Asymptotic dissipation rate in turbulence

Siegfried Grossmann
Fachbereich Physik, Philipps-Universität, Renthof 6, D-35032 Marburg, Germany
(Received 8 August 1994)

Intermittency corrections $\delta\zeta$ to the scaling behavior of the structure function in the inertial range imply an asymptotic decrease ($\propto \mathrm{Re}^{-\kappa}$) of the dissipation rate divided by the energy input rate as a function of the Reynolds number with $\kappa \propto \delta\zeta$. Data analysis favors $\kappa=0$. It is the classical exponent $\zeta=\frac{2}{3}$ that guarantees an asymptotically Re-independent dissipation-input ratio. Alternatively, intermittency may imply a nonuniversal viscous-inertial crossover in the structure function together with a Re-dependent amplitude $b(\mathrm{Re})$, scaling with a small, negative exponent, as measured recently.

PACS number(s): 47.27.$-$i, 05.45.$+$b

There has been an ongoing discussion on intermittency in the inertial range of turbulent fluid flow. After a longer period of describing it with models [1–4], supported by various measurements [5] and numerical simulations [6], recent results have shed new light on the mechanism and the physical implications of the spatiotemporal fluctuations of the turbulent activity, i.e., intermittency, from the point of view of the Navier-Stokes equations [7–13].

While there is no doubt that there is much intermittent fluctuation in the viscous subrange (VSR) (for experimental support, cf. [14]), leading to large and measurable deviations of the probability density function for the velocity fluctuations from the Gaussian form, there is an increasing indication that there might be no measurable effect of these fluctuations on the scaling behavior in the inertial subrange (ISR), provided that this latter is well enough developed, i.e., for large Reynolds numbers Re [10,11]. This seems to be supported by large wind-tunnel and atmospheric measurements [15–17].

The purpose of this Brief Report is to indicate another possibility to experimentally clarify the presence or absence of intermittency corrections in the scaling behavior of the ISR structure functions or spectra. It is based on the interesting observation [18] that the mean dissipation rate ε can be related to the energy input rate U^3L^{-1} by means of the often studied second order structure function $D(r)=\langle[\mathbf{u}(\mathbf{x}+\mathbf{r})-\mathbf{u}(\mathbf{x})]^2\rangle$. The dissipation rate is sensitive to the small scale properties of the flow field, whereas the energy input is sensitive to the large scale properties. Connecting both by the structure function should be a good method to identify scaling corrections in $D(r)$. Moreover, there is a careful analysis of experimental (grid turbulence) data [19], which can be used to compare with the theory.

Lohse [18] studies the dissipation rate in terms of the energy input, $c_\varepsilon(\mathrm{Re})=\varepsilon/U^3L^{-1}$. Here U is defined as $u_{1,\mathrm{rms}}$, the rms of one component of the velocity field fluctuations, and L is the outer scale. More specifically, L shall be that scale above which the spatial correlation of the velocity field is lost. $\langle\mathbf{u}(r)\cdot\mathbf{u}(0)\rangle\approx 0$, $r\gtrsim L$. Thus $D(L)$ reaches its plateau value $2\langle\mathbf{u}^2\rangle$ or $6\,u_{1,\mathrm{rms}}^2$. Calculating $D(L)$ in its scaling form, which depends on ε, then leads to the desired relation for $c_\varepsilon(\mathrm{Re})$. Since the main contribution is expected to come from the classical Kolmogorov behavior [20], $D(L)$ is calculated in [18] on the basis of a mean field closure of the Navier-Stokes dynamics, as derived in [21]. This implies that fluctuation effects are neglected in $D(r)$. Since the results find support from the data [19], this already indicates that intermittency corrections in $D(r)$ might be small or even missing for large Re.

