Nonuniqueness of the Lorentz-Dirac equation with the free-particle asymptotic condition

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I show the nonuniqueness of the Lorentz-Dirac equation with the asymptotic condition of vanishing acceleration at the distant future, by studying the one-dimensional nonrelativistic motion of a charge in the presence of a potential step. As a minor result, I also show that, for position-dependent forces, the fact that the trajectory of the charge crosses a point in which the force diverges does not prevent the Lorentz-Dirac equation from having physical solutions.

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I. INTRODUCTION

The great majority of physicists agree that the motion of a classical elementary charged particle (say an electron) in the presence of its own radiation field is governed by the Lorentz-Dirac equation (LDE) [1,2]. Nevertheless some controversy persists nowadays due to the well-known strange features of this equation, namely, runaways and preacceleration.

The main characteristic of the LDE is the fact that it is a third-order differential equation for the position of the particle. If only initial position and velocity are fixed, this non-Newtonian character of the LDE introduces an indetermination which is overcome by imposing an additional condition. As one is obviously interested in physical solutions, the most accepted condition in a collision problem requires the motion of the particle to evolve to a free-particle motion as time goes to infinity [2]. And this asymptotic condition just makes it extremely difficult to obtain explicitly the trajectory of the charge in the presence of a given force field, both analytically [2–6] and numerically [7–9].

In particular it has been shown [8,9] that the problem of two unlike charges in a head-on collision has no physical solution. This is not very important in itself because the two charges will in fact collide with a non-null (although very small) impact parameter and people think that in this case a solution exists. The reason is that, as it has been argued [5,8–10], a problem arises due to the preacceleration if the particle "knows" that it is going to cross a point where the force diverges, which will not happen if the impact parameter does not exactly vanish.

In the present article I am going to analyze a relatively simple problem, namely, the one-dimensional motion of a nonrelativistic charge in the presence of a potential step. I shall consider both an ideal abrupt potential step and a more realistic smooth one. The results I want to point out are the two following. In the first place, we shall see that, in the abrupt case, the existence of a point in the actual trajectory of the charge where the force diverges does not preclude the existence of a physical solution, contrary to what was indicated above. That is, another particular feature of the Coulomb force, not shared by the potential step, is responsible for the nonexistence of physical trajectories. Second, and this is the most important result of the paper, the equation can admit more than one solution, which means that the prescription of vanishing acceleration at the distant future is not sufficient to determine the actual trajectory of the charge (this fact has already been reported on in Refs. [11,12]).

Before going on to the calculations I want to make some remarks. The asymptotic condition is added to the LDE in order to avoid the runaway behavior of the solution. Yet such a prescription gives rise to the preacceleration phenomenon. From a purely classical point of view this is unacceptable and the equation should be discarded. The reason the equation is accepted lies in the fact that the preacceleration affects time intervals of the order of $\tau=2e^2/(3mc^3)$, a very tiny quantity even for an electron ($\tau_{\text{electron}}=6.266\,33\times10^{-24}$ s). And this time interval, it is argued [2], involves microscopic processes that are to be explained by a quantum theory and not by a classical one.

The problem I am putting forward in this paper bears the same characteristic: its observation involves microscopic quantities and consequently does not pose any trouble if we are studying a situation in which a classical theory can be applied.

However, this situation poses a dilemma: either the classical electrodynamics is self-contradictory or the LDE is not the correct equation. And this has nothing to do with saying that classical electrodynamics does not fit the microscopic dynamics. In my opinion the first alternative of the dilemma has not been strictly proved and consequently I think we should seek another equation of motion for the charge. From my point of view, the regular publication of papers dealing with all these problems is evidence of the dissatisfaction existing with the LDE.

As I said above I am going to study the one-dimensional motion of a charge in the presence of a potential step. The equation of motion is the nonrelativistic version of the LDE, the so-called Abraham-Lorentz equation (ALE),

$$a=f+\tau\dot{a}\ , \quad f=-\frac{d}{dx}\varphi(x) \tag{1}$$

where a is the acceleration, $f=F/m$, F being the force acting on the particle, and m its mass. I will consider first, in Sec. II, an abrupt potential step in the form

$$\varphi(x)=\begin{cases}\varphi_0>0 & \text{if } x>0\\ 0 & \text{if } x<0\ ,\end{cases} \tag{2}$$

which leads to

$$f(x)=-\varphi_0\delta(x)\ . \tag{3}$$

Here one must impose $\varphi_0 \ll c^2$ for the nonrelativistic equation to be valid.

Next, in Sec. III I will study a smooth potential step

$$\varphi(x)=\begin{cases}\varphi_0 & \text{if } x>l\\ \dfrac{\varphi_0 x}{l} & \text{if } 0\le x\le l\\ 0 & \text{if } x<0\end{cases} \tag{4}$$

leading to

$$f(x)=\begin{cases}0 & \text{if } x>l\\ -\dfrac{\varphi_0}{l} & \text{if } 0<x<l\\ 0 & \text{if } x<0\ .\end{cases} \tag{5}$$

This more realistic case will require $l \gg c\tau$ (see at the end of Sec. V). However, I shall admit smaller values of l in order to attain the situation of (2) by means of the limit $l \to 0$, $f_0 \to \infty$, and $f_0 l \equiv \varphi_0 = \text{const}$, and thus to compare with the abrupt case, which is undertaken in Sec. IV. For the sake of clarity I also make a comparison with the nonradiative solutions. Finally, the nonuniqueness of the LDE, which is the main result of this paper, is analyzed in Sec. V. A brief discussion is presented in Sec. VI and a number of technical points are relegated to the Appendixes.

In the following the particle is considered to come from the left with an asymptotic velocity $v(-\infty)=v_{\text{in}}$. For the sake of simplicity I will assume that it reaches the point $x=0$ at $t=0$.

II. THE ABRUPT POTENTIAL STEP

A. Nonradiative solution

Energy conservation gives the following well-known solution.

(1) If $E_{\text{in}} \equiv \frac{1}{2}v_{\text{in}}^2 > \varphi_0$ the particle crosses the origin and

$$v^2(t)=\begin{cases}v_{\text{in}}^2 & \text{if } t<0\\ v_{\text{in}}^2-2\varphi_0 & \text{if } t>0\ .\end{cases} \tag{6}$$

(2) If $E_{\text{in}} < \varphi_0$ the particle cannot cross the origin and

$$v(t)=\begin{cases}v_{\text{in}} & \text{if } t<0\\ -v_{\text{in}} & \text{if } t>0\ .\end{cases} \tag{7}$$

B. Radiative solution

Let me first study the case in which the particle crosses the origin. For $t\neq 0$, $a=\tau\dot{a}$ and then

$$a(t)=\begin{cases}a_0 e^{t/\tau} & \text{if } t<0\\ b e^{t/\tau} & \text{if } t>0\ ,\end{cases} \tag{8}$$

with a_0 and b constants. The asymptotic condition, $a(\infty)=0$, imposes $b=0$.

