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Statistics of a passive scalar advected by a large-scale two-dimensional velocity field: Analytic solution

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Steady statistics of a passive scalar advected by a random two-dimensional flow of an incompressible fluid is described in the range of scales between the correlation length of the flow and the diffusion scale. This corresponds to the so-called Batchelor regime where the velocity is replaced by its large-scale gradient. The probability distribution of the scalar in the locally comoving reference frame is expressed via the probability distribution of the line stretching rate. The description of line stretching can be reduced to a classical problem of the product of many random matrices with a unit determinant. We have found the change of variables that allows one to map the matrix problem onto a scalar one and to thereby prove the central limit theorem for the stretching rate statistics. The proof is valid for any finite correlation time of the velocity field. Whatever the statistics of the velocity field, the statistics of the passive scalar (averaged over time locally in space) is shown to approach Gaussian statistics with increase in the Péclet number Pe (the pumping-to-diffusion scale ratio). The first $n < \ln \mathrm{Pe}$ simultaneous correlation functions are expressed via the flux of the square of the scalar and only one factor depending on the velocity field: the mean stretching rate, which can be calculated analytically in limiting cases. Non-Gaussian tails of the probability distributions at finite Pe are found to be exponential.

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I. INTRODUCTION

Description of a small-scale statistics of a passive scalar advected by a large-scale solenoidal velocity field is a special problem in turbulence theory. This problem was treated consistently from the very beginning and some exact results have been obtained, which is quite unusual for a turbulence problem. Batchelor [1] found exactly the form of the double correlation function in the case of the external velocity field being so slow that it does not change during the time of the spectral transfer of the scalar from the external scale to the diffusion scale. Then Kraichnan [2] obtained many of exact results in the opposite limit of a velocity field δ-correlated in time. Those results are valid for space of any dimension. We consider the two-dimensional problem and show that it has some special features that allow one to develop an analytical theory further and get some additional qualitative and even quantitative results.

We assume the velocity field to contain motions from some interval of scales while the statistics of the scalar will be considered for smaller scales beyond this interval. A steady turbulence with a constant supply of the passive scalar is considered. We wish to find the statistics of the passive scalar θ in the convective interval of scales, i.e., for scales that are less than both the velocity correlation scale and the scale of the scalar supply L, and larger than the diffusion scale r_{dif}. Since the scalar is a tracer in the velocity field, then in order to find the statistics of the scalar one should study the statistics of line stretching first.

To get rid of the uniform sweeping and concentrate on the stretching process we use a locally comoving reference frame [3–5]. All averaging is temporal. The source of the scalar is assumed to be δ-correlated in the comoving frame. As we show in Sec. II, the correlation functions of the scalar could be expressed via the integrals of the correlation functions of the external force along the trajectories of the fluid particles. For example, the value of the pair correlation function $\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\rangle$ averaged over time in the locally comoving frame is shown to be equal to the flux P_2 of θ^2 multiplied by the time τ_* that is necessary for the distance between two fluid particles to increase from $R(0) = |\mathbf{r}_1 - \mathbf{r}_2|$ to $R(\tau_*) = L$. This makes it possible to reduce our problem to the equation

$$\dot{\mathbf{R}}(t) + \hat{\sigma}(t)\mathbf{R}(t) = \mathbf{0}\,, \tag{1.1}$$

where $\mathbf{R}$ is the two-dimensional vector describing the separation of two points and $\hat{\sigma}(t)$ is a 2×2 matrix of the velocity derivatives randomly varying with time and having some *correlation time* τ. Due to incompressibility this matrix is traceless. The main value of interest is the rate of line stretching $\lambda(t) = t^{-1}\ln[R(t)/R(0)]$.

