Soliton evolution and radiation loss for the Korteweg–de Vries equation

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The time-dependent behavior of solutions of the Korteweg–de Vries (KdV) equation for nonsoliton initial conditions is considered. While the exact solution of the KdV equation can in principle be obtained using the inverse scattering transform, in practice it can be extremely difficult to obtain information about a solution's transient evolution by this method. As an alternative, we present here an approximate method for investigating this transient evolution which is based upon the conservation laws associated with the KdV equation. Initial conditions which form one or two solitons are considered, and the resulting approximate evolution is found to be in good agreement with the numerical solution of the KdV equation. Justification for the approximations employed is also given by way of the linearized inverse scattering solution of the KdV equation. In addition, the final soliton state determined from the approximate equations agrees very well with the final state determined from the exact inverse scattering transform solution.

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I. INTRODUCTION

The Korteweg–de Vries equation on the infinite line

$$\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0\ , \quad -\infty<x<\infty\ , \quad t>0 \qquad (1)$$

together with the initial condition

$$u(x,0)=U_0(x), \quad -\infty<x<\infty\ , \qquad (2)$$

where $U_0(x)$ is square integrable, is the simplest of a series of equations which have an exact solution via the inverse scattering transform [1,2]. This method converts the Korteweg–de Vries equation into a linear eigenvalue problem for a set of scattering data and a linear integral equation (the Marchenko equation) for the time evolution. The eigenvalue problem is Schrödinger's equation from quantum mechanics together with the boundary condition that $u\to 0$ as $x\to\pm\infty$. It can be shown from the inverse scattering solution that a general initial condition (2) evolves into a fixed number of solitons plus decaying radiation [2].

A soliton is the particular solution

$$u=a\,\mathrm{sech}^2\left[\frac{a}{2}\right]^{1/2}(x-2at) \qquad (3)$$

of the Korteweg–de Vries equation (1) for which the nonlinearity and dispersion balance one another. The solitons formed from the initial condition (2) are given by the point spectrum of the eigenvalue problem and hence the final state can be easily obtained from the inverse scattering solution since any radiation formed decays as $t\to\infty$. However, the time evolution of the initial condition into the soliton(s) is not easily determined via the inverse scattering method as this time evolution is determined by solving the Marchenko integral equation, which in general is nontrivial. As a consequence of this, the transient evolution of the solution from the initial condition to the final soliton state is difficult to calculate.

It is this time evolution of the soliton(s) from the initial condition which forms the focus of the present work. Rather than using the inverse scattering method, the time evolution will be studied using the conservation laws associated with the Korteweg–de Vries equation (1). The particular initial condition $U_0(x)=A\,\mathrm{sech}^2(x/b)$ will be used as a simple, specific example since the calculations are relatively simple, and, furthermore, for this U_0, the point spectrum of Schrödinger's equation can be found explicitly [1].

The Korteweg–de Vries equation (1) possesses an infinite number of conservation laws and conserved densities [1,2]. Berezin and Karpman [3] used these conservation laws to determine the N solitons formed from the initial condition (2). The long-time solution was assumed to consist of N solitons and the initial and final values of the N lowest-order conserved densities were equated to determine the N unknown amplitudes of the final solitons. The calculated amplitudes were found to be in very good agreement with the inverse scattering solution.

This method of approximate solution suffers from a number of drawbacks, however. The most severe is that the time evolution of the initial condition (2) into the N solitons is not determined, as only the long-time asymptotic state is evaluated. Also, as the initial conserved densities are equated to the final conserved densities for N solitons, the amount of the conserved densities associated with dispersive radiation is not taken into account. It is found from numerical solutions of the Korteweg–de Vries equation with the initial condition (2) that dispersive radiation is generated as well as solitons. The lack of inclusion of radiation leads to discrepancies between the inverse scattering and approximate solutions. Finally, the number N of solitons into which the initial condition evolves is not known *a priori,* but is determined from the inverse scattering solution. In the present work, it is shown how information which describes the time evolution of the solution from its initial condition into soliton(s) can be determined from the conservation laws alone. Furthermore, it is shown how the effect of dispersive radiation can be included via the conservation laws and how the dispersive radiation, if it carries away positive mass from the lead soliton, can cause additional solitons to be formed. It is this determination of how much radiation is shed as the soliton evolves that is the major component of the present work.

Another reason for the approach presented here is that this method for investigating the evolution of an initial condition for the Korteweg–de Vries (KdV) equation can be applied to equations for which there is no inverse scattering solution. For example, in [4] the evolution of an initial condition for the mKdV (modified Korteweg de Vries) equation, in which the nonlinear term uu_x in the Korteweg–de Vries equation (1) is replaced by $u^n u_x$, is considered. The mKdV equation can be solved using the inverse scattering transform only for the cases $n=1$ and $n=2$. The approximate conservation equations are found to predict in a straightforward manner the instability of the solitary wave solution for $n \geq 4$, which was found previously only through very detailed analysis [5]. For $1 \leq n < 4$, the solitary wave solutions are found to be stable, again in agreement with [5]. The further information found by [4] in the case of the mKdV equation is the existence of a critical initial amplitude of the soliton for $n \geq 4$. Below this critical initial amplitude, the soliton decays into dispersive radiation alone; above this critical initial amplitude, the amplitude of the soliton blows up. This blowup has been observed previously in numerical simulations of the mKdV equation [5], but the determination of the critical amplitude is new.

The approximate method outlined in the present work is also being applied to perturbed KdV equations describing, for example, the motion of a soliton on a fluid of varying depth or the effect of kinematic viscosity on the motion of a soliton. The method outlined in the present work is expected to be particularly useful in the case in which the soliton moves on a fluid of decreasing depth. In this case, a second (and possibly more) soliton(s) is (are) generated behind the original soliton, a situation which has been observed in numerical simulations of the perturbed KdV equation [6]. The formation and evolution of the second soliton, which is extremely difficult to determine via inverse scattering techniques, should be describable by the method outlined in Sec. IV.

Applications connected with other evolution equations are also expected to benefit from improved methods for studying the interaction between solitons and dispersive radiation, and provide added motivation for the present work. One application of current interest is the modeling of optical solitons using the nonlinear Schrödinger (NLS) equation, and related problems involving perturbed or coupled NLS equations. The behavior of solutions in such cases has typically been explored in the past via soliton perturbation theory, which yields equations for the evolution of the soliton parameters. A straightforward application of soliton perturbation theory, however, does not include interaction effects between the soliton and the dispersive radiation.