Now, writing $\delta(x)\equiv\delta(x(t))=\delta(t)/|v(0)|$ and integrating the ALE (1) over an infinitesimal neighborhood of $t=0$ I get

$$\frac{\varphi_0}{v_0}+\tau a_0=0\ , \tag{9}$$

where $v_0=v(0)$. The solution is meaningful only if $v_0>0$. The velocity is given by

$$v(t)=\begin{cases}v_{\text{in}}+a_0\tau e^{t/\tau} & \text{if } t<0\\ v_0\equiv v_{\text{in}}+a_0\tau & \text{if } t>0\ .\end{cases} \tag{10}$$

Equation (9) may therefore be written

$$\varphi_0+a_0\tau v_{\text{in}}+(a_0\tau)^2=0\ . \tag{11}$$

This equation admits real solutions if

$$E_{\text{in}}\equiv\tfrac{1}{2}v_{\text{in}}^2>2\varphi_0\ . \tag{12}$$

Consequently, the particle crosses the origin if its initial energy is at least twice the potential step height, that is, twice the nonradiative threshold. If this is the case Eq. (11) has two solutions,

$$a_0\tau=\frac{-v_{\text{in}}\pm(v_{\text{in}}^2+4\varphi_0)^{1/2}}{2}\ , \tag{13}$$

giving

$$v_0=\frac{v_{\text{in}}}{2}\pm\tfrac{1}{2}(v_{\text{in}}^2-4\varphi_0)^{1/2}\ . \tag{14}$$

Both values of v_0 are positive, which means that the two solutions are valid. This shows the nonuniqueness of the LDE with the asymptotic prescription, a deeper analysis of which is relegated to a forthcoming section. As concerns the threshold energy for the particle to cross the origin, the fact that it has a larger value than in the nonradiative case is essentially due to the preacceleration phenomenon. Since the acceleration is continuous and the particle has to reach the origin with a negative acceleration, the deceleration must begin at earlier times. As a consequence, the particle arrives at the origin with a smaller velocity than it had at $t=-\infty$, that is, v_{in}. It is then reasonable to find that, in order to cross the origin, the particle needs to "begin" its motion with a larger velocity than in the nonradiative case.

If $E_{\text{in}}<2\varphi_0$, Eq. (11) has no solution. This means that the charge cannot cross the origin and then must return. Taking into account that the velocity is a continuous function of time, the only way this is possible is arriving at $x=0$ with zero velocity. Consequently, in this case, $v_0=0$ and $a_0\tau=-v_{\text{in}}$. This result will be obtained more convincingly as the limit case of a smooth potential step.

III. THE SMOOTH POTENTIAL STEP

In this section I study the motion of the charge in the presence of the potential given by Eq. (4). Again the case in which the charge reaches the region $x > l$, that is, $v(\infty) > 0$, and the case in which the charge is repelled by the force, $v(\infty) < 0$, have to be considered separately. In both cases I will denote t_1 the time at which the charge reaches $x = l$ or $x = 0$, respectively. [The case $v(x = l) = 0$ should be studied by using a smoothed force law, which is not undertaken in the present paper.]

A. Nonradiative solution

The motion is very well known. There are two cases.

1. $E_{\rm in} > \varphi_0$

The charge enters the region $x = l$ and

$$a(t) = \begin{cases} 0 & \text{if } t < 0 \\ -f_0 & \text{if } t \in [0, t_1] \\ 0 & \text{if } t > t_1 \,, \end{cases} \tag{15a}$$

$$v(t) = \begin{cases} v_{\rm in} & \text{if } t < 0 \\ v_{\rm in} - f_0 t & \text{if } t \in [0, t_1] \\ v_{\rm in} - f_0 t_1 \equiv v_1 & \text{if } t > t_1 \,, \end{cases} \tag{15b}$$

$$x(t) = \begin{cases} v_{\rm in} t & \text{if } t < 0 \\ v_{\rm in} t - \frac{1}{2} f_0 t^2 & \text{if } t \in [0, t_1] \\ l + v_1 (t - t_1) & \text{if } t > t_1 \,. \end{cases} \tag{15c}$$

In these expressions the time t_1 is determined by the condition

$$v_{\rm in} t_1 - \tfrac{1}{2} f_0 t_1^2 = l \,, \tag{16}$$

which has two solutions, the largest one corresponding to the time at which the charge would pass back through $x = l$ if the force did not vanish for $x > l$. Consequently,

$$t_1 = \frac{1}{f_0} [v_{\rm in} - (v_{\rm in}^2 - 2\varphi_0)^{1/2}] \tag{17}$$

and

$$v(t_1) \equiv v_1 = v_{\rm in} - f_0 t_1 = (v_{\rm in}^2 - 2\varphi_0)^{1/2} \,, \tag{18}$$

giving the same velocity for $t > t_1$ as in the abrupt case.

2. $E_{\rm in} < \varphi_0$

Equation (16) has no real solutions and the particle is compelled to return before arriving at $x = l$. Now t_1 is the time at which the charge passes back through $x = 0$,

$$0 = v_{\rm in} t_1 - \tfrac{1}{2} f_0 t_1^2 \,. \tag{19}$$

I thus obtain

$$t_1 = \frac{2 v_{\rm in}}{f_0} \tag{20}$$

and

$$v_1 = -v_{\rm in} \,. \tag{21}$$

The final velocity is equal to the initial one but in the opposite direction.

B. Radiative solution

It will be convenient for the last part of this paper to integrate Eq. (1) in each time interval, by making use of the following expression:

$$a(t_b) = a(t_a) e^{(t_b - t_a)/\tau} - (e^{t_b/\tau}/\tau) \int_{t_a}^{t_b} f(s) e^{-s/\tau} ds \,, \tag{22}$$

and then imposing the continuity of x, v, and a at $t = 0$ and $t = t_1$. I denote $v(0) = v_0$ and $v(t_1) = v_1$. For the sake of simplicity I introduce from the beginning the asymptotic condition $a(\infty) = 0$, which means $a(t) = 0$ $\forall t \geq t_1$. Again two cases have to be considered.

1. *The charge reaches the region $x > l$*

The solution of the motion is given by

$$a(t) = \begin{cases} a_0 e^{t/\tau} & \text{if } t < 0 \\ (a_0 + f_0) e^{t/\tau} - f_0 & \text{if } t \in [0, t_1] \\ 0 & \text{if } t > t_1 \,, \end{cases} \tag{23a}$$

$$v(t) = \begin{cases} v_{\rm in} + a_0 \tau e^{t/\tau} & \text{if } t < 0 \\ v_0 - f_0 t + (a_0 + f_0)\tau (e^{t/\tau} - 1) & \text{if } t \in [0, t_1] \\ v_1 & \text{if } t > t_1 \,, \end{cases} \tag{23b}$$

$$x(t) = \begin{cases} v_{\rm in} t + a_0 \tau^2 (e^{t/\tau} - 1) & \text{if } t < 0 \\ [v_0 - (f_0 + a_0)\tau] t - \frac{1}{2} f_0 t^2 + (a_0 + f_0)\tau^2 (e^{t/\tau} - 1) & \text{if } t \in [0, t_1] \\ l + v_1 (t - t_1) & \text{if } t > t_1 \,. \end{cases} \tag{23c}$$

The boundary conditions yield

$$a(t_1)\equiv 0=(a_0+f_0)e^{t_1/\tau}-f_0\ , \tag{24a}$$

$$v(0)\equiv v_0=v_{\rm in}+a_0\tau\ , \tag{24b}$$

$$v(t_1)\equiv v_1=v_0-f_0t_1+(a_0+f_0)\tau(e^{t_1/\tau}-1)\ , \tag{24c}$$

$$x(t_1)\equiv l=[v_0-(a_0+f_0)\tau]t_1-\tfrac{1}{2}f_0t_1^2+(a_0+f_0)\tau^2(e^{t_1/\tau}-1)\ . \tag{24d}$$

One must solve for a_0, v_0, v_1, and t_1. The simplest way to do it is to express everything in terms of t_1,

$$a_0=-f_0(1-e^{-t_1/\tau})\ , \tag{25a}$$

$$v_0=v_{\rm in}-f_0\tau(1-e^{-t_1/\tau})\ , \tag{25b}$$

$$v_1=v_{\rm in}-f_0t_1\ , \tag{25c}$$

$$l=(v_{\rm in}-f_0\tau)t_1-\tfrac{1}{2}f_0t_1^2+f_0\tau^2(1-e^{-t_1/\tau})\ . \tag{25d}$$

Note that the expression for v_1 is just the same as in the nonradiative case [Eq. (15b)]. However, the corresponding numerical values are different since t_1 takes different values in the two cases.