Studying the pair relative dispersion of the Lagrangian tracers is by itself of great importance for describing spatially nonuniform situations such as a pollutant spreading out into atmospheric turbulence. It is quite well known

[6] how difficult it is to make some definite statements about the statistics of $\lambda(t)$ and even about the mean value $\bar\lambda = \lim_{t\to\infty}\lambda(t)$ which is usually called the Lyapunov exponent. The difficulties stem from the matrix character of Eq. (1.1). If this equation were scalar, then it could be solved immediately: $\lambda(t) = -\int_0^t \sigma(t')dt'/t$. Due to the central limit theorem, the value $\lambda(t)$ at $t \gg \tau$ would have to have Gaussian statistics with the mean $-\langle\sigma\rangle$ and with the variance $\int\langle\sigma(t')\sigma(0)\rangle\, dt'/t$ which decreases with time.

The situation is essentially the same as in the scalar case if the matrix $\hat\sigma(t)$ is δ-correlated in time. In this case, the rate of stretching has Gaussian statistics with a mean which is determined solely by the pair correlation function: $\bar\lambda = \int \mathrm{tr}\, \hat s(t')\hat s(0)dt'/4$ where the so-called strain $\hat s$ is the symmetric part of the matrix $\hat\sigma$. A rigorous mathematical proof of the central limit theorem for $\lambda(t)$ in the δ-correlated case can be found in [7]. The statistics of the scalar is also easy to analyze in this case (this was briefly explained in [4] and is described in Sec. III A and Appendix C, subsection 1 in more detail).

Fortunately, the opposite limit of a long-correlated velocity field (with the correlation time τ larger than the typical turnover time and smaller than the transfer time τ_*) can be analytically solved in two dimensions (2D), too. We have found $\bar\lambda = \mathrm{Re}\left\langle\sqrt{\mathrm{tr}\,\hat\sigma^2/2}\right\rangle$—see Sec. III B. Note that in this case both the symmetric and antisymmetric parts of $\hat\sigma$ determine $\bar\lambda$. Since the central limit theorem for λ statistics is readily proven in the slow case as well, it is tempting to assume that the theorem is valid for an arbitrary value of $\bar\lambda\tau$.

The consideration of the general (and most physically interesting) case of a velocity field with the correlation time τ comparable with the turnover time requires more sophisticated formalism. The fact that the matrices $\hat\sigma(t_1)$ and $\hat\sigma(t_2)$ do not commute prohibits a straightforward expression of $R(t)$ as an exponent of some integral of $\hat\sigma$ (it could be written only via a time-ordered exponent). The value $\lambda(t)$ is not an immediate object of the central limit theorem. All correlation functions of $\hat\sigma(t)$ as well as its antisymmetric part determining vorticity should be taken into account in the calculation of $\bar\lambda$.

To analyze the general case, we suggest in Sec. V a nonlinear substitution (specific to 2D) that enables one to write $R(t)$ as a plain exponent in terms of the new variables. A substitution of that kind was first introduced in the theory of magnetism [8] and it has proven useful in different problems which reduce to time-ordered exponents of 2×2 matrices [9]. In the general case, those variables cannot be expressed analytically via the original $\hat\sigma(t)$, yet some important properties could be established. For example, we can prove the finiteness of the correlation time of the random process in the new variables if τ is finite. This allows us to establish the central limit theorem for $\lambda(t)$, since now it can be represented as an integral of some scalar quantity (nontrivially expressed through $\hat\sigma$) with a finite correlation time. And, what is probably more important, the substitution allows one to evaluate the correlation time of the stretching rate fluctuations, which is generally different from the correlation time of the velocity field. Our goal was also to find out whether some anomalies are possible at $\bar\lambda\tau \simeq 1$ that prevent estimating $\bar\lambda$ by interpolation between limiting cases. For the particular case of Gaussian velocity statistics with arbitrary correlation time, this problem is reduced in Sec. V B to finding the ground state of some not very complicated quantum-mechanical system. That made it possible to calculate $\bar\lambda$ numerically for the different values of the correlation time and for the different vorticity-strain ratios by using a pocket calculator. The Lyapunov exponent has quite a simple behavior which agrees with intuitive expectations: $\bar\lambda$ monotonically grows with τ until $\bar\lambda\tau \simeq 1$ and then the dependence is saturated; $\bar\lambda$ monotonically decreases as a function of the vorticity-strain ratio. Such a regular behavior enables one to use, instead of numerics, a simple interpolation formula explained in Sec. III. The formula expresses $\bar\lambda$ in terms of the mean strain S, mean vorticity Ω, and the correlation time τ:

$$\bar\lambda = S\tanh\frac{S\tau}{1+\Omega\tau}\ . \tag{1.2}$$

That formula satisfies all possible asymptotics and should give a reliable estimate for any possible statistics of the velocity field.