A variational method for partially including such effects in the NLS equation has been demonstrated [7]. This variational method includes a local interaction between the soliton and the dispersive radiation, but makes no provision for the propagation of radiation away from the vicinity of the soliton. This variational method has been used to study a number of applications such as pulse propagation in birefringent optical fibers [8–12], and has been shown to provide a more accurate description of the solution behavior. Recently it has been reported that it also provides an efficient method for modeling soliton dragging logic gates (high-speed optical switches which employ solitons) [13]; in particular, the variational method proved to be the most accurate among the several approximate methods tested.

While inclusion of the local interaction between a soliton and dispersive radiation gives an improvement over standard soliton perturbation theory, there are still differences between these approximate solutions and numerical solutions. These differences are clearly due to the propagation of dispersive radiation, which causes a permanent momentum loss in the soliton. For additional improvements in the approximations, it is necessary to include this effect. The KdV equation is the obvious first choice for study since its solitons have fewer parameters, and the local interaction between the soliton and the dispersive radiation is relatively simple.

II. CONSERVATION LAWS AND THE FORMATION OF ONE SOLITON

Let us consider the classical initial value problem for the Korteweg–de Vries equation on the infinite line $-\infty < x < \infty$, which consists of the Korteweg–de Vries equation (1) together with the initial condition (2). An approximate solution of this problem will now be found by using the conservation laws associated with the Korteweg–de Vries equation. In this section, we shall consider the case when only one soliton is formed from the initial condition (2). The case when two solitons are formed is considered in Sec. IV.

It can easily be found that the Korteweg–de Vries equation (1) has the following two conservation laws: conservation of mass

$$\frac{\partial}{\partial t}(u)+\frac{\partial}{\partial x}(3u^2+u_{xx})=0 \tag{4}$$

and conservation of momentum

$$\frac{\partial}{\partial t}(\tfrac{1}{2}u^2)+\frac{\partial}{\partial x}(2u^3+uu_{xx}-\tfrac{1}{2}u_x^2)=0 . \tag{5}$$

These two conservation laws are the two lowest-order conservation laws of the infinite set of conservation laws for the Korteweg–de Vries equation. To be specific, we consider the particular initial condition

$$U_0(x)=A\,\mathrm{sech}^2(x/b)\ . \tag{6}$$

As was stated in the Introduction, this initial condition is chosen for simplicity and because the Schrödinger equation eigenvalue problem can be solved explicitly in this case, so that the conservation law results can be compared with those from inverse scattering.

It is known from inverse scattering that the initial condition will evolve into a soliton or solitons plus dispersive radiation. We shall therefore seek a solution of the form

$$u=u_0+u_1\ , \tag{7}$$

where

$$u_0=a\,\mathrm{sech}^2(\theta/\beta)\ , \tag{8}$$

$$\theta=x-\xi(t)\ ,\quad \xi'(t)=V(t)\ . \tag{9}$$

Here $\xi(t)$ is the position of the "soliton" maximum, $V(t)$ is the "soliton" speed, and the amplitude a and the width β are functions of time t. From the initial condition (6) and upon taking $u_1(x,0)=0$, we see that

$$a(0)=A \quad\text{and}\quad \beta(0)=b\ . \tag{10}$$

Comparing (8) with the soliton solution (3), we see that the first-order solution u_0 is assumed to be a time varying solitonlike solution. The solution u_0 is only an exact soliton if $a=2\beta^{-2}$, in which case $V=2a=4\beta^{-2}$ and a and β are constants [see (3)]. It is expected, both from the inverse scattering solution [1,2] and from noting that the soliton has positive velocity while small amplitude linear dispersive radiation must have negative group velocity, that as $t\to\infty$ u_0 will approach an exact soliton. The function u_1 is assumed to incorporate the dispersive radiation and to have small amplitude compared with u_0. It is difficult to precisely quantify this criterion, of course, since no explicit solution for the dispersive radiation is available. The validity of this small amplitude assumption can be verified from numerical solutions of the Korteweg–de Vries equation with the initial condition (6); an example solution is shown in Fig. 1. Alternatively, in general terms one can determine that the dispersive radiation is small from the inverse scattering solution, since from the general theory the amplitude of the dispersive radiation goes to 0 as $t\to\infty$ [2]. Note that u_1 also incorporates the small transient changes of shape necessary for the evolution of the initial pulse into a soliton. Thus, u_1 accommodates any transient deviation of the pulse shape away from the hyperbolic secant form assumed in (8).

Substituting the assumed form of solution (7) into the Korteweg–de Vries equation (1), we obtain, on neglecting terms of order u_1^2,

$$u_{1t}+6(u_0u_1)_x+u_{1xxx}=-a'\mathrm{sech}^2(\theta/\beta)+2a\beta^{-3}(4-\beta\theta\beta'-\beta^2V)\mathrm{sech}^2(\theta/\beta)\tanh(\theta/\beta)$$
$$+12a\beta^{-3}(a\beta^2-2)\mathrm{sech}^4(\theta/\beta)\tanh(\theta/\beta)\ . \tag{11}$$

The error terms on the right-hand side of (11) arise since the three-parameter family of solutions (8) is (of course) too simplistic to capture all of the complicated pointwise behavior of the KdV equation. By working with integrals of u_1, however, we expect many of these pointwise deviations to cancel one another. For example, integrating (11) from $-\infty$ to ∞, we obtain

$$\frac{d}{dt}\int_{-\infty}^{\infty}u_1dx=-2(a\beta)'\ . \tag{12}$$

In addition, from (8), we find that

$$\frac{d}{dt}\int_{-\infty}^{\infty}u_0dx=2(a\beta)'\ . \tag{13}$$

Hence, the integrated form of the mass conservation equation (4) shows that the mass lost by the "soliton" u_0 is gained by the dispersive radiation u_1, as expected. The key problem is therefore to determine precisely how much mass goes into the dispersive radiation, or equivalently how much mass is lost by the "soliton" u_0. It will be seen in what follows that including only the integrated effect of the dispersive radiation (i.e., the total amount of mass shed) is sufficient to give a quite accurate representation of the effect of the dispersive radiation upon the evolving pulse. In particular, if one is not interested in the specific details of the radiation but only in the evolving pulse (which is the most likely case), it is not necessary to determine the dispersive radiation in a pointwise sense.