Now the problem reduces to finding a positive solution for t_1 from Eq. (25d) in such a way that v_0 and v_1 are also positive. The nonexistence of such a solution would mean that the particle stops down before reaching $x=l$ and then is compelled to reverse its run, a situation that will be treated below.

My first aim is to study the solutions of this equation in terms of l in order to compare with the abrupt case ($l\to 0$, $f_0\to\infty$, and $f_0l\equiv\varphi_0=$const). To do this let me introduce the following parameters:

$$\eta=\frac{E_{\rm in}}{\varphi_0}\equiv\frac{\frac{1}{2}v_{\rm in}^2}{f_0l}\ ,\quad \xi=\frac{v_{\rm in}t_1}{l}\ ,\quad \delta=\frac{l}{\tau v_{\rm in}}\ . \tag{26}$$

Now Eq. (25d) can be written as

$$1=\Lambda_\delta(\xi;\eta)\equiv\xi-\frac{1}{4\eta}\xi^2[1+h(\delta\xi)]\ , \tag{27}$$

where

$$h(u)=2\frac{u+e^{-u}-1}{u^2}\ ,\quad u=\delta\xi=\frac{t_1}{\tau}\ . \tag{28}$$

I show in Appendix A that $\Lambda_\delta(\xi;\eta)$ are functions similar to but different from a parabola, and lying between Λ_0 and Λ_∞, these being

$$\Lambda_0(\xi;\eta)=\xi-\frac{1}{2\eta}\xi^2\ , \tag{29}$$

$$\Lambda_\infty(\xi;\eta)=\xi-\frac{1}{4\eta}\xi^2\ . \tag{30}$$

Moreover, as a function of δ, Λ_δ decreases for each ξ from Λ_∞ to Λ_0 (see Fig. 1). A first result concerning the existence of solutions can already be stated as follows.

(1) If $1\le\eta/2$ there is a solution of (27) for every value of δ, and then for every value of l. This situation corresponds to $E_{\rm in}>2\varphi_0$, which is the condition obtained for $l=0$.

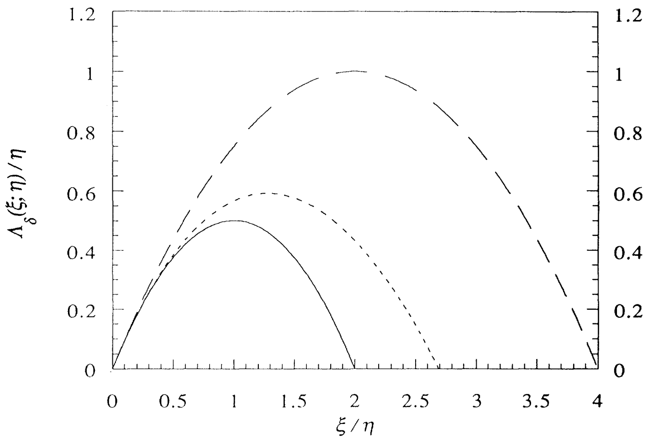

FIG. 1. $\Lambda_\delta(\xi;\eta)/\eta$ vs ξ/η, for three values of δ: ——, $\delta=0$; – –, $\delta=\infty$; and ---, $\delta=1,\eta=1$.

(2) If $\eta/2<1\le\eta$ a solution exists only if δ is larger than a certain value $\delta_{\rm th}(\eta)$. In particular, there is no solution for $\delta=0$.

(3) If $\eta<1$ no solution exists in any case and the particle returns. This corresponds to $E_{\rm in}<\varphi_0$, the nonradiative condition for the particle to be repelled.

In case (2) the value $\delta_{\rm th}(\eta)$ is given by the value of δ at which $1=\max\Lambda_\delta$. Denoting by $\xi_m(\delta)$ the position of the maximum, I can write

$$1=\Lambda_{\delta_{\rm th}(\eta)}(\xi_m(\delta_{\rm th}(\eta));\eta)\ . \tag{31}$$

Now, if $\delta>\delta_{\rm th}(\eta)$ I get a solution and if $\delta<\delta_{\rm th}(\eta)$ the charge is compelled to return. Recall that I have fixed the potential height, which amounts to f_0 being smaller for larger values of l. One can see that $\delta_{\rm th}$ is a decreasing function of η. Differentiating in Eq. (31) with respect to η I get

$$0=\left.\left[\frac{\partial\Lambda_\delta}{\partial\delta}\frac{d\delta_{\rm th}}{d\eta}+\frac{\partial\Lambda_\delta}{\partial\eta}+\frac{\partial\Lambda_\delta}{\partial\xi}\frac{\partial\xi_m}{\partial\delta}\frac{d\delta_{\rm th}}{d\eta}\right]\right|_{\substack{\xi=\xi_m\\ \delta=\delta_{\rm th}}}\ , \tag{32}$$

and, as $(\partial\Lambda_\delta/\partial\xi)|_{\xi_m}=0$,

$$\frac{d\delta_{\rm th}}{d\eta}=-\left.\frac{\partial\Lambda_\delta/\partial\eta}{\partial\Lambda_\delta/\partial\delta}\right|_{\substack{\xi=\xi_m\\ \delta=\delta_{\rm th}}}=\frac{1+h(\delta\xi_m)}{\xi_m\eta h'(\delta\xi_m)}<0\ , \tag{33}$$

where I have made use of properties (b) and (c) of $h(u)$ explained in Appendix A.

The validity of the foregoing analysis still requires that $v(0)$ and $v(t_1)$ be positive. To show that this is the case, note first that

$$v_0-v_1=f_0\tau(u-1+e^{-u})>0\ , \tag{34}$$

which says that the velocity is always smaller at $x=l$ than at $x=0$. As regards v_1, it can be seen in Fig. 1 that if Eq. (27) has solutions, at least one of them (note there might be two) fulfills $\xi<2\eta=v_{\rm in}^2/f_0l$. Therefore, I have

$$v_1 = v_{in} - f_0 t_1 \equiv v_{in} - \frac{f_0 l}{v_{in}} \xi > 0 \ , \tag{35}$$

and the analysis presented above is valid. However, an inspection of Fig. 1 indicates that for some values of η one can find not just one but two solutions fulfilling $\xi < 2\eta$, (that is, $v_1 > 0$). Due to its importance, this point will be analyzed in a specific section.