Gaussianicity of the stretching rate is an asymptotic property at $t\to\infty$. Since we are interested in the stretching that provides for the spectral transfer of the passive scalar from L to r_{dif} then we always consider a finite time. Measured at any finite time, the probability distribution function (PDF) $P(\lambda)$ has generally non-Gaussian tails which we show to be exponential for any velocity field (Sec. IV A).

Extending the result of Furstenberg [10] for a finite τ, we get the positiveness of $\bar\lambda$, so the average stretching is exponential in time for any velocity field. Consequently, the stretching time τ_* logarithmically depends on distances and so the pair correlation function is

$$\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\rangle = \frac{P_2}{\bar\lambda}\ln\frac{L}{|\mathbf{r}_1-\mathbf{r}_2|}\ . \tag{1.3}$$

That expression is valid for $|\mathbf{r}_1-\mathbf{r}_2| = r_{12} \ll L$. On the other hand, we neglect diffusion which is possible for r_{12} sufficiently large to make the typical stretching time $\bar\lambda^{-1}$ much less than the diffusion time r_{12}^2/κ. Introducing $r_{dif} = (\kappa/\bar\lambda)^{1/2}$, the last condition could be written as $r_{12} \gg r_{dif}$. The pair correlation function $\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\rangle$ logarithmically increases as r_{12} decreases until $r_{12}\simeq r_{dif}$; at smaller distances, the growth saturates: with the logarithmic accuracy, $\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\rangle \approx \langle\theta^2\rangle = P_2\bar\lambda^{-1}\ln(L/r_{dif})$.

Section IV B describes the statistics of the passive scalar. The PDF $P\{\theta\}$ is generally non-Gaussian for any finite Pe. It is interesting to find whether the non-Gaussianicity is related to the finiteness of the convective range or is inherent in the passive scalar dynamics and present even at the limit of infinite Pe. Here we show that the non-Gaussianicity is not an intrinsic property of the advection but rather appears either due to a nonergodic nature of the flow or as a result of an interplay between the advection and diffusion. By a straightforward calculation, we obtain the expressions for the high-order correlation functions (this was briefly explained in [4]). The logarithmic case we are dealing with is substantially simpler than cases with powerlike correlation functions usually encountered in turbulence problems.

One should distinguish between the statistics of the products and of the differences of θ in different spatial points. The statistics of the products is especially simple in our logarithmic regime. We show that, as long as all the distances between the points are much less than L, the mean value of the product of θs in n points is given by the reducible parts (i.e., is expressed via the pair products) until $n < \ln(L/r)$, where r is either the smallest distance between the points or r_{dif} depending on which is larger. In particular, $\langle\theta^{2n}\rangle = (2n-1)!!\langle\theta^2\rangle^n$ for $n \ll \ln(L/r_{dif}) = \ln \mathrm{Pe}$. The reason for such Wick decoupling of the first n moments is simply the fact that reducible parts contain more large logarithmic factors than nonreducible parts do. That means that $P\{\theta\}$ has a Gaussian core (that describes the first moments) and remote non-Gaussian tails (that describe moments with $n \gg \ln \mathrm{Pe}$). We show that the tails are exponential (see also [5]). Let us emphasize that this is true for the statistics of the products taken at the points that are separated by however small distances.