Since dispersive radiation is quickly shed by the soliton, we assume that ahead of and in the vicinity of the soliton u_1 quickly decays to zero. (It is possible that there will be some small error in this approximation just after $t=0$, however.) For this to be true it is necessary that the soliton position be correct. This can most easily

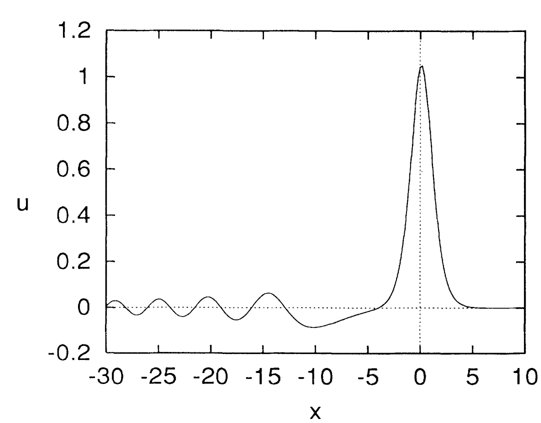

FIG. 1. Numerical solution of the KdV equation using the initial condition $U_0(x)=A\,\mathrm{sech}^2(x/b)$ with $A=1.25$ and $b=1$, showing the smallness of the dispersive radiation and how it is quickly left behind by the soliton. The solution is plotted at $t=3$; in this and in subsequent figures all plotted quantities are dimensionless.

be seen from Eq. (11), since a homogeneous solution of (11) is easily found to be

$$u_1 = u_{0x} = -4\kappa^3 \operatorname{sech}^2(\kappa x - 4\kappa^2 t)\tanh(\kappa x - 4\kappa^2 t) , \quad (14)$$

when $a = 2\beta^{-2} = 2\kappa^2$ is the soliton amplitude. Adding a small amount of this solution to u_0 can be thought of merely as representing a small shift in the soliton position. If the soliton position has been accurately determined, however, then u_1 quickly decays to zero ahead of and in the vicinity of the soliton. Further justification for this is given in Sec. III.

Assuming this to be the case, we integrate (11) from ξ to ∞ [or equivalently integrate the mass conservation law (4) directly over this range, assuming that only u_0 contributes] and obtain

$$(a\beta)' = 3a^2 - Va - 2a\beta^{-2} . \quad (15)$$

As before, we work with an integral of the equation since this allows the pointwise errors in the approximation (8) to cancel one another. In addition, if we denote the mass present in the linear dispersive radiation by

$$M = \int_{-\infty}^{\infty} u_1 dx , \quad (16)$$

then from (12) we see that mass conservation gives

$$\frac{d}{dt}(2a\beta) + \frac{dM}{dt} = 0 . \quad (17)$$

We have thus completed the consideration of mass conservation and have two equations for the unknowns a, β, M, and V. We next consider momentum conservation.

From the momentum conservation equation (5) and the expansion for u (7), we see that the momentum density is

$$\tfrac{1}{2}u^2 = \tfrac{1}{2}u_0^2 + u_0 u_1 + \tfrac{1}{2}u_1^2 . \quad (18)$$

Note, however, that since u_1 is assumed to be zero in the vicinity of the soliton, no contribution comes from the cross term $u_0 u_1$. The term $u_1^2/2$ is second order in u_1, but can produce a significant contribution to the momentum; since the contribution from this term is not restricted to points near the soliton, it can become large if the region over which u_1 is nonzero becomes large. The amount of momentum in the radiation contributed from this term will be approximated below. If this term is neglected, however, integrating the momentum conservation equation (5) from $-\infty$ to ∞ gives to first order

$$\frac{d}{dt}(a^2\beta) = 0 , \quad (19)$$

upon using (8).

To complete the ordinary differential equation (ODE) system describing the approximate solution of the KdV equation (1), we need an expression for the velocity V. Unfortunately, none of the standard conservation laws for the KdV equation involves the velocity. It is well known that each conservation law is associated with an invariance of the KdV equation, and that the lowest conservation laws are associated with physical invariances [1]. For example, conservation of momentum is associated with the invariance of the equation (or the Lagrangian) with respect to translations in x, and conservation of energy is associated with the invariance with respect to translations in t. In addition to these invariances, however, there are two additional invariances which give rise to equations involving moments of two conserved densities. The first additional invariance is Galilean invariance, which gives rise to the equation

$$\frac{d}{dt}\int_{-\infty}^{\infty} xu\, dx = \int_{-\infty}^{\infty} 3u^2 dx . \quad (20)$$

This equation is difficult to use to determine an expression for the velocity because it is clear that the mass of the soliton, $\int u\, dx$, is not constant.

The second additional invariance of the KdV equation is scale invariance, and this gives rise to the equation

$$\frac{d}{dt}\int_{-\infty}^{\infty} xu^2 dx = \int_{-\infty}^{\infty} [4u^3 - 3(u_x)^2] dx . \quad (21)$$

This equation can be used to determine an expression for V, since the largest contribution comes from the vicinity of the soliton; as explained above, any contributions to this from u_1 are of second order. In addition, momentum is very nearly constant [see (19)]. If we neglect these second-order contributions, and substitute the leading-order soliton (8) into (21), we obtain an expression for the velocity

$$\frac{d\xi}{dt} \equiv V = \frac{16}{5}a - \frac{12}{5\beta^2} . \quad (22)$$

Note that for large times $2\beta^{-2} \to a$, so that $V \to 2a$, the velocity of a soliton [see (3)].