2. *The charge is repelled*

In the situation denoted above as (3) the charge is being acted upon by the force until it reaches back to the origin. The equations are the same as before, (23a)–(23c) and (24a)–(24c), except for the expressions concerning $x(t)$ at $t \geq t_1$. These are written now

$$x(t) = v_1(t - t_1) \ , \quad t > t_1 \tag{36a}$$

$$x(t_1) = 0 = [v_0 - (a_0 + f_0)\tau]t_1 - \tfrac{1}{2} f_0 t_1^2 + (a_0 + f_0)\tau^2(e^{t_1/\tau} - 1) \ , \tag{36b}$$

and finally Eq. (27) reads

$$0 = \Lambda_\delta(\xi;\eta) \ . \tag{37}$$

Obviously this equation has two solutions (see Fig. 1), $\xi_0 = 0$ and $\xi_1 > 0$, corresponding to the two times at which the charge passes through the origin. Note that $\xi_1 > 2\eta$ and so $v_1 < 0$. The LDE has one and only one solution in this case.

IV. THE LIMIT OF ABRUPT POTENTIAL STEP

This situation is attained by taking the limit $l \to 0$ while keeping $\varphi_0 = f_0 l$ constant. In particular we have to take the limit $\delta \to 0$ in Eq. (27), which means that the values of ξ satisfy $\Lambda_0(\xi;\eta) = 1$. I obtain the following.

(i) If $1 < \eta/2$, that is, $E_{in} > 2\varphi_0$, there are two different solutions,

$$\xi_\pm = \eta \pm (\eta^2 - 2\eta)^{1/2} \ , \tag{38}$$

from which, with the aid of Eqs. (25) and (26), one can reproduce the results of Sec. II B, Eqs. (13) and (14) (note that $l \to 0$ leads to $t_1 \to 0$).

(ii) If $1 > \eta/2$, that is, $E_{in} < 2\varphi_0$, there are no solutions of Eq. (27), which means that the particle returns towards the left. Consequently, ξ is now the solution of Eq. (37), that is, $\xi = 2\eta$. Finally I get, from Eqs. (25) and (26), $v_0 = v_1 = 0$ and $a_0 \tau = -v_{in}$, reproducing again the results obtained for the abrupt case in Sec. II B.

V. NONUNIQUENESS OF SOLUTIONS

It has been seen that in the case $l \to 0$ two solutions appear that are physically acceptable. Let me analyze this for an arbitrary value of l. The problem arises when studying the solutions of Eq. (25d) or Eq. (27) in the case where the particle penetrates the $x > l$ region. The point is that v_1 is non-negative for any value of ξ smaller than 2η, as one can easily deduce from Eqs. (25c) and (26). And one observe in Fig. 1 that for any finite δ the max-

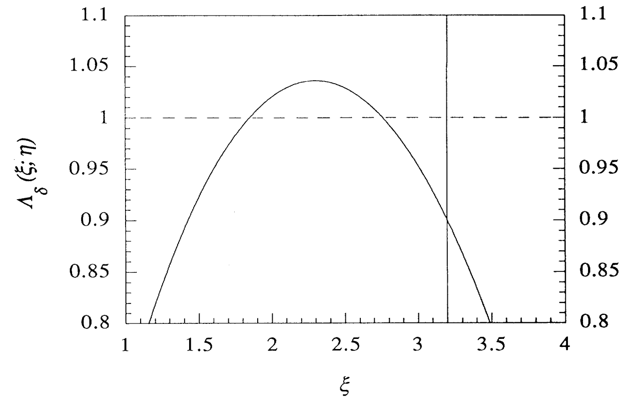

FIG. 2. $\Lambda_\delta(\xi;\eta)$ vs ξ for $\eta = 1.6$ and $\delta = 1$ showing the existence of two physical solutions ($\xi < 2\eta = 3.2$) of $1 = \Lambda_\delta(\xi;\eta)$: $\xi_1 \cong 1.84$ and $\xi_2 \cong 2.75$.

imum of $\Lambda_\delta(\xi;\eta)$ lies on the left of 2η. Consequently, if η and δ are such that 1 lies between $\Lambda_\delta(2\eta;\eta)$ and $\max\Lambda_\delta$, then there are two values of ξ fulfilling Eq. (27) and giving $v_1 > 0$ (see Fig. 2). Moreover, whatever the solution is, the particle cannot have passed $x = l$ at a prior time for this would require negative velocities at some time interval before t_1. And this is inconsistent with v_1 being positive, since the acceleration is never positive, as can be seen from Eqs. (23a) and (25a).

Now the question arises as to the origin of this double solution. The answer is easy: the asymptotic prescription of vanishing acceleration at remote future does not serve to completely determine the physical solution. In other words, there can be more than one solution of the third-order differential equation (1) for which the acceleration goes to zero as $t \to \infty$. To see this clearly I am going to analyze how the acceleration at t_1 depends on the various parameters of the problem. Note that the asymptotic condition amounts in this case to imposing $a(t_1) = 0$ because for $t > t_1$ the solution is given by $a(t) = a(t_1)\exp[(t - t_1)/\tau]$. Equation (24a) yields

$$a(t_1) = (a_0 + f_0)e^{t_1/\tau} - f_0 \ . \tag{39}$$

Now I should have to express t_1 in terms of a_0. However, it is easier to use t_1 as the independent variable and to write a_0 in terms of it. By using Eqs. (24b) and (24d) I obtain

$$a_0 + f_0 = \frac{l - (v_{in} - f_0\tau)t_1 + \frac{1}{2} f_0 t_1^2}{\tau^2(e^{t_1/\tau} - 1)} \ , \tag{40}$$

and then

$$a(t_1) = f_0 \frac{\Delta - (\nu - 1)u + \frac{1}{2}u^2 - 1 + e^{-u}}{1 - e^{-u}} = f_0 \rho(u) \ , \tag{41}$$

where I have set

$$\Delta = \frac{l}{f_0\tau^2} \ , \quad \nu = \frac{v_{in}}{f_0\tau} \ , \quad u = \frac{t_1}{\tau} \ . \tag{42}$$

Recall that the largest allowed value of t_1 for the particle not to be repelled is v_{in}/f_0, which is written in terms of the new parameters as

$$0 \leq u \leq \nu \ . \tag{43}$$

I want to know how $\rho(u)$ behaves for $u \in [0,\nu]$. A detailed analysis of this can be found in Appendix B. In the following I present the results.

In any case ρ behaves as $u^2/2$ as $u \to \infty$. For $\Delta=0$ it is strictly increasing if $\nu<2$ and presents one unique minimum if $\nu>2$. It cuts the u axis at a point on the right of $u=\nu$ and its derivative at $u=\nu$ is positive. For $\Delta \neq 0$ ρ behaves like Δ/u as $u \to 0$, but if Δ is small enough, for $u>0$ ρ is very similar to its value for $\Delta=0$, although slightly larger.