In contrast, the statistics of the differences depends on whether the distance is in the convective interval ($r \gg r_{dif}$) or in the diffusion interval ($r \ll r_{dif}$). The point is that the mean values $\langle(\theta_1-\theta_2)^n\rangle$ are logarithmic only for the distances $L \gg r_{12} \gg r_{dif}$. For example, $\langle(\theta_1-\theta_2)^2\rangle = 2\langle\theta^2\rangle - 2\langle\theta_1\theta_2\rangle = 2P_2\bar{\lambda}^{-1}\ln(r_{12}/r_{dif})$. Let us consider $\langle(\theta_1-\theta_2)^4\rangle = 2\langle\theta^4\rangle + 6\langle\theta_1^2\theta_2^2\rangle - 4\langle\theta_1\theta_2(\theta_1^2+\theta_2^2)\rangle$. Substituting here the mean values of the products, one finds $3\big\langle(\theta_1-\theta_2)^2\big\rangle^2$. Thus in the convective interval Wick decoupling is valid while at the diffusion interval the reducible contributions cancel. As a result, the statistics of the differences is getting more Gaussian as r_{12}/L decreases yet as the distance r_{12} approaches the diffusion scale the statistics again starts to deviate from Gaussian. In the diffusion interval, one can easily find $\langle(\theta_1-\theta_2)^2\rangle = P_2 r_{12}^2/4\kappa$. To determine higher moments and describe the statistics of the differences in the diffusion interval one needs further studies.

As far as the dependence on the velocity field is concerned, the Gaussian parts of the scalar distributions [giving lower moments with $n < \ln(L/r)$] are determined solely by the value $\bar{\lambda}$. It is remarkable that the non-Gaussian exponential tails are also universal and are, up to a dimensionless factor (depending on $\hat{\sigma}$ statistics), determined by the mean value $\bar{\lambda}$ and variance Δ. Note also that (1.3) is true at small scales for any large-scale turbulent velocity field, even one containing some long-living vortices that could trap the passive scalar for a long time, locally suppressing stretching. Note that we obtain the result on asymptotic Gaussian statistics by only temporal averaging. If there are separate space regions with different values of the pumping or the mean stretching rate (the flow is nonergodic) and if one desired to average with respect to such a superensemble then Gaussian statistics is lost while the logarithmic dependencies of the correlation functions persist. Yet a single-probe measurement should reveal the probability distribution function with a Gaussian core and exponential tails.

The subject of a local Gaussian statistics of the passive scalar was considered as a bit confusing due to the existence of an infinite number of integrals of motion $I_n = \int \theta^n(r)\, d^2r$ in the undamped unforced case. It was supposed [2] that the fluxes P_n of I_n should determine high-order correlation functions so that the statistics should depend on the pumping that determines P_n. It is not the case for logarithmic correlation functions. We shall show in Sec. IV C that the fluxes of I_n for $n > 2$ are not constant in the convective interval. This happens due to an effect of "distributed pumping" [11]: high-order integrals are pumped even in the convective interval of scales due to nonzero correlation functions of the force with lower powers of θ. As a result, the whole set of the correlation functions (until $n \simeq \ln \mathrm{Pe}$) is solely determined by the pair correlation function, that is, by the values P_2 and $\bar{\lambda}$.

The paper is organized as follows: the first part (Secs. II–IV) contains all physical statements supplied by a moderate mathematical formalism; those who need more mathematical strictness and quantitative precision can find those at the second part that includes Sec. V and the Appendixes.

II. FORMULATION OF THE PROBLEM

We formulate the problem following Ref. [4]. The advection of the scalar field $\theta(t,\mathbf{r})$ is governed by the following equation:

$$\dot{\theta} + u_\alpha \nabla_\alpha \theta = \phi + \kappa\Delta\theta, \tag{2.1}$$

where $\mathbf{u}(t,\mathbf{r})$ is the external velocity field and $\phi(t,\mathbf{r})$ is the external source which we assume to be random functions of t and $\mathbf{r}$. We assume that the source ϕ is correlated on a scale L. This means, e.g., that the pair correlation function of the source $\langle\phi(\mathbf{r}_1,t_1)\phi(\mathbf{r}_2,t_2)\rangle = \Xi(t_1-t_2, r_{12})$ as a function of the argument r_{12} decays on the scale L. The same behavior is assumed for high-order correlation functions of the source. The velocity field might be multiscale; its smallest scale is assumed to be larger than or of the order of L. We consider a statistical steady state with a source provided by ϕ and a small-scale sink due to diffusion with diffusivity κ (note that most of the results below are independent of the particular form of the sink). The mechanism of spectral transfer from the pump to the sink due to stretching by an inhomogeneous velocity field is clearly explained in [1,2].