The system describing the evolution of the solution of the Korteweg–de Vries equation (1) with initial condition (6) is then (15), (17), (19), and (22). This system gives a solution for the pulse amplitude $a(t)$ which has final amplitudes which are in good agreement with the final amplitudes obtained from the exact solution of the Korteweg–de Vries equation, even after the initial amplitude A or width b are increased past the point where two solitons form out of the initial profile. (In this case, the above solution describes the larger soliton.) Here the final steady state is independent of the expression used for the velocity V, and is given by

$$a = \left[\frac{A^2 b}{\sqrt{2}}\right]^{2/3} , \quad (23)$$

while the exact solution given by inverse scattering is

$$a = \frac{1}{2b^2}[(1+4Ab^2)^{1/2} - (2n-1)]^2 \quad (24)$$

(see [1]). Here n is the number of the soliton formed, $1 \le n \le N$, where N is the total number of solitons generated. For given A and b, N is determined by

$$N = \text{(largest integer)} \le \tfrac{1}{2}[(1+4Ab^2)^{1/2} + 1] . \quad (25)$$

A comparison of the approximate result (23), the exact inverse scattering result (24), and the result obtained by using the N lowest conservation laws [3] for the amplitude of the largest (first) soliton is shown in Fig. 2. Note

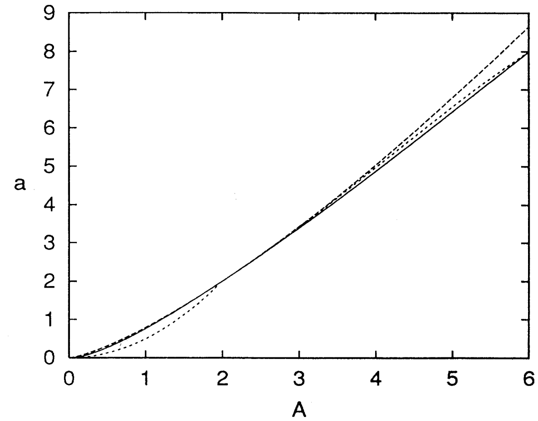

FIG. 2. Comparison between the final steady soliton amplitudes as given by the solution of the approximate conservation equations: ---, result from [3]; — — —, result from present work (assuming constant momentum); ——, inverse scattering solution (24) with $n=1$.

that for $0 \leq A \leq 2$ the result obtained from the lowest conservation law (i.e., mass conservation) underestimates the final amplitude, because the pulse sheds a considerable amount of dispersive radiation which carries away negative (relative) mass. In contrast to previous work, however, the present method also gives an approximation for the time evolution for the pulse. A comparison between this approximate solution and the numerical solution is given in Fig. 3, and shows reasonable agreement.

The accuracy of the approximate solution can be improved somewhat, however, by including the momentum loss from the soliton. While this improvement is not essential here, it will be shown to be so in Sec. IV when the formation of a second soliton from the mass lost by the first is considered. In many perturbed KdV problems, it has been shown that mass and momentum leak away from the soliton through the formation of a nonzero shelf behind it (i.e., a region where the solution is relatively flat; see [14]). This region can be seen forming near $x=-10$ in Fig. 1. This effect can also be seen in the solution of Eq. (11). Taking $u_1=u_1(\theta)$ and assuming that a and β are slowly varying, Eq. (11) for u_1 can be integrated once. Upon choosing the constant of integration so that $u_1 \to 0$ as $\theta \to \infty$, we find that u_1 approaches a nonzero value as $\theta \to -\infty$. Thus, if we seek a solution for u as a perturbed soliton, we find that there is mass and momentum loss far behind the soliton. (Note that the loss in mass is first order, while the loss in momentum is second order, however.) It is this that gives rise to the dispersive radiation.

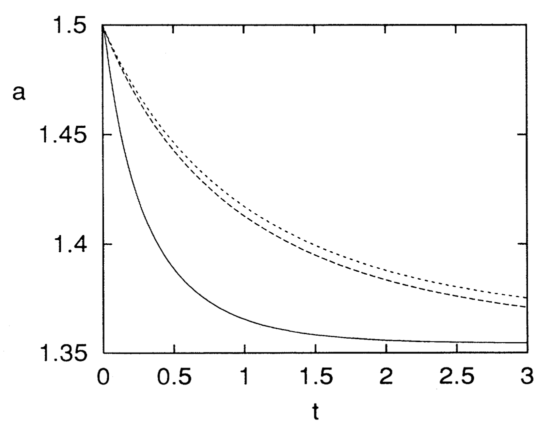

FIG. 3. Comparison between the time evolution of the soliton amplitude a as given by the solution of the approximate conservation equations and that obtained from the numerical solution of the KdV equation: ——, numerical solution of the KdV equation; — — —, using velocity (22) with momentum equation (30); ---, using velocity (22) with constant momentum (19). Here $A=1.5$ and $b=1$.

The momentum associated with this leaking radiation can be estimated as follows. We suppose that the mass loss from the "soliton" is due to the solution approaching a small nonzero value (which we denote by u_∞) in the shelf, which we assume occurs near $x=-L$. The shelf position is also assumed to follow at a fixed distance behind the soliton, so that it moves with the soliton velocity. Integrating the mass conservation equation (4) from $-L$ to ∞, we obtain

$$\frac{d}{dt}\int_{-L}^{\infty} u\,dx = -Vu_\infty , \tag{26}$$

after neglecting terms quadratic and higher in u_∞ and assuming that spatial derivatives in the shelf region are small. Comparing (26) with (17), we see that we can set

$$\frac{dM}{dt} = Vu_\infty . \tag{27}$$

Similarly, integrating the momentum conservation equation (5), we obtain

$$\frac{d}{dt}\int_{-L}^{\infty} \tfrac{1}{2}u^2 dx = -\tfrac{1}{2}Vu_\infty^2 , \tag{28}$$

which becomes, upon using (27),

$$\frac{d}{dt}\int_{-L}^{\infty} \tfrac{1}{2}u^2 dx = -\frac{M'^2}{2V} . \tag{29}$$

This gives an approximate expression for the rate of momentum loss by the soliton.

From the approximate expression for the velocity (22), we see that it is possible for V to be zero. This will introduce an artificial singularity into the above expression (29) for the momentum loss. It is reasonable to use another expression for the velocity V which does not have a singularity, however, as long as it approaches the correct value when the solution is close to a soliton. First of all, the rate of momentum loss (29) is only intended to be valid when the solution is close to a soliton, i.e., when $a=2\beta^{-2}$ and $V=2a$. In addition, in the worst case a different expression for V will produce errors only for a finite time; since the rate of momentum loss is small, the total integrated error in the momentum will also be small. One obvious replacement for the velocity in the rate of momentum loss (29) is $V=2a$. Alternatively, one can use $V=4(a^2\beta/4)^{2/3}$. This latter expression has less variation since $a^2\beta$ is approximately constant. No significant quantitative difference between results obtained with these two expressions [and with (22) when it is nonsingular] was observed, however, and so the simpler of the two, $V=2a$, will be used in (29). Finally, then, this gives the required expression for the momentum being shed from the "soliton." Including this momentum loss term in the momentum equation (19), we therefore have

$$\frac{d}{dt}(a^2\beta) = -\frac{3M'^2}{8a} . \tag{30}$$

Technically speaking, when the momentum varies the expression for the velocity V given by (22) is not correct. If the effect of variable momentum is included, an additional term appears which is proportional to ξ, the position of the soliton. Since the KdV equation is translation invariant, this is not correct. The momentum loss is therefore neglected in determining the velocity. In addition, since the momentum loss is small this is, in any event, a small effect.