Now the behavior of ρ can be understood in terms of two functions $\Delta_+(\nu)$ and $\Delta_-(\nu)$ defined in Appendix B [see Eqs. (B14), (B15a), and (B15b)] and shown in Fig. 3. If, for given ν, $\Delta \in (0,\Delta_-(\nu))$ ρ has only one zero on the left of ν. The other zero lies on its right. When $\Delta \to \Delta_-(\nu)$ from below this second zero approaches ν. If $\Delta \in (\Delta_-(\nu),\Delta_+(\nu))$ the two zeros lie within the interval $(0,\nu)$. For Δ going to $\Delta_+(\nu)$ from below the two zeros tend to collapse in a single one. This zero coincides with the minimum of ρ. If $\Delta > \Delta_+(\nu)$ ρ has no zeros.

Finally, recalling that the physical solutions are the solutions for which $\rho=0$, one can conclude the following: if the pair (ν,Δ) lies, region I of the plane ν-Δ in Fig. 3, there exists just one physical solution; if (ν,Δ) lies in region III the charge cannot reach the point $x=l$ and is repelled; and finally, if (ν,Δ) lies in region II there exist two physical solutions, which is the main result of this paper.

In Fig. 4 I present the five typical situations, $\Delta<\Delta_-$, $\Delta=\Delta_-$, $\Delta \in (\Delta_-,\Delta_+)$, $\Delta=\Delta_+$, and $\Delta>\Delta_+$, for $\nu=1$, Δ_- and Δ_+ being $\Delta_- \cong 0.132\,12$ and $\Delta_+ \cong 0.272\,03$. In particular, if $\nu=1$ and $\Delta=0.2$, the two solutions for u are $u^{(1)} \cong 0.269\,65$ and $u^{(2)} \cong 0.875\,70$. The acceleration at $t=0$ can be calculated from Eq. (39) or (40) giving $a_0^{(1)} \cong -0.236\,36 f_0$ and $a_0^{(2)} \cong -0.583\,43 f_0$. The velocities at $t=0$ and t_1 take the values $v_0^{(1)} \cong 0.763\,64 f_0\tau$, $v_0^{(2)} \cong 0.416\,57 f_0\tau$, $v_1^{(1)} \cong 0.730\,34 f_0\tau$, and $v_1^{(2)} \cong 0.124\,29 f_0\tau$. In Fig. 5 the time evolution of the acceleration, the velocity, and the position for both solutions are shown along with the nonradiative ones.

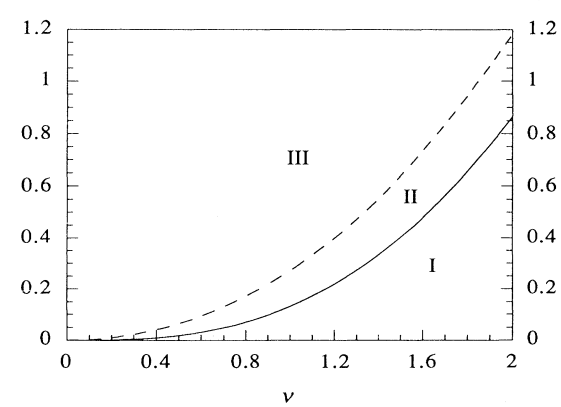

FIG. 3. Δ_+ (— — —) and Δ_- (——) delimiting the region of existence of two solutions of the LDE.

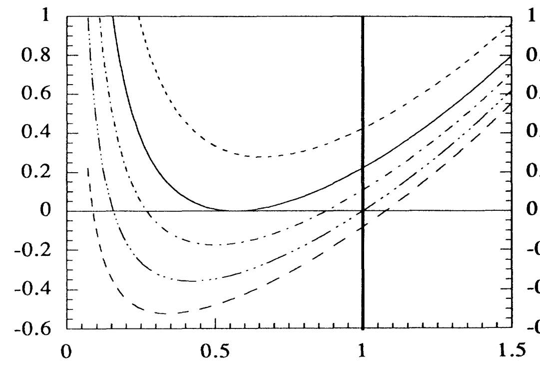

FIG. 4. Five typical situations of $\rho(u)$ showing the zeros that represent physical solutions of the LDE. $\nu=1$ and the values of Δ are — — —, 0.08; – · · · –, $0.132\,12 \equiv \Delta_-$; -·-·-·, 0.2; ——, $0.272\,03 \equiv \Delta_+$; and ---, 0.4.

To end this section I am going to analyze the order of magnitude of the *separation* between the two solutions. As can be seen in Eq. (14) the two solutions of the abrupt case can be quite different, depending on the initial energy. However, in macroscopic and even mesoscopic situations, having such an abrupt potential step is not possible and we should impose some restrictions on the possible values of the various parameters. As we shall see, if this is done, the two solutions will be too close to each other so as to be distinguished at a nonmicroscopic level. The reason is the astonishing smallness of τ as compared to any time scale occurring in the classical world ($\tau=6.266\,33 \times 10^{-24}$ s). We can therefore think that the time intervals taking place in our system should be much larger than τ, in particular, $u=t_1/\tau \gg 1$. Moreover, the length of the barrier, l, should be large as compared to the classical radius $c\tau$ ($=1.878\,51$ F) and *a fortiori*, $l \gg v_{in}\tau$. Also the distance the particle travels due to the action of the force in a time of the order of τ will be very small with respect to l. We arrive at the following relations:

$$u \gg 1 \ , \tag{44a}$$

$$\Delta \equiv \frac{l}{f_0\tau^2} \gg 1 \ , \quad \frac{\Delta}{\nu} \equiv \frac{l}{v_{in}\tau} \gg 1 \ . \tag{44b}$$

Note that, whereas $\Delta \gg \nu$, in general $\nu^2/\Delta = 2\eta$ can be arbitrary.

The first point is to see that, in the case where the particle penetrates the $x>l$ region, condition (44a) is a consequence of (44b). If we write Eq. (25d) with the parameters defined in (42) we get

$$\Delta = (\nu-1)u - \tfrac{1}{2}u^2 + 1 - e^{-u} \equiv \Psi(u) \ . \tag{45}$$

Now, I will write the same equation for the nonradiative case. The nonradiative variables are denoted with an overbar):

$$\Delta = \nu\bar{u} - \tfrac{1}{2}\bar{u}^2 \equiv \bar{\Psi}(\bar{u}) \ . \tag{46}$$

Since for $u>0$, $\Psi(u) < \bar{\Psi}(\bar{u})$, and considering for the moment the solution of (47) with positive slope, we deduce from Eq. (46) that $u > \bar{u}$, $\Delta < \nu\bar{u}$, and finally, $u > \bar{u} > \Delta/\nu \gg 1$. Obviously, the other solution (the