To eliminate homogeneous sweeping, we pass to the reference frame locally comoving with the fluid at some point $\mathbf{r} = \mathbf{0}$ (here and below $\mathbf{r}$ denotes the radius vector of a point in the comoving frame). This corresponds to introducing the quasi-Lagrangian velocities $\mathbf{v}(t,\mathbf{r})$ related to the initial Eulerian ones as $\mathbf{u}(t,\mathbf{r}) = \mathbf{v}\left(t, \mathbf{r} - \int^t \mathbf{v}(0,t')dt'\right)$. We aim at finding simultaneous correlation functions of θ which are the same for both sets of variables. Equation (2.1) takes the form

$$\dot{\theta} + (v^\alpha - v_0^\alpha)\nabla_\alpha \theta = \phi + \kappa\Delta\theta, \quad \mathbf{v}_0 = \mathbf{v}(t,0). \tag{2.2}$$

We will study the correlation functions of the scalar at different spatial points separated by a distance that is smaller than the correlation length L. We thus consider the points in space with the distances from the zero point (where the sweeping is excluded) to be also much smaller than the typical scale of velocity variations. This allows one to expand the difference $v^\alpha(\mathbf{r}) - v^\alpha(0) = \sigma^{\alpha\beta} r^\beta$. Here $\hat{\sigma}(t)$ is the matrix of the velocity derivatives which contains symmetric (strain) and antisymmetric (vorticity) parts:

$$\hat{\sigma} = \hat{s} + \hat{\sigma}_a = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} + \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix} . \tag{2.3}$$

We assume that $a(t), b(t)$, and $c(t)$ are independent random processes with zero means and $\langle a^2\rangle = \langle b^2\rangle$. The strain is determined by a, b and the vorticity by c.

The resulting equation for $\theta(t,\mathbf{r})$ is

$$\dot{\theta}(t,\mathbf{r}) + \sigma_{\alpha\beta}(t) r_\beta \nabla_\alpha \theta(t,\mathbf{r}) = \phi(t,\mathbf{r}) + \kappa\Delta\theta(t,\mathbf{r}) \ . \tag{2.4}$$

A formal solution of (2.4) is written in Appendix A. It is shown there that as long as we are going to consider the correlation functions of the scalar at distances large compared to $r_{dif} \equiv \sqrt{\kappa/\bar{\lambda}}$, the diffusion term can be neglected. We thus omit for a while the term $\kappa\Delta\theta$; we shall bring that term back in Sec. IV C in considering the conservation laws and in Appendix A in considering one-point statistics of $\theta(t,\mathbf{r})$. A formal solution of the equation

$$\dot{\theta}(t,\mathbf{r}) + \sigma_{\alpha\beta}(t) r_\beta \nabla_\alpha \theta(t,\mathbf{r}) = \phi(t,\mathbf{r})$$

can be written as follows:

$$\theta(t,\mathbf{r}) = \int_0^\infty dt' \phi\Big(t - t', \hat{W}(t, t-t')\mathbf{r}\Big) , \tag{2.5}$$

where the matrix $\hat{W}$ should satisfy the following equation:

$$\partial_t \hat{W}(t,t') + \hat{W}(t,t')\hat{\sigma}(t) = 0 \ . \tag{2.6}$$

The initial condition is $\hat{W}(t,t) = \hat{1}$. The solution of (2.6) can be written in the following form:

$$\hat{W}(t,t') = \tilde{T}\exp\Big(-\int_{t'}^{t} dt_1\, \hat{\sigma}(t_1)\Big) , \tag{2.7}$$

where $\tilde{T}$ designates the antichronologically ordered exponent.