The system governing the evolution of the solution of the Korteweg–de Vries equation including momentum loss with initial condition (6) is then (15), (17), (22), and (30). This system is solved numerically using a fourth-order Runge-Kutta scheme and the results are compared with those obtained from a numerical solution of the Korteweg–de Vries equation using the pseudospectral method of [15]. A comparison between the numerical solution of the KdV equation and the above system both with and without momentum loss is shown in Fig. 3. From the figure it is seen that including momentum loss gives a slight improvement in the accuracy of the approximation.

As mentioned previously, the accuracy of the transient behavior of the ODE system depends most sensitively upon the expression used for the velocity V in the equation for the mass loss (15). The calculated expression (22) gives reasonably good results, but by changing the constants slightly they can be substantially improved.

The general form of the right-hand side of (22), as well as other approximations for V (such as that obtained from the moment of mass conservation equation), suggests a velocity expression of the form

$$\frac{d\xi}{dt} \equiv V = ca + \frac{4-2c}{\beta^2} . \tag{31}$$

The constants in this expression are chosen so that $V \to 2a$ as $t \to \infty$ (c is still arbitrary even with this constraint). Note, however, that the expression calculated using the moment of the momentum, (22), gives $c=\frac{16}{5}$. If we assume that the momentum is constant, i.e., (19) is satisfied, then the steady-state solution of the ODE system occurs when $a=2\kappa^2$ and $\beta=1/\kappa$, where $a^2\beta=4\kappa^3=A^2b$. The constant momentum expression allows us to eliminate β from (15), i.e., $\beta=4\kappa^3/a^2$. Then using the above expression for the velocity (31), (15) becomes

$$\frac{da}{dt} = \frac{c-3}{4\kappa^3}\left[a^4 - \frac{a^7}{8\kappa^6}\right] . \tag{32}$$

The steady state of this equation is clearly $a=2\kappa^2$, and linearizing about this solution with $a=2\kappa^2+\hat{a}$, we obtain

$$\frac{d\hat{a}}{dt} = -6\kappa^3(c-3)\hat{a} , \tag{33}$$

so that the steady state is stable only if $c>3$. Also, the rate of approach to the steady state strongly depends upon the value of c appearing in the modified velocity expression (31). Evidently, the velocity depends somewhat sensitively upon deviations from the exact soliton shape.

To determine a better value of c, the KdV equation and the ODE system were both solved using $A=1.25$ and $b=1$ and c was adjusted until the transient behavior of the two solutions was close. This occurred for a value of c of approximately $\frac{7}{2}$, and this was used as an alternate value in the numerical simulations. As described below, this value gives much better results for other initial conditions than the value $c=\frac{16}{5}$.

Figure 4 shows a comparison between the final steady soliton amplitude a obtained from the conservation equations (15), (17), (31) [with both $c=\frac{16}{5}$, or equivalently (22),

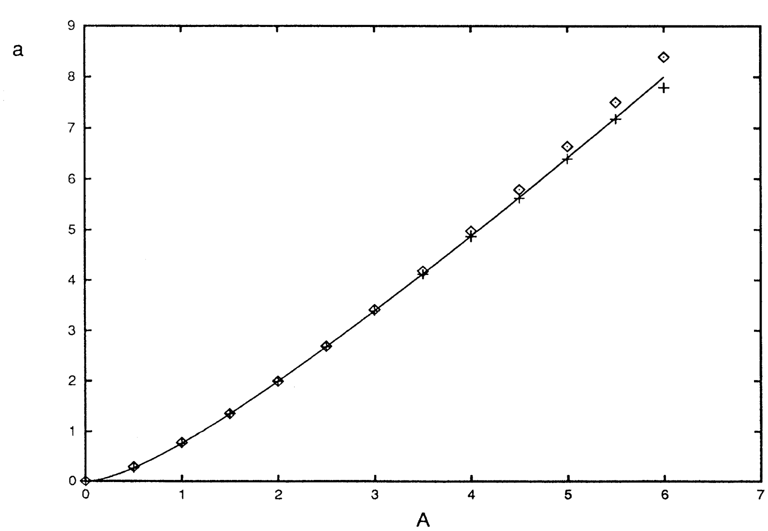

Fig. 4. Comparison between the final steady soliton amplitudes as given by the solution of the approximate conservation equations: ◇, using velocity (22); +, using velocity (31) with $c=\frac{7}{2}$; —— from the inverse scattering solution (24) with $n=1$.

and $c=\frac{7}{2}$], and (30), and the largest soliton amplitude obtained from inverse scattering theory [(24) with $n=1$] for A in the range 0 to 6 and $b=1$. For $0<A\leq 2$, one soliton forms, while for $2<A\leq 6$, two solitons form. It can be seen that the agreement between the approximate equations and inverse scattering theory is good up to $A=6$. Around $A=6$, the two solutions start to diverge, which is not unexpected since to obtain the approximate equations the assumption was made that the mass lost by the soliton is small. As three solitons start to form after $A=6$, this assumption clearly starts to break down. The interesting point about Fig. 4 is that the amplitude of the largest soliton is well predicted even when two solitons are formed. The prediction of the evolution of the second soliton is dealt with in Sec. IV.