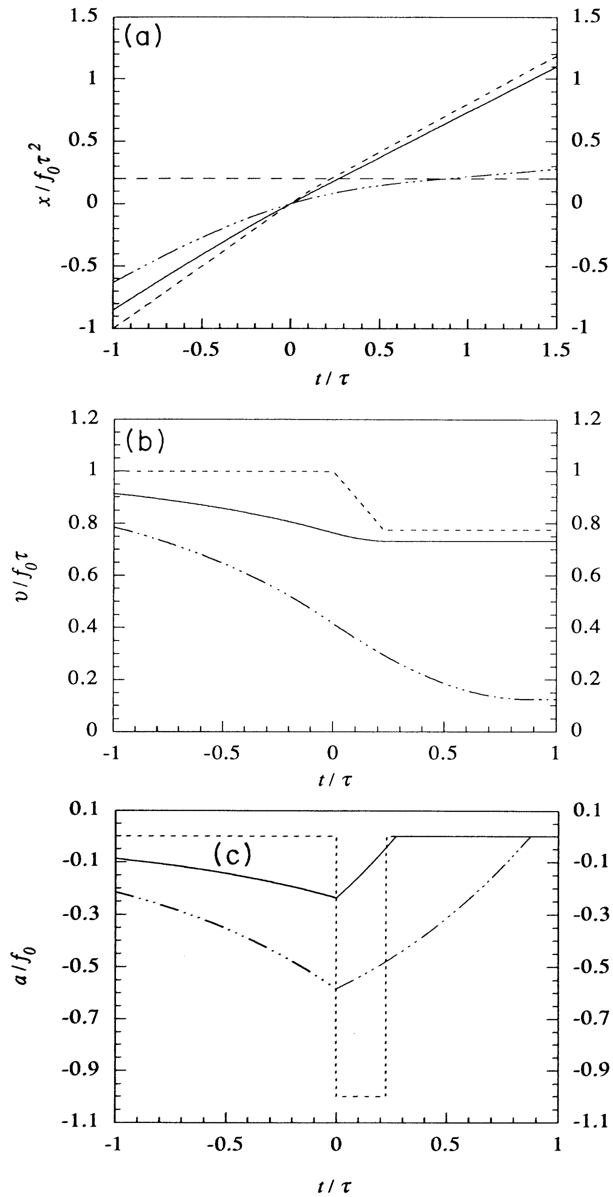

FIG. 5. The two solutions of the LDE vs t/τ, along with the nonradiative solution, for $\nu=1$ and $\Delta=0.2$: ——, $a_0^{(1)} \cong -0.236\,36 f_0$; – · · · –, $a_0^{(2)} \cong -0.583\,43 f_0$; ---, nonradiative solution; (a) position, (b) velocity, and (c) acceleration.

value of u with negative slope), if it exists, is even greater. Moreover, if Eq. (45) has a solution, so does Eq. (46) and then $\Delta < \max(\nu\bar{u} - \bar{u}^2/2) = \nu^2/2$. One thus concludes that $\nu^2 > 2\Delta \gg 1$.

If the particle is made to turn back, it is evident that the value of Δ (which has to be larger than a certain threshold value) does not influence the other parameters. In this case $u \gg 1$ implies $\nu \gg 1$. To see it I will write the equation for u,

$$0 = (\nu - 1)u - \tfrac{1}{2}u^2 + 1 - e^{-u} \ . \tag{47}$$

Solving for ν,

$$\nu = \frac{u}{2} + \left[\frac{u - 1 + e^{-u}}{u} \right] > \frac{u}{2} \ , \tag{48}$$

which proves the assertion.

In conclusion, one has the following restrictions:

$$u \gg 1 \ , \quad \bar{u} \gg 1 \ , \quad \Delta \gg \nu \ , \quad \nu \gg 1 \ . \tag{49}$$

To become aware of the relative order of magnitude of the various quantities corresponding to the two solutions it suffices to calculate the width of region II in Fig. 3 when conditions (49) hold. From Eqs. (B14) and (B23) I have

$$\Delta_+(\nu) - \Delta_-(\nu) = \tfrac{1}{2} + O(e^{-\nu}) \ . \tag{50}$$

Therefore the interval δl of values of l for which there exist two solutions is smaller than $f_0\tau^2/2$. Thus, taking into account the nonrelativistic condition $f_0\tau \ll c$, I get

$$\delta l < \tfrac{1}{2} f_0 \tau^2 \ll \tfrac{1}{2} c\tau \ , \tag{51}$$

which is an extremely tiny quantity.

The same analysis can be made for the values of the initial velocity $v_{\rm in}$. I must solve for ν in terms of Δ for each curve $\Delta_+(\nu)$ and $\Delta_-(\nu)$. Let $\nu_+(\Delta)$ and $\nu_-(\Delta)$ be the corresponding expressions. From Eqs. (B14) and (B23) I get

$$\frac{\delta v_{\rm in}}{v_{\rm in}} = \frac{\nu_+ - \nu_-}{\nu} \cong \frac{1}{4\Delta} \ll 1 \ . \tag{52}$$

This can be achieved by neglecting the $e^{-\nu}$ dependence in Eqs. (B14) and (B23), and then obtaining ν as a function of Δ_+ and Δ_-, respectively.

Finally it remains to study the difference between the values of the time t_1 at which each solution arrives at $x = l$. The largest interval length of values of u corresponds to the difference between ν, the upper limit value of u [see (43)], and the other value of u at which the curve $\Psi(u)$ [see Eq. (45)] attains the same value as it takes at $u = \nu$. Taking into account that both values are close to ν I can in first approximation neglect the e^{-u} term in Eq. (45) and then get two approximate solutions for u. I obtain $\delta u \approx 2$ and so

$$\delta t_1 \approx 2\tau \ . \tag{53}$$

Consequently, one can see that, if conditions stated in (49) hold, the two solutions, when existing, are very close to one another, and one might attribute the nonuniqueness of the LDE to the necessity of resorting to quantum physics whenever such small time and length intervals are involved.

VI. SUMMARY AND DISCUSSION

As the main result of the paper, I have shown that, in the one-dimensional motion of a radiating charged particle in the presence of a potential step (whether abrupt or smooth), and for certain values of the parameters involved, two different trajectories with a vanishing acceleration at the distant future fulfill the ALE (the nonrelativistic version of the LDE). This means that in general, the LDE, complemented with the asymptotic condition, does not have a unique solution, or, in other words, does not predict the actual motion of the charge. And my opinion is that overcoming this trouble is not simply a matter of *selecting* one out of the various possible solutions.

Accordingly, I conclude that the LDE, along with the asymptotic condition, does not constitute an appropriate *fundamental equation* for the motion of charged particles in classical electrodynamics. Once one arrives at this conclusion, two attitudes can be adopted: either classical electrodynamics is inconsistent because the LDE is an inevitable consequence of the theory, or it is a well-behaved theory and simply the correct equation governing the motion of charged particles has not yet been found. In fact, a number of alternative equations has been proposed during this century, although none of them was found to be satisfactory [13].

At this point I should like to bring to the reader's attention the already well-known difficulties of the LDE. This equation by itself admits an infinity of solutions because of its third-order differential equation character. If one allows the acceleration to be zero before the force sets in, two difficulties arise: (i) the charge runs away when the force stops acting upon it, and (ii) if the force has a constant direction, the acceleration goes in the opposite direction [14].

These two problems can be overcome by imposing the asymptotic condition I have referred to above. But now another difficultly arises, namely, the well-known preacceleration. I might say that this is in some sense the smallest problem one can attain with the LDE. It is in fact a very tiny one, since it affects times of the order of 10^{-24} s, which involves processes at the level of subatomic particles (see a discussion on this point in Ref. [9]).

The situation with the nonuniqueness of the equation is similar, since the two solutions found in this paper are very close to each other, to the extent that the separation between them takes on microscopic values, at least for a realistic potential. This situation supports the first alternative referred to above in the sense that classical electrodynamics governs correctly the dynamics of charged particles in the presence of electromagnetic radiation but only if microscopic processes are not involved; otherwise it would be necessary to make use of a quantum theory.

