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Remark on "Some simple chaotic flows"

William G. Hoover
Department of Applied Science, University of California at Davis/Livermore, and Lawrence Livermore National Laboratory, Livermore, California 94551-7808
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Sprott [Phys. Rev. E **50**, 647 (1994)] developed a computerized search method and applied it to generate *millions* of sets of coupled chaotic ordinary differential equations. Each of the sets contained three coupled quadratic equations. Of these, the simplest, Sprott's case A, is the set of equations that Posch, Hoover, and Vesely [Phys. Rev. A **33**, 4253 (1986)] used to describe a single one-dimensional Nosé-Hoover oscillator. In this sense, the Nosé-Hoover oscillator system is the simplest mechanical system exhibiting chaos.

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During a relatively exhaustive stochastic computer search of millions of special cases, Sprott [1] found, and then displayed, five sets of three coupled first-order ordinary differential equations having the simplest possible structure consistent with chaos: "five terms and two nonlinearities." Of these, only a single set, his case A, has the fundamental dynamical property of *time reversibility*. With a slight change of notation, Sprott's unique case is

$$\dot{x}=p;\quad \dot{p}=-x-\zeta p;\quad \dot{\zeta}=p^2-1\ .$$

It is interesting that this set of equations, the simplest found in an arduous numerical search, is the set of equations describing a Nosé-Hoover oscillator [2,3]. The set was first used to investigate the utility of Nosé's thermostating ideas [4] a decade ago.

In the canonical-ensemble oscillator interpretation, x and p are, respectively, coordinate and momentum, while ζ is a *time-reversible* friction coefficient, which imposes a canonical temperature $\langle p^2\rangle \equiv \mathrm{mkT}$ on the oscillator. The most general form of the Nosé-Hoover oscillator equations differs from Sprott's case through the introduction of a relaxation time τ;

$$\dot{x}=p;\quad \dot{p}=-x-\zeta p;\quad \dot{\zeta}=[p^2-1]/\tau^2\ .$$

Sprott's Lyapunov exponents for this case [1] agree well with those we found earlier in our comprehensive investigation of Nosé-Hoover oscillator dynamics. See Fig. 13 of Ref. [3]. The phase-space attractor dimension of exactly three is directly traceable to the Nosé-Hoover oscillator's Hamiltonian heritage, which is not apparent from the equations' structure, but which was discussed in detail, in terms of Nosé's time scaling, in 1985 [2]. This Hamiltonian background also shows that ζ can be interpreted as a generalized momentum, so that ζ changes sign in the time-reversed motion.

The use of *integral feedback* to control the kinetic energy is a device that can be applied at (Gibbs ensembles), near (Green-Kubo theory), and far away from equilibrium (nonequilibrium molecular dynamics), as has by now been demonstrated with many examples [5]. The original *dissipative* system using Nosé-Hoover feedback was the Galton staircase [6], the one-dimensional field-driven motion of a thermostated particle in a periodic potential

$$\dot{x}=p;\quad \dot{p}=E-\sin x-\zeta p;\quad \dot{\zeta}=p^2-1\ .$$

With the field E turned off, this example is indistinguishable from a thermostated pendulum, and reduces again to the Nosé-Hoover oscillator when the oscillation amplitude is small. A host of few-body and many-body dissipative systems have since been treated with this same thermostat idea.

Sprott's work is an extremely interesting example of the ways in which computers and mathematics can converge upon, and provide insight into, problems with physical content.

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