Figure 5 shows a comparison between the time evolution of the amplitude a of the soliton as predicted by the conservation equations (15), (17), and either (30) or (19), and the results obtained from a numerical solution (pseudospectral) of the Korteweg–de Vries equation for $A=1.5$ and $b=1$. In both approximate solutions the velocity (31) was used with $c=\frac{7}{2}$. It can be seen that the two solutions are in quite good agreement, and that the results obtained by including momentum loss are slightly better than those with constant momentum. The momentum loss becomes important when a second soliton is produced as the inclusion of this term allows momentum to flow into the second soliton. Figures 6 and 7 show similar plots when the initial conditions are $A=1.0$ and 2.5, both for $b=1$. In these figures only approximate solutions with momentum loss are shown. Again, using $c=\frac{7}{2}$ in (31) is seen to give much better results than using (22).

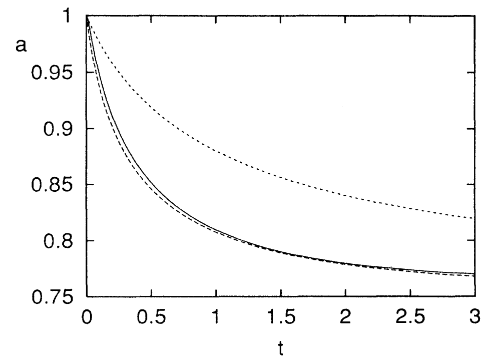

FIG. 6. Comparison between the time evolution of the soliton amplitude a as given by the solution of the approximate conservation equations and that obtained from the numerical solution of the KdV equation: — — —, using velocity (31); ---, using velocity (22) with $c=\frac{7}{2}$; ——, numerical solution of the KdV equation. Here $A=1.0$ and $b=1$.

III. LINEARIZED INVERSE SCATTERING SOLUTION FOR ONE SOLITON

While it is not possible to obtain a full transient solution using the inverse scattering transform, it is possible to obtain some information from the linearized version of this solution when the amount of dispersive radiation is small. In particular, this linearized solution provides additional justification for the approximations used in the preceding section. The idea is to linearize the KdV equation (1) about a single soliton by substituting

$$u=u_s+\eta \tag{34}$$

into the KdV equation, where

$$u_s=2\kappa^2\text{sech}^2(\kappa x-4\kappa^3 t-\kappa x_0)\equiv 2\kappa^2\text{sech}^2\xi \tag{35}$$

is an exact soliton solution and $|\eta|\ll 1$. This gives the linearized equation

$$\frac{\partial\eta}{\partial t}+6\frac{\partial(u_s\eta)}{\partial x}+\frac{\partial^3\eta}{\partial x^3}=0\ . \tag{36}$$

Equation (36) is still relatively complicated, but its ex-

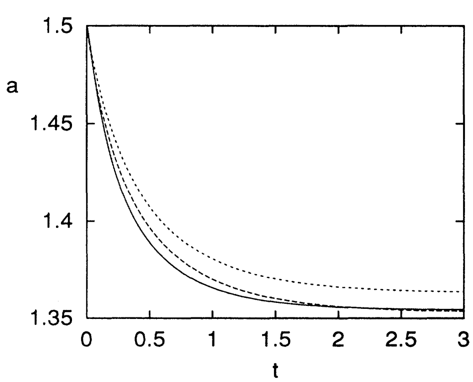

FIG. 5. Comparison between the time evolution of the soliton amplitude a as given by the solution of the approximate conservation equations and that obtained from the numerical solution of the KdV equation: ——, numerical solution of the KdV equation; — — — using velocity (31) with $c=\frac{7}{2}$ and with momentum equation (30); ---, using velocity (31) with $c=\frac{7}{2}$ and with constant momentum (19). Here $A=1.5$ and $b=1$.

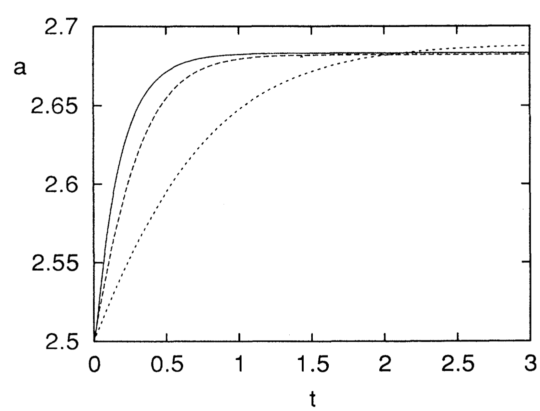

FIG. 7. Comparison between the time evolution of the soliton amplitude a as given by the solution of the approximate conservation equations and that obtained from the numerical solution of the KdV equation: — — —, using velocity (31); ---, using velocity (22) with $c=\frac{7}{2}$; ——, numerical solution of the KdV equation. Here $A=2.5$ and $b=1$.

act solution can be written explicitly [16] as

$$\eta=f_{xxx}+4f_x\kappa^2+8\kappa^3 f\,\mathrm{sech}^2\xi\tanh\xi-8\kappa^2 f_x\mathrm{sech}^2\xi-4\kappa f_{xx}\tanh\xi+C\,\mathrm{sech}^2\xi\tanh\xi\ , \tag{37}$$

where f is any solution of the *linear* KdV equation

$$\frac{\partial f}{\partial t}+\frac{\partial^3 f}{\partial x^3}=0 \tag{38}$$

and C is an arbitrary constant. This constant C can be interpreted merely as a small change in the position x_0 in the single soliton solution (35); since the initial condition and the soliton are both even about their center positions, however, we may take both $x_0=0$ and $C=0$.

Alternatively, the linearized solution (37) can be written as

$$\eta=\frac{\partial}{\partial x}[f_{xx}+4\kappa^2 f-4\kappa^2 f\,\mathrm{sech}^2\xi-4\kappa f_x\tanh\xi]\ , \tag{39}$$

or as

$$u_s\eta=\frac{\partial^2}{\partial x^2}(u_s f_x)-\frac{\partial}{\partial x}(u_s^2 f)\ . \tag{40}$$

These forms of the solution show that

$$\int_{-\infty}^{\infty}\eta\,dx=4\kappa^2 f|_{-\infty}^{+\infty}\quad\text{and}\quad\int_{-\infty}^{\infty}u_s\eta\,dx=0 \tag{41}$$

if $f_x\to 0$ as $x\to\pm\infty$. Thus, for bounded f's, this linearized solution can only be used to consider soliton perturbations which have no net change in momentum. This is equivalent to saying that the linearized solution η cannot be used to alter the amplitude of the soliton u_s.

