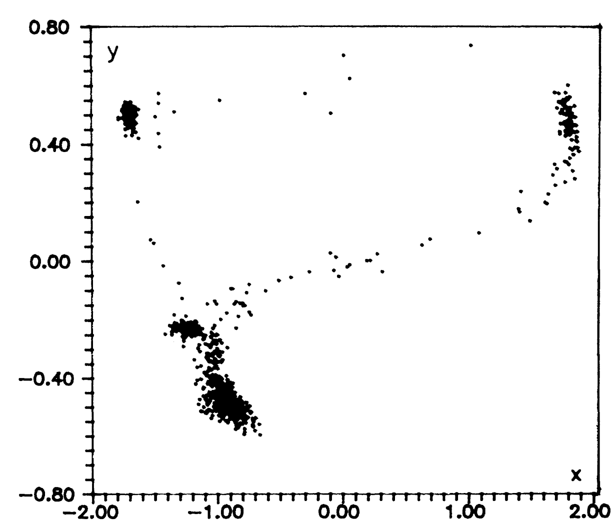

FIG. 8. Strange nonchaotic attractor of the BVP oscillator with $D=0.3$.

tractor is strange and nonchaotic. A strange nonchaotic attractor has been observed in the BVP equation when an appropriate Gaussian white noise is added to the system instead of chaos [5].

To conclude, we have shown here that there is a possibility of conversion of a strange and chaotic attractor to a strange but nonchaotic attractor by adding an appropriate chaotic signal and noise term in both the logistic map and BVP equation. To these systems chaos from deterministic dynamical systems is added. However, a chaotic solution from the same system is added to the logistic map (3), whereas a chaotic signal from a different dynamical system is added to the BVP equation. When a chaotic solution is added to these systems, a strange nonchaotic attractor is found to replace the strange chaotic attractor for a range of coupling strength. A strange nonchaotic attractor is also observed when Gaussian noise is added instead of a chaotic signal. Though the controlled orbit still appears complex, it is structurally stable and small errors in an initial condition will not have a strong effect on the long time prediction. Further, chaos is not always an unwanted phenomenon. It can be utilized for useful purposes. In such a case, conversion of a chaotic attractor to a strange nonchaotic attractor may provide much better predictability.

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