By using (2.5), the correlation functions of the scalar can be rewritten in terms of the known correlation functions of the pumping. The correlation time of the pumping in the Lagrangian frame τ_ϕ^L will be assumed to be much less than $\bar{\lambda}^{-1}$ (which, in a typical case, is of the order of the turnover time of the vortices of size L). Since the pair correlation function of the source in the frame moving with the fluid

$$\Xi\left(t_1 - t_2, \left|\mathbf{r}_1 - \mathbf{r}_2 + \int_{t_1}^{t_2} \mathbf{v}(0,t)\, dt\right|\right)$$

decays as a function of $(t_1 - t_2)$ due to both arguments, then $\tau_\phi^L = \min\{\tau_\phi^E, L/V\}$, where τ_ϕ^E is the correlation time in the Eulerian frame and V is the mean turbulent velocity. Further consideration will be valid if either the Eulerian correlation time of the pumping is much less than the turnover time of L eddies or the mean turbulent velocity is much larger than the typical velocity of L eddies. The latter could be the case if, due to the inverse energy cascade in 2D (at scales larger than L), the mean turbulent velocity V that sweeps the scalar is determined by the largest scale while the strain and the vorticity (that determine $\bar{\lambda}$) are determined by the eddies with scale L. We thus write $\langle\phi(\mathbf{r}_1,t_1)\phi(\mathbf{r}_2,t_2)\rangle = P_2\xi_2(r_{12})\delta(t_1 - t_2)$, where the function $\xi_2(r_{12})$ describes spatial correlations of the pumping; $\xi_2(0) = 1$. The constant P_2 has the physical meaning of the production rate of θ^2.

The simultaneous pair correlation function of the scalar is written as follows:

$$\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\rangle = P_2\left\langle \int_0^\infty dt\, \xi_2(\,|\,\hat{W}(0,-t)(\mathbf{r}_1 - \mathbf{r}_2)\,|\,)\right\rangle_\sigma , \tag{2.8}$$

where $\langle\cdots\rangle_\sigma$ denotes the average over the statistics of σ. Averaging with respect to both the external velocity and external source is implied on the left-hand side of (2.8). We suggest that the statistics is homogeneous in time; it enabled us to take the $\langle\theta\theta\rangle$ correlator in (2.8) at zero time. The only unknown function in this expression is $\hat{W}(0,-t)$. The function ξ_2 is determined by the statistics of the source ϕ. We can put simply $\xi_2(x) = 1$ for $x < L$ and $\xi_2(x) = 0$ for $x > L$ [the account of any shape of $\xi_2(x)$ will give the same results with a logarithmic accuracy]. In this case,

$$\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\rangle = P_2\langle t(r_{12})\rangle_\sigma = P_2\tau_*(r_{12}) , \tag{2.9}$$

where $t(r_{12})$ is the time necessary for two points to increase their distance from r_{12} to L under the action of the transfer matrix $\hat{W}(0,-t)$ and τ_* is $t(r_{12})$ averaged over the statistics of $\hat{\sigma}$. It is quite natural that the pair correlation function is proportional to the time of separation: imagine a "heater" ϕ of size L, then the values θ_1 and θ_1 of the "temperature" are correlated until the cold fluid comes from outside into one of the points.

Since it is the modulus $|\hat{W}\mathbf{r}|$ that enters (2.8) then it is useful to represent $\hat{\sigma}$ as a sum of its symmetric part $\hat{s}$ responsible for the stretching and antisymmetric part $\hat{\sigma}_a$ that describes rotation. We get $\hat{W}$ in the form of the product $\hat{W} = \hat{W}_s\hat{W}_a$, where the multipliers satisfy the following separate equations:

$$\dot{\hat{W}}_a + \hat{W}_a\hat{\sigma}_a = 0, \qquad \dot{\hat{W}}_s + \hat{W}_s\hat{\tilde{s}} = 0, \tag{2.10}$$

with $\hat{\tilde{s}} = \hat{W}_a\hat{s}\hat{W}_a^T$. The first equation is immediately integrated

$$\hat{W}_a(t,t') = \exp\left(-i\hat{\sigma}_y \int_{t'}^{t} c d\tilde{t}\right) \ , \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \tag{2.11}$$