This allows us to use the linearized solution to determine the approximate final soliton state from the initial condition. The initial condition $A\,\mathrm{sech}^2(x/b)$ is split into two parts, a soliton part and a part which is the linearized perturbation,

$$A\,\mathrm{sech}^2(x/b)=2\kappa^2\mathrm{sech}^2\kappa x+\eta\ . \tag{42}$$

The linearized perturbation η eventually disperses, leaving only the soliton. Since $\int u_s\eta\,dx=0$, however, if we multiply Eq. (42) by u_s and integrate, we obtain (letting $x=b\zeta$ in the integral)

$$Ab^2\int_{-\infty}^{\infty}\mathrm{sech}^2(\kappa b\zeta)\mathrm{sech}^2\zeta\,d\zeta=\tfrac{8}{3}\kappa b\ , \tag{43}$$

which can be regarded as an equation for κ given A and b, which determines the final soliton state from a given initial condition.

While (37) or (39) cannot be solved explicitly for $f(x,0)$, it can be seen that $f(x,0)$ is an odd function in x, and that

$$4\kappa^2 f|_{-\infty}^{+\infty}=2Ab-4\kappa\equiv\Delta m\ . \tag{44}$$

Since f is odd, this means that

$$4\kappa^2 f(+\infty,t)=-4\kappa^2 f(-\infty,t)=Ab-2\kappa\equiv\tfrac{1}{2}\Delta m\ . \tag{45}$$

This linearized solution is now seen to separate the perturbation into two parts, one of which is essentially a shift of the soliton position and another which is dispersive radiation that does not stay localized near the soliton. This justifies the division which was used in Sec. II. The result follows from integrating (37) to obtain

$$\int_{x_s}^{\infty}\eta\,dx=\tfrac{1}{2}\Delta m-f_{xx}|_{x_s}$$

and (46)

$$\int_{-\infty}^{x_s}\eta\,dx=\tfrac{1}{2}\Delta m+f_{xx}|_{x_s}\ .$$

where x_s is the position of the soliton maximum, $\xi=0$. Next, all that is necessary is to determine f_{xx} at the unperturbed soliton position x_s, which means solving the time-dependent problem for f, Eq. (38). As mentioned above, however, it is not possible to obtain a simple solution for $f(x,0)$. One can approximate this solution, however, by taking

$$f(x,0)=a\tanh\kappa x \tag{47}$$

and choosing the constant a so that the amount of mass represented by this $f(x,0)$ is correct, as given by Eq. (44). This condition gives $a=\Delta m/8\kappa^2$, since $a\equiv f(+\infty,0)$.

Whichever initial condition is used, the solution of Eq. (38) can be written as

$$f(x,t)=a+\int_{-\infty}^{\infty}\mathrm{Ai}(\zeta)[f((3t)^{1/3}(\eta-\zeta),0)-a]d\zeta\ , \tag{48}$$

where $\mathrm{Ai}(z)$ is the Airy function and $x=(3t)^{1/3}\eta$. For large t the term inside the square brackets asymptotes to 0 for $\eta>\xi$ and $-2a$ for $\eta<\xi$, since $f(x,t)\to\pm a$ as $t\to\pm\infty$. Thus, for large t,

$$f(x,t)\sim a-2a\int_{\eta}^{\infty}\mathrm{Ai}(\zeta)d\zeta=a-2a\int_{x/(3t)^{1/3}}^{\infty}\mathrm{Ai}(\zeta)d\zeta\ . \tag{49}$$

As expected, the linearized solution quickly loses almost all information from the initial condition as it disperses, becoming a similarity solution. The only information retained is the total amount of mass in the initial condition.

We need the behavior of f and f_{xx} in the vicinity of the soliton position x_s, where $\dot{x}_s=4\kappa^2$. Since the soliton moves much faster than the dispersive radiation spreads (t versus $t^{1/3}$), very quickly we have $f\sim a$ in the vicinity of the soliton, and f_x and $f_{xx}\to 0$. We therefore see from Eq. (46) that, apart from a short initial transient, the mass in the linearized solution η divides itself up into two equal halves: one half in the part of the solution from $-\infty$ to the unperturbed soliton position x_s and the other half in the part of the solution from the soliton position x_s to $+\infty$.

Still more can be said, however, if we use $f\to a$ in the vicinity of the soliton. This immediately gives

$$\eta\sim 8a\kappa^3\mathrm{sech}^2\xi\tanh\xi=\kappa\Delta m\,\mathrm{sech}^2\xi\tanh\xi\ , \tag{50}$$

so that

$$u \sim 2\kappa^2 \mathrm{sech}^2\xi + \kappa\Delta m \, \mathrm{sech}^2\xi \tanh\xi$$
$$\approx 2\kappa^2 \mathrm{sech}^2 \left[\xi - \frac{\Delta m}{4\kappa^2} \right] + \cdots \, . \quad (51)$$

From this we see that the mass change from the unperturbed soliton position x_s to $+\infty$ is really the result of a small shift in the position of the soliton. This means, of course, that there is an equal and opposite mass change just to the left of x_s, since the total change of mass in the vicinity of a soliton associated with a shift in position is zero. Thus, the total amount of mass which goes into the dispersive radiation is really equal to the total amount of mass in the linearized solution, Δm, as expected, and not one-half of this value.

In addition, if the shift in the soliton position is taken into account, then the linearized perturbation in the vicinity of the soliton will vanish completely for sufficiently large time. This is precisely the assumption which was made in Sec. II.

IV. TWO SOLITONS

In Sec. II, the evolution of the largest soliton created from the initial condition (6) was found from approximate equations derived from the mass and momentum conservation equations for the Korteweg–de Vries equation. It was found that Eqs. (15), (17), (31) with $c=\frac{7}{2}$, and (30) described the soliton of largest amplitude quite well, even when two solitons formed from the initial condition. Equations describing the evolution of the second soliton formed from the initial condition (6) when $2 < Ab^2 \le 6$ will now be found. For $Ab^2 > 6$, three or more solitons are formed. However, the results derived from the conservation equations lose accuracy as Ab^2 increases above 6, and so the formation of three or more solitons will not be dealt with in the present work.