In this Brief Report I use another parametrization for the second order structure function, which allows one to include scaling corrections if they are present. This generalizes Ref. [18]. Take the Batchelor parametrization [22] and generalize it to include intermittency

$$D(r)=\frac{\varepsilon}{3\nu}\frac{r^2}{[1+(r/a\eta)^2]^{(2-\zeta)/2}} . \tag{1}$$

Here η is the Kolmogorov length $(\nu^3/\varepsilon)^{1/4}$ and a, measuring the crossover scale from the VSR to the ISR in terms of η, $r_{\mathrm{crossover}}=a\eta$, is related to the structure function amplitude in the ISR, $b=a^{4/3}/3$. The exponent ζ denotes the ISR scaling exponent and may deviate from $\frac{2}{3}$, $\zeta=\frac{2}{3}+\delta\zeta$. The scaling of $D(r)$ in the VSR and ISR are limiting cases of (1),

$$D(r)=\begin{cases}(\varepsilon/3\nu)r^2, & r\ll a\eta & \text{(2a)}\\ b\varepsilon^{2/3}r^{2/3}(r/a\eta)^{\delta\zeta}, & r\gg a\eta . & \text{(2b)}\end{cases}$$

Use now $D(L)=D_\infty u_{1,\mathrm{rms}}^2$. The value of D_∞ should be near $D_\infty=6$. It may differ from this, since the plateau value of $D(r)$, $r\to\infty$, for which $\mathbf{u}(r)$ is decorrelated from $\mathbf{u}(0)$, may differ from $D(r)$ at $r=L$, depending on the precise choice of L. Also geometry effects may influence the value of D_∞; e.g., the aspect ratio, or anisotropies. Now put $r=L$ in (1), insert $\varepsilon=c_\varepsilon u_{1,\mathrm{rms}}^3L^{-1}$, and introduce the Reynolds number based on the relevant scales, $\mathrm{Re}=u_{1,\mathrm{rms}}L/\nu$. Then

$$c_\varepsilon=(3D_\infty/\mathrm{Re})[1+(L/a\eta)^2]^{(2-\zeta)/2} . \tag{3}$$

This immediately reproduces the small Re behavior [18,23] (cf. [24] for laminar flows also),

$$c_\varepsilon=3D_\infty/\mathrm{Re}=18/\mathrm{Re}, \quad L/\eta\lesssim a . \tag{4}$$

1063-651X/95/51(6)/6275(3)/$06.00

In general, the relation holds,

$$L/\eta = L\varepsilon^{1/4}/\nu^{3/4} = c_\varepsilon^{1/4}\,\mathrm{Re}^{3/4}\ ; \tag{5}$$

thus

$$c_\varepsilon = (3D_\infty/\mathrm{Re})[1 + a^{-2}c_\varepsilon^{1/2}\,\mathrm{Re}^{3/2}]^{(2-\zeta)/2}\ . \tag{6}$$

If Re becomes large, one obtains, after decomposing $\zeta = \frac{2}{3} + \delta\zeta$,

$$c_\varepsilon = \varepsilon/U^3L^{-1} = C\,\mathrm{Re}^{-\kappa},\quad L/\eta \gg a\ , \tag{7}$$

where the scaling exponent κ is to a good approximation proportional to the scaling corrections $\delta\zeta$.

$$\kappa = (\tfrac{9}{8})\delta\zeta/(1 + 3\delta\zeta/8)\ . \tag{8}$$

The prefactor C only weakly depends on the scaling correction $\delta\zeta$,

$$C = (3D_\infty/a^{(4/3)-\delta\zeta})^{12/(8+3\delta\zeta)}\ . \tag{9}$$

If one neglects the small influence of $\delta\zeta$ on C and uses $D_\infty = 6$, the corresponding constant in Ref. [18] is, of course, reproduced, $C(\delta\zeta = 0) = (6/b)^{3/2} = 0.60$. The data [19] seem to prefer $C(0) \approx 1$; other authors (cf. [23]) obtain $C(0) = 0.76$.

Equation (7) shows that the presence or absence of scaling corrections in the structure function is directly visible in the asymptotic Re dependence of the dissipation rate in terms of the energy input rate. In this sense scaling corrections of $D(r)$ in the ISR have an immediate, observable, low order influence and are not only small corrections.