Since $|\hat{W}_a \mathbf{r}| = r$ then it is $\hat{W}_s$ that should be substituted in (2.8) but $\hat{W}_s$ actually depends on the effective strain $\hat{\tilde{s}}$ which is determined by the whole set $a(t)$, $b(t)$, and $c(t)$.

III. LIMITING CASES OF A RAPID AND SLOW STRAIN

Here we show how Eq. (2.6) can be directly solved in the limiting cases. It is more convenient for us to consider the equivalent problem of the behavior of the vector

$$\mathbf{R}(t) = \hat{W}(0,-t)\mathbf{r} \tag{3.1}$$

which determines, e.g., the pair correlation function of θ via (2.8). Differentiating $\mathbf{R}(t)$ and using (2.6) we get (1.1). Here we aim at finding the probability distribution function $P(t,\lambda)$ for $\lambda = (1/t)\ln[R(t)/r]$.

A. Shortly correlated velocity field

If the correlation time τ of the velocity derivatives is much less than the turnover time (which is of the order of the inverse mean strain or vorticity) then the random matrix $\hat{\sigma}$ can be considered as δ-correlated in time. This case was solved by Kraichnan [2] for a sparse distribution of sheets of the passive scalar. One can discretize Eq. (2.6) and represent the solution as a product of random matrices from SL(2, $\mathbb{R}$). For a δ-correlated case, those matrices are independent, which allowed Furstenberg [10] to prove the positivity of the Lyapunov exponent and La Page [7,6] to prove the central limit theorem: For all generic initial vectors $\mathbf{r}$, the function

$$\Delta(t,\mathbf{r}) = \frac{1}{t}\left\langle\left(\ln\frac{R(t)}{r} - t\bar{\lambda}\right)^2\right\rangle$$

converges with increasing t to a constant $\Delta > 0$ independent of t and $\mathbf{r}$; the value $[\ln(R/r) - t\bar{\lambda}]/\sqrt{\Delta t}$ converges in distribution to a Gaussian standard random variable.

Our formalism allows for a compact description of that case. Let us calculate the averages $\langle R^{2n}(t)\rangle = \langle[\mathbf{r}^T \hat{W}^T(0,-t)\hat{W}(0,-t)\mathbf{r}]^n\rangle$. Substituting here the expression (2.7) for $\hat{W}(0,-t)$ expanded in powers of $\hat{\sigma}(t')$ and calculating the difference between the instants t and $t+\Delta t$ one gets

$$\Delta\langle R^{2n}(t)\rangle = n\left\langle \mathbf{R}^T(t)\cdot\int\!\!\int dt_1 dt_2 2\hat{s}(t_1)\hat{s}(t_2)\mathbf{R}(t)R^{2n-2}\right\rangle + \frac{n(n-1)}{2}\left\langle\left(\mathbf{R}^T(t)\cdot\int dt_1 2\hat{s}(t_1)\mathbf{R}(t)\right)R^{2n-4}\left(\mathbf{R}^T(t)\cdot\int dt_2 2\hat{s}(t_2)\mathbf{R}(t)\right)\right\rangle,$$

where t_1 and t_2 run between $-t-\Delta t$ and $-t$. We have chosen Δt sufficiently small to allow the expansion in $\tilde{T}$ exponents and sufficiently large to neglect the correlations between $s(t_1)$ and $s(t_2)$ so that $\tau \ll \Delta t \ll S^{-1}$. The terms determined by the irreducible correlation functions of $\hat{\sigma}$ are small in τ/τ_*, since they contain additional restrictions for the region of integration over times t_i. Accounting for the tensor structure $\langle s_{\alpha\beta}(t_1)s_{\gamma\delta}(t_2)\rangle = D_s\delta(t_1-t_2)(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta})$ one gets the equation $\Delta\langle R^{2n}(t)\rangle = \langle R^{2n}(t)\rangle\Delta t D_s 2n(n+1)$ where