This attitude seems to be correct. However, a new question arises: is it really impossible to find in classical electrodynamics an equation of motion devoid of conceptual difficulties? Note that I do not mean an equation of motion capable of explaining the actual dynamics of charges, but simply a completely well-behaved equation of motion. Were the answer yes, one should admit self-inconsistency of classical electrodynamics, but in my opinion this is still an open question, which needs to be clarified.

Finally, as a minor result, I have obtained a physical solution (two in fact) of the LDE in the presence of a divergent force (abrupt case). This is in contrast with a claim made in Refs. [5–9] that the presence of such a divergence would prevent the existence of physical solutions in the case of head-on collisions of unlike charges.

ACKNOWLEDGMENT

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APPENDIX A

Let me begin studying the function $h(u)$ [see Eq. (28)]. First of all a series expansion of Eq. (28) readily leads to

$$h(0)=1\ ,\quad h'(0)=-\tfrac{1}{3}\ . \tag{A1}$$

Now I am going to prove the following properties, valid for $u>0$: (a) $h(u)<2/u$; (b) $h(u)>2/(u+2)>0$; (c) $h'(u)<0$.

Proof.

(a) $u>0 \Longrightarrow u-(1-e^{-u})<u \Longrightarrow h(u)<2/u$.

(b) Let $\mu(u)\equiv u^2[(u+2)h(u)-2]$. It is easy to see that $\mu(0)=0$, $\mu'(0)=0$, and $\mu''(u)=2ue^{-u}>0$. This means that $\mu(u)$ is a positive function for $u>0$ and then, as $u^2>0$, $h(u)>2/(u+2)$.

(c) $h'(u)=[2-(u+2)h(u)]/u<0$ according to (b).

Property (c) says that h is a decreasing function for $u>0$. Since $h(0)=1$ I deduce

$$h(u)<1 \quad \forall u>0\ . \tag{A2}$$

Now, let me introduce

$$D(\xi,\delta)=\Lambda_\delta(\xi;\eta)-\Lambda_0(\xi;\eta)=\frac{\xi^2}{4\eta}[1-h(\delta\xi)]\ . \tag{A3}$$

D is an increasing function of its two variables since

$$\frac{\partial D}{\partial \xi}=\frac{\xi}{2\eta}[1-h(\delta\xi)]-\frac{\xi^2}{4\eta}\delta h'(\delta\xi)>0 \tag{A4}$$

and

$$\frac{\partial D}{\partial \delta}=-\frac{\xi^3}{4\eta}h'(\delta\xi)>0\ , \tag{A5}$$

where I have made use of properties (A1), (A2), and (c). Consequently, Λ_δ describes a family of curves lying between Λ_0 and Λ_∞ in such a way that the distance between Λ_0 and Λ_δ increases with both δ and ξ (see Fig. 1).

APPENDIX B

Consider the function

$$\alpha=\rho+1=\frac{\Delta-(\nu-1)u+\frac{1}{2}u^2}{1-e^{-u}}\ . \tag{B1}$$

Throughout this appendix the variable u takes only positive values. In the following I am going to show that α, and then ρ, exhibits at most one minimum at a positive value of u. Note that α behaves as $u^2/2$ for large u, whereas its behavior for small u depends on the values of both Δ and ν,

$$\alpha(u)\underset{u\to 0}{\sim}\frac{\Delta}{u}[1+O(u)]-(\nu-1)[1+O(u)]+\frac{u}{2}+O(u^2)\ . \tag{B2}$$

Now, to study the extrema of α, I calculate its derivative

$$\alpha'=\frac{f(u)}{(1-e^{-u})^2}\ , \tag{B3}$$

where

$$f(u)=-\Delta e^{-u}+f_\nu(u)\ , \tag{B4}$$

$$f_\nu(u)=-(\nu-1)\sigma(u)+f_0(u)\ , \tag{B5}$$

$$\sigma(u)=1-e^{-u}-ue^{-u}\ , \tag{B6}$$

$$f_0(u)=u-ue^{-u}\left[1+\frac{u}{2}\right] . \tag{B7}$$

Consider first the case $\Delta=0$. The functions σ and f_0 are easily seen to have the following properties:

$$f_0(0)=f_0'(0)=0\ ,\quad f_0''(0)=1\ ,\quad f_0'(u)>0\ , \tag{B8a}$$

$$f_0(u)\underset{u\to\infty}{\sim}u\ ,\quad f_0(u)\underset{u\to 0}{\sim}\frac{u^2}{2}+O(u^4)\ , \tag{B8b}$$

$$\sigma(0)=\sigma'(0)=0\ ,\quad \sigma'(u)>0\ , \tag{B8c}$$

$$\sigma(\infty)=1\ ,\quad \sigma(u)\underset{u\to 0}{\sim}\frac{u^2}{2}-\frac{u^3}{3}+O(u^4)\ . \tag{B8d}$$

The behavior of f_ν close to the origin will also be useful,

$$f_\nu(u)\underset{u\to 0}{\sim}u^2\left[1-\frac{\nu}{2}\right]+O(u^3)\ . \tag{B9}$$

All this leads to the following.

(a) If $\nu\le 1$, then $f_\nu'(u)>0$. Consequently, f_ν is a monotonically increasing function vanishing at the origin. As for ρ, one can see from Eqs. (B1) and (B3) that $\rho'>0$. Finally, $\rho(0)=-\nu<0$.

(b) If $\nu>1$ the analysis is more complicated. If I write

$$f_\nu''=-(\nu-1)\sigma''+f_0''=e^{-u}\theta\ , \tag{B10}$$

with θ given by

$$\theta(u)=-\frac{u^2}{2}+u\nu+2-\nu\ , \tag{B11}$$

the following two cases are to be distinguished.

(i) $1<\nu\le 2$. $\theta(u)$ is an inverted parabola crossing the u axis at a positive value u_2. Then, for $u<u_2$, $f_\nu''>0$ and f_ν' increases; and for $u>u_2, f_\nu''<0$ and f' decreases. As $f_\nu'(0)=0$ and $f_\nu'(\infty)=1$ this means that f_ν' must be positive for $u>0$, and as $f_\nu(0)=0$ I get that f_ν is always positive. Consequently, ρ is again monotonically increasing.

(ii) $\nu>2$. In this case there exist two positive values of u, say u_1 and u_2, at which θ vanishes. Then, $f_\nu''<0$ for both $u<u_1$ and $u>u_2$, and $f_\nu''>0$ for $u_1<u<u_2$. As $f_\nu'(0)=0$ and $f_\nu'(\infty)=1$ I finally conclude that f_ν' is negative up to a certain value of u, say u_0, and then becomes positive. Consequently, f_ν has one maximum at $u=0$, where the function is zero, and one minimum at u_0. Taking into account the behavior of f_ν at infinity, all this means that it must vanish at some unique positive value of u larger than u_0. Finally, from Eqs. (B1), (B3), and (B9) I obtain $\rho'(0)=(1-\nu/2)<0$, which says that ρ' vanishes at only one point, showing the existence of a unique minimum.