As found in Sec. II, Eqs. (15), (17), (31), and (30) describe the evolution of the largest soliton quite well. Hence, the amplitude a_1 and width β_1 of this "soliton" are given by

$$\frac{d}{dt}(2a_1\beta_1) + \frac{dM}{dt} = 0 \, , \quad (52)$$

$$\frac{d}{dt}(a_1^2\beta_1) = -\frac{3M'^2}{8a_1} \, , \quad (53)$$

$$\frac{d}{dt}(a_1\beta_1) = 3a_1^2 - V_1 a_1 - 2a_1\beta_1^{-2} \, , \quad (54)$$

$$V_1 = \frac{7}{2}a_1 - \frac{3}{\beta_1^2} \, . \quad (55)$$

Next, equations for the evolution of the second, smaller "soliton" need to be found.

The second "soliton" could be assumed to have the profile (8), as was done for the first "soliton." This was not found to give physically valid solutions, however, as either a_2 or β_2 were found to go negative. This is because the second "soliton" is being built up from zero amplitude at $t=0$. The profile (8) then gives too much freedom and either the amplitude or width can become nonphysical. This is related to the question of what value to give β_2 at $t=0$. When a_2 is zero, β_2 can have any value. If the initial amplitude was nonzero, the profile (8) could be assumed. To overcome these problems, the second "soliton" will be assumed to have the soliton profile

$$a_2 \mathrm{sech}^2 \left[\frac{a_2}{2} \right]^{1/2} [x - \xi_2(t)] \, , \quad (56)$$

where

$$\xi_2'(t) = 2a_2 \quad (57)$$

for all time. The second "soliton" is different from the first "soliton" in that it is formed from a zero initial condition, so that $a_2=0$ at $t=0$.

The point at which two solitons form presents itself in the solution of Eqs. (52) to (55) for the larger "soliton" by the change of sign of M. As the point $Ab^2=2$ is crossed, M changes from a negative to a positive quantity. Hence for $Ab^2 > 2$, the lead "soliton" leaves behind a positive amount of mass, which can form a soliton, as it is known from inverse scattering theory that any positive mass will result in the formation of a soliton. As a first approximation, it will be assumed that all of this positive mass goes into forming the second soliton. Hence from (4) and (56), mass conservation for the second "soliton" gives

$$2\frac{d}{dt}(2a_2)^{1/2} = \frac{dM}{dt} \, , \quad (58)$$

upon integrating the mass density u over the second "soliton" (which can be taken to be from $-\infty$ to ∞). This equation is then the equation governing the evolution of the second soliton. As there is only one free parameter in (56), only one equation is needed for the second "soliton" and momentum conservation does not need to be considered.

Figure 8 shows a comparison between the final steady amplitude of the second soliton as found from a numerical solution of the approximate equations (52) to (55) and (58), and the amplitude of the second soliton as predicted by inverse scattering theory [(24) with $n=2$] for $2 < A \le 6$ (the range for which two solitons are formed) and $b=1$. For comparison, the result obtained using the velocity expression (22) for the first soliton is also included. It can be seen that the comparison is very good considering the approximations which have been made. Similar to the result for the lead soliton, the agreement gets worse as $A=6$ is approached, as then the mass being left behind by the lead soliton is not small, and the approximations of the present work start to break down. For $A > 6$, three solitons are formed.

It was assumed above that all of the mass being left behind by the lead "soliton" was used to generate the second soliton. Also, in deriving (58), it was assumed that the second soliton does not produce dispersive radiation, where in fact it will. To account for this radiation, however, a second parameter is needed in the assumed profile for the second "soliton" and then momentum conservation must be used to give a second equation for this parameter, as was done for the lead "soliton." The results shown in Fig. 8 indicate that this radiation is not

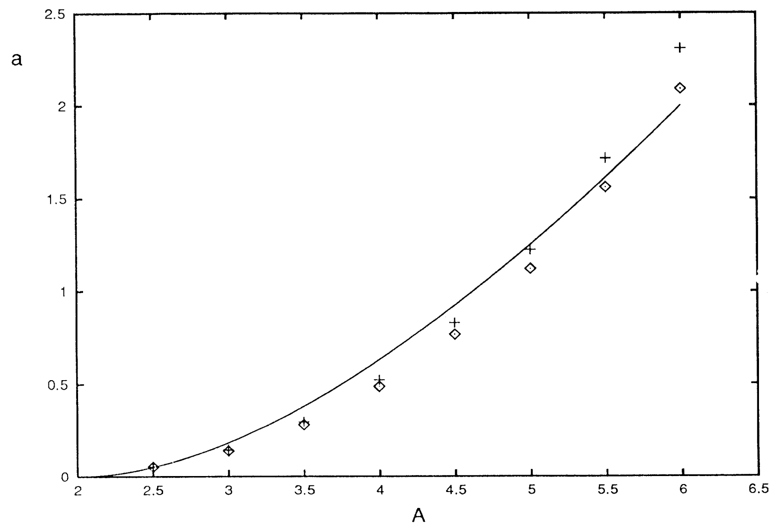

FIG. 8. Comparison between the final steady amplitude of the second soliton as given by the solution of the approximate conservation equations: ◇, using velocity (22); +, using velocity (55); ——, from the inverse scattering solution (24) with $n=2$.

significant until near the point at which a third soliton is formed (for $A>6$). In analogy with what happens at $A=2$ for the lead "soliton," it is expected that the mass left behind the second "soliton" will become positive at $A=6$, so that the third soliton can form.

V. SUMMARY AND CONCLUSIONS

We have examined the time-dependent behavior of solutions of the Korteweg–de Vries equation. While in principle the exact inverse scattering solution of the KdV equation can be used to determine such transient behavior, in practice it is difficult to do so. As an alternative, here we have used the conservation laws associated with the KdV equation to determine a system of ordinary differential equations which approximately describe the time evolution of solutions. We have considered initial conditions which form one or two solitons, and have obtained good agreement between the solutions of the approximate equations and the full numerical solution of the KdV equation.

Key to the analysis was the use of the conservation laws to determine how the mass and momentum from the initial condition split into two parts, one part associated with the evolving pulse which eventually forms the soliton and the other associated with the dispersive radiation. In addition, an approximate expression for the velocity of the pulse is needed, and the transient behavior is found to depend somewhat sensitively on the expression used for this velocity.

In addition, we have presented a linearized inverse scattering analysis of the transient evolution. This linearized analysis is consistent with, and provides additional justification for, the analysis done using the conservation laws.

ACKNOWLEDGMENTS

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