$$D_s = \frac{1}{8}\int \operatorname{tr}\left[\langle\hat{\sigma}(t)\hat{\sigma}(0)\rangle + \langle\hat{\sigma}(t)\hat{\sigma}^T(0)\rangle\right]dt = \frac{1}{4}\int \operatorname{tr}\langle\hat{s}(t)\hat{s}(0)\rangle\, dt. \tag{3.2}$$

Its solution

$$\langle R^{2n}(t)\rangle = r^{2n}e^{D_s 2n(n+1)t} \tag{3.3}$$

exactly corresponds to the average

$$\langle R^{2n}(t)\rangle = r^{2n}\int P(t,\lambda)e^{2n\lambda t}\,d\lambda$$

with the Gaussian probability function

$$P(t,\lambda) = \sqrt{t/2\pi\bar{\lambda}}\exp[-(\lambda-\bar{\lambda})^2 t/2\bar{\lambda}]\,, \tag{3.4}$$

leading to $\bar{\lambda} = \Delta = D_s$. We would like to stress that in the white noise limit the formula (3.4) is exact for arbitrary t. The alternative method to get this result can be found in [14]; the development of that method is used in Sec. V.

The above calculations are valid until $n\bar{\lambda}\Delta t \ll 1$. Since it should be true that $\Delta t > \tau$ then (3.3) is valid for the moments with $n < (\bar{\lambda}\tau)^{-1}$. Therefore the probability distribution function $P(t,\lambda)$ has non-Gaussian corrections due to a finiteness of the ratio t/τ. In a formal limit of a δ-correlated strain, $P(t,\lambda)$ is Gaussian everywhere for any finite time t.

B. Slow stretching and 1D localization

It is worth noting that the vorticity gave no contribution in the δ-correlated case. This was already clear from (2.10) and (2.11) since the correlation functions of $\hat{\tilde{s}}$ coincide with those of $\hat{s}$ in this case. For a finite correlation time, the vorticity plays an essential role suppressing stretching due to the rotation of a fluid element with respect to the axis of expansion and contraction. Let us illustrate this by considering the simplest case of a time-independent velocity field. Following Batchelor [1] we consider a solution $\theta(\mathbf{r},t) = A\sin[\mathbf{k}(t)\cdot\mathbf{r}]$ and get $k(t) = k(0)\exp\left(t\sqrt{a^2+b^2-c^2}\right)$. That formula is valid if $\tau \gg t$. Everywhere in this paper we are interested in the opposite case, $t \gg \tau$, when a universal statistics could appear.

An account of variations with time (even slow ones) is quite difficult in a general case. Still, the case of a slow velocity field also can be exactly solved. We assume that the matrix $\hat{\sigma}(t)$ does not change substantially during a typical turnover time. We differentiate Eq. (1.1) to get $\dot{\mathbf{R}} = -\hat{\sigma}\mathbf{R}$ with respect to time and neglect $\dot{\sigma}$ in comparison with σ^2. And here a little miracle happens: because of incompressibility, the matrix $\hat{\sigma}$ is traceless so that its square is proportional to the unit matrix in the 2D case. We thus come to the scalar equation instead of the matrix one; this scalar equation can be written in the form

$$\partial_t^2(R_x + iR_y) = (a^2 + b^2 - c^2)(R_x + iR_y) \; . \tag{3.5}$$

One can consider (3.5) as a Schrödinger equation for a particle in a random potential $U = a^2 + b^2 - c^2$; time plays the role of coordinate. We thus encounter a problem of the type considered in the 1D localization theory. We should find the behavior of the solution of (3.5) under initial conditions given at $t = 0$. It is similar to the computation of a 1D sample resistivity in the Abrikosov-Ryzhkin formulation (see [15] and [9] for