If I use a realistic estimate for the expected scaling correction, $\delta\zeta = 0.03$, the scaling exponent in (7) is $\kappa = 0.0334$. (This choice, by the way, corresponds to the exponent $\mu = 0.25$ for the decay of the dissipation correlation in the log-normal model [1].) This leads to a considerable reduction of the asymptotic dissipation rate, which should be measurable. $c_\varepsilon(\delta\zeta)/c_\varepsilon(0) = \mathrm{Re}^{-\kappa}$ is reduced by about 25–45 % if Re is increased from 10^4 through 10^8. Even if $\delta\zeta = 0.02$ only, thus $\kappa = 0.0223$, there is 20–33 % less dissipation. In Fig. 1 the dissipation rate reduction is compared graphically with the classical plateau behavior as well as with the data [19].

The difference between the Effinger-Grossmann [21] parametrization for $D(L)$ used in [18] and the extended Batchelor parametrization (1) used here leads to a small difference for $c_\varepsilon(\mathrm{Re})$ in the transition range between small and large Re behavior, i.e., between (4) and (7). These differences are nonmeasurably small. One might argue, however, that the correction $\propto (r/\eta)^{\delta\zeta}$ in (2b) is not acceptable physically, since at least for r near L there should be no such correction. Thus the effect of intermittency should better read $(r/L)^{\delta\zeta}$. Then instead of a Re-independent amplitude b, one would have to use another ISR coefficient $B = (\frac{1}{3})a^{(4/3)-\delta\zeta}(L/\eta)^{\delta\zeta}$. This can be interpreted by two different alternatives: Either the amplitude in the ISR structure function (2b), which is now B, depends on Re; or, if one assumes the amplitude to be constant, the crossover a must be taken as Re dependent, $a = a(\mathrm{Re})$, with

$$a = (3B)^{3/(4-3\delta\zeta)}(\eta/L)^{3\delta\zeta/(4-3\delta\zeta)}\ . \tag{10}$$

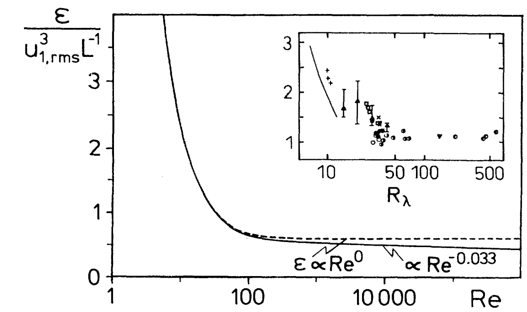

FIG. 1. Dissipation rate ε in terms of the energy input rate without and with intermittency corrections $\delta\zeta = 0.03$ in the ISR. The inset shows the data (grid turbulence) as analyzed by Sreenivasan [19]. The case $\delta\zeta = 0$ according to [18] with asymptote $c_\varepsilon = (6/b)^{3/2} = 0.60$. If $\delta\zeta \neq 0$ the curve represents Eq. (6) with $D_\infty = 6$ and $a = (3b)^{3/4}$, $b = 8.4$.

If according to the first alternative B is taken as Re dependent via $(L/\eta)^{\delta\zeta}$, one again finds (7) for c_ε, i.e., an asymptotic decrease of the dissipation rate. The pure rewriting of (2b) with $(r/L)^{\delta\zeta}$ instead of $(r/\eta)^{\delta\zeta}$ is thus irrelevant for the large Re behavior of the dissipation rate. Let us therefore check the consequences of a Re-dependent crossover $a(\mathrm{Re})$ but with a constant ISR amplitude, which then is to be chosen as $B = b = 8.4$. Now the structure function

$$D(r) = b\varepsilon^{2/3}r^{2/3}(r/L)^{\delta\zeta} \tag{11}$$

contains only weak intermittency effects in the large eddies. At the outer scale $r \approx L$ itself, there is no intermittency contribution left at all.