The case $\Delta>0$ is now easy to solve. The idea is based on the relation (B4), where $\Delta e^{-\nu}$ is a strictly decreasing function. Let u_ν be the point at which f_ν vanishes (in the case $\nu>2$, u_ν is the positive value). For $u\le u_\nu$, $f(u)<0$ and, for $u\to\infty$, $f(u)\sim f_\nu(u)$. I then conclude that there exists a point $u_z>u_\nu$ for which $f(u_z)=0$. I am going to prove now that u_z is the unique zero of f. Since $u_z>u_\nu$, $f_\nu(u_z)>0$. If $u>u_z$, $f_\nu(u)-f(u)=\Delta e^{-u}<\Delta e^{-u_z}=f_\nu(u_z)-f(u_z)$, and then $f(u)>f_\nu(u)-f_\nu(u_z)>0$ because, as I said above, f_ν is an increasing function for $u\ge u_\nu$. On the other hand, if $u_\nu<u<u_z$ (this affects only the case $\nu>2$), $f_\nu(u)-f(u)=\Delta e^{-u}>\Delta e^{-u_z}=f_\nu(u_z)-f(u_z)$. Then, $f(u)<f_\nu(u)-f_\nu(u_z)<0$, the last inequality being due again to the increasing behavior of f_ν for $u>u_\nu$. This ends the proof of the uniqueness of u_z, which amounts to saying that ρ has a unique minimum.

In the following I am going to study how the location of the positive zeros of ρ varies with Δ and ν. For the sake of clarity I initially consider a fixed value of ν, thus studying the dependence with Δ. This is easy to do if I notes that ρ grows up monotonically for each value of ν as Δ increases from zero. And this growth is continuous except at $u=0$. For $\Delta=0$ it is easy to show that ρ has one zero on the right of ν, the slope being positive at this point. To see this I will calculate ρ and ρ' at $u=\nu$ with $\Delta=0$. From Eq. (B1) it follows that

$$\rho(\nu)=\frac{\nu-(\nu^2/2)-1+e^{-\nu}}{1-e^{-\nu}}=\frac{r(\nu)}{1-e^{-\nu}}\ . \tag{B12}$$

The function $r(\nu)$ is easily seen to be negative for all positive values of ν. On the other hand,

$$\rho'(u)|_{u=\nu}=\frac{1+e^{-\nu}(\nu^2/2-\nu-1)}{(1-e^{-\nu})^2}\ , \tag{B13}$$

which is always a positive function.

Finally, taking into account the behavior of ρ at the origin, one can conclude that, for Δ small enough but different from zero, $\rho(\nu)$ cuts the ν axis at two points, one of them being smaller and the other larger than ν. Moreover it always reaches the point $u=\nu$ in the growing part of the curve.

For every fixed u, when Δ grows up, the function ρ always increases. At a certain value of Δ, say Δ_-, it exactly vanishes at $u=\nu$. For $\Delta>\Delta_-$ the two zeros of ρ are on the left of ν. At a certain later value of Δ, Δ_+, ρ cuts the ν axis at just one point that coincides with its minimum. For $\Delta>\Delta_+$ there are no zeros of ρ. A typical situation can be visualized in Fig. 4 in the main text.

It remains to calculate the functions Δ_+ and Δ_-. The value of Δ_- is very easy to obtain from the condition $\rho(\nu)=0$. I get

$$\Delta_-=\frac{(\nu-1)^2}{2}+\frac{1}{2}-e^{-\nu}\ . \tag{B14}$$

Δ_- is an increasing function of ν. As concerns Δ_+ it is defined by the conditions $\rho'(u_m)=0$ and $\rho(u_m)=0$. If I write in Eq. (B1) $\alpha=f(u;\Delta)/E(u)$, the former conditions are equivalent to

$$f(u_m;\Delta_+)=E(u_m)\ , \tag{B15a}$$

$$f'(u_m;\Delta_+)=E'(u_m)\ . \tag{B15b}$$

It is not possible to get an explicit expression of Δ_+ in terms of ν. But it is easy to prove the following properties:

$$\frac{d}{d\nu}\Delta_+(\nu)>0 \ , \tag{B16a}$$

$$\frac{d}{d\nu}[\Delta_+(\nu)-\Delta_-(\nu)]>0 \ . \tag{B16b}$$

To see it I first solve for u_m in Eq. (B15b),

$$-(\nu-1)+u_m=e^{-u_m} \ . \tag{B17}$$

This equation has one unique positive solution $u_m(\nu)$ fulfilling

$$u_m(0)=0 \ , \quad u_m'(\nu)=\frac{1}{1+e^{-u_m}}>0 \ . \tag{B18}$$

Finally I introduce this value into Eq. (B15a) and calculate Δ_+,

$$\Delta_+(\nu)=(\nu-1)u_m(\nu)-\frac{u_m^2(\nu)}{2}+1-e^{-u_m(\nu)}$$
$$\equiv\bar{\Delta}(u_m(\nu),\nu) \ . \tag{B19}$$

Now $\Delta_+(0)=0$ and

$$\Delta'_+(\nu)=\frac{\partial\bar{\Delta}}{\partial u_m}\frac{\partial u_m}{\partial \nu}+\frac{\partial\bar{\Delta}}{\partial \nu}=u_m>0 \ , \tag{B20}$$

where I have used Eq. (B17). This proves (B16a). To prove (B16b) I will set

$$\Delta'_+-\Delta'_-=u_m-(\nu-1+e^{-\nu}) \ . \tag{B21}$$

Since u_m satisfies Eq. (B17) I get

$$\Delta'_+-\Delta'_-=e^{-u_m}-e^{-\nu} \quad \text{and} \quad \nu-u_m=1-e^{-u_m}>0 \ , \tag{B22}$$

which ends the proof of (B16b). A plot of the functions Δ_+ and Δ_- can be found in Fig. 3 in the main text.

I finally give an approximate expression of Δ_+ in the limit $\nu\gg1$. From (B17) I get $u_m=\nu-1+O(e^{-\nu})$, and putting it into Eq. (19).

$$\Delta_+=\frac{(\nu-1)^2}{2}+1+O(e^{-\nu}) \ . \tag{B23}$$

[1] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **167**, 148 (1938).

[2] F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965).

[3] C. J. Eliezer, Rev. Mod. Phys. **19**, 147 (1947).

[4] G. N. Plass, Rev. Mod. Phys. **33**, 37 (1961).

[5] P. A. Clavier, Phys. Rev. **124**, 616 (1961).

[6] E. Comay, Found. Phys. Lett. **3**, 221 (1990).

[7] J. Huschilt and W. E. Baylis, Phys. Rev. D **13**, 3256 (1976).

[8] W. E. Baylis and J. Huschilt, Phys. Rev. D **13**, 3262 (1976).

[9] S. Parrot, Found. Phys. **23**, 1093 (1993).

[10] J. Huschilt and W. E. Baylis, Phys. Rev. D **17**, 985 (1978).

[11] W. E. Baylis and J. Huschilt, Phys. Rev. D **13**, 3237 (1976).

[12] S. F. Gull, in *The Electron, New Theory and Experiment*, edited by D. Hestenes and A. Weingartshofer (Reidel, Dordrecht, 1991).

[13] The number of papers dealing with this issue is too large to give here an exhaustive list. Those readers interested in it may consult, for example, J. M. Aguirregabiria, A. Chamorro, and M. A. Valle, Phys. Lett. **89A**, 114 (1982); L. Herrera, Phys. Lett. A **145**, 143 (1990); G. W. Ford and R. F. O'Connell, *ibid.* **174**, 182 (1993), and references therein.

[14] R. Blanco, Phys. Lett. A **169**, 115 (1992).