If this asymptotic form (11) is used instead of (2b), the same analysis as before immediately leads to

$$c_\varepsilon = (D_\infty/b)^{3/2}\ . \tag{12}$$

Thus, c_ε is now independent of Re. The price to pay for this asymptotic Re independence of the dissipation rate in terms of the energy input despite scaling corrections in the structure function is a nonuniversal Re dependence of the VSR-ISR crossover scale a, in addition to the Re dependence via η. From (10) one gets

$$r_{\mathrm{crossover}}/\eta = a(\mathrm{Re}) = A\,\mathrm{Re}^{-\alpha}\ . \tag{13}$$

Here $\alpha = 9\delta\zeta/(16 - 12\delta\zeta)$ and $A(\delta\zeta) = (3b)^{3/(4-3\delta\zeta)}(b/D_\infty)^{9\delta\zeta/(32-24\delta\zeta)} \approx A(0) = (3b)^{3/4}$. The corresponding structure functions are displayed in Fig. 2. It would be interesting to analyze the data with the generalized Batchelor parametrization (1), including its scaling corrections using a fixed and Re-independent value for b but a nonuniversal crossover behavior. Since $\alpha \approx (\frac{9}{16})\delta\zeta \approx 0.017$ is small, this might not be too clearly visible in contrast to the quite apparent asymptotic decrease of ε according to (7). But it leads at fixed r/η to a systematic decrease of the ISR amplitude of the structure function for increasing Re. One easily denotes this in Fig. 2. Such an effect seems to have been found recently; see [16], Fig. 3, roughly proportional to $\mathrm{Re}^{-0.05}$. From

(1) with (13) one finds that the amplitude at fixed r/η in the ISR should decrease $\propto \mathrm{Re}^{-3\delta\zeta/4}$. This possibly explains the not yet understood experimental observation in Ref. [16] as an effect of the VSR-ISR crossover. It is thus due to the competition between viscosity and nonlinearity in the transport of energy [8,9], but does not indicate pure ISR intermittency.

Equation (13) describes intermittency effects as concentrated on the VSR-ISR crossover, but being only small as r approaches L. This situation is very compatible with our results when solving the Navier-Stokes equation for high Re with a reduced wave vector set approximation [8–10].

Let me point out that this idea of explaining the experimentally found $b = b(\mathrm{Re})$ (cf. [16]) by a systematic crossover shift does not depend on the generalized Batchelor parametrization. Even if ζ has the classical value $\zeta = \frac{2}{3}$, a systematic shift of the crossover $a = a(\mathrm{Re}) = A\ \mathrm{Re}^{-\alpha}$ implies that for fixed r/η the ISR coefficient of $D(r)$ decreases $\propto \mathrm{Re}^{-4\alpha/3}$. Now, α plays the role of an independent exponent, not derivable from $\delta\zeta$. Taking the measured [16] value $4\alpha/3 \approx 0.05$, one gets the estimate $\alpha \approx 0.035$. If Re is increased by a factor of, say, 10^3, the crossover factor a is reduced by $\approx 10^{-0.1} \approx 0.8$. This may well have escaped previous attention.

To summarize, it is not only the correlation function $\langle \delta\varepsilon(r)\delta\varepsilon(0)\rangle$ which is a useful object for deciding upon intermittency effects, but this can also be concluded from the mean dissipation rate ε itself, if one studies its behavior for large Re. Its experimentally found approach to a constant level near $\varepsilon \approx u_{1,\mathrm{rms}}^3 L^{-1}$ seems to be an argument against ISR scaling corrections in the structure function.

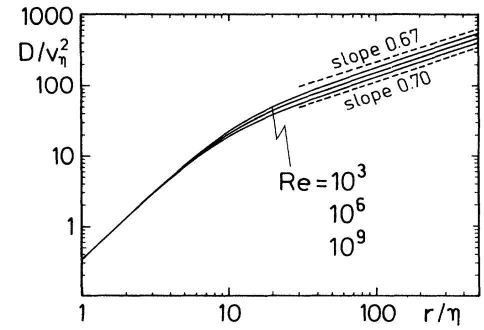

FIG. 2. Nonuniversal VSR-ISR transition from the generalized Batchelor parametrization (1) with the rigorous VSR scaling (2a) and the assumed ISR scaling (11), taking $b = 8.4$ and $\delta\zeta = 0.03$. The crossover position $a(Re)$ is according to (13), $A = 11.9$, $\alpha = 0.017$, for various Re.

Progress in this exciting riddle about the presence or absence of inertial range scaling corrections evidently has to come either from experiment (further high resolution measurements) or from theory, provided it is Navier-Stokes based. Resorting to models will hardly be convincing, since by general argument a dimensionally correct model, after including statistics, inevitably will lead to scaling corrections, without ever referring to the Navier-Stokes dynamics; see, e.g., [25].

Stimulating discussions with Detlef Lohse are gratefully acknowledged.

[1] A. N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962); A. M. Obukhov, *ibid.* **13**, 77 (1962).
[2] B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).
[3] U. Frisch, P. Sulem, and M. Nelkin, J. Fluid Mech. **87**, 719 (1978); G. Parisi, U. Frisch, in *Turbulence and Predictability of Geophysical Fluid Dynamics,* edited by M. Ghil, R. Benzi, and G. Parisi (North-Holland, Amsterdam, 1985), p. 84.
[4] C. Meneveau and K. R. Sreenivasan, Phys. Rev. Lett. **59**, 1424 (1987).
[5] F. Anselmet, Y. Gagne, E. J. Hopfinger, and R. A. Antonia, J. Fluid Mech. **140**, 63 (1984); R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, F. Massaioli, and S. Succi, Phys. Rev. E **48**, R29 (1993); G. Stolovitzky, K. R. Sreenivasan, and A. Juneja, *ibid.* **48**, R33 (1993); **48**, R3217 (1993).
[6] A. Vincent and M. Meneguzzi, J. Fluid Mech. **225**, 1 (1991).
[7] V. I. Belinicher and V. S. L'vov, Zh. Eksp. Teor. Fiz. **93**, 533 (1987) [Sov. Phys. JETP **66**, 303 (1987)].
[8] S. Grossmann and D. Lohse, Z. Phys. B **89**, 11 (1992); D. Lohse and S. Grossmann, Physica A **194**, 519 (1993).
[9] S. Grossmann and D. Lohse, Phys. Fluids A **6**, 611 (1994).
[10] S. Grossmann and D. Lohse, Phys. Rev. E **50**, 2784 (1994).
[11] S. Grossmann, D. Lohse, V. S. L'vov, and I. Procaccia, Phys. Rev. Lett. **73**, 432 (1994).
[12] B. Castaing, Y. Gagne, and M. Marchand, Physica D **68**, 387 (1993).
[13] V. S. L'vov and I. Procaccia, Phys. Rev. Lett. **74**, 3 (1995).
[14] C. Meneveau and K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991); G. K. Batchelor and A. A. Townsend, Proc. R. Soc. London Ser. A **199**, 82 (1962).
[15] A. Praskovsky, Phys. Fluids A **4**, 2589 (1992).
[16] A. Praskovsky and S. Oncley, Phys. Fluids **6**, 2886 (1994).
[17] T. Katsuyama, Y. Horiuchi, and K. Nagata, Phys. Rev. E **49**, 4052 (1994).
[18] D. Lohse, Phys. Rev. Lett. **73**, 3223 (1994).
[19] K. R. Sreenivasan, Phys. Fluids **27**, 1048 (1984).
[20] A. N. Kolmogorov, C. R. Acad. Sci. USSR **30**, 299 (1941); A. M. Obukhov, *ibid.* **32**, 19 (1941); C. F. von Weizsäcker, Z. Phys. **124**, 614 (1948); W. Heisenberg, *ibid.* **124**, 628 (1948); L. Onsager, Phys. Rev. **68**, 286 (1945).
[21] H. Effinger and S. Grossmann, Z. Phys. B **66**, 289 (1987).
[22] G. K. Batchelor, Proc. Cambridge Philos. Soc. **47**, 359 (1951).
[23] J. C. Rotta, *Turbulente Strömungen* (Teubner, Stuttgart, 1972), p. 116ff.
[24] L. D. Landau and E. M. Lifschitz, *Fluid Dynamics* (Pergamon, Oxford, 1987).
[25] A. Erzan, S. Grossmann, and A. Hernández-Machado, J. Phys. A **20**, 3913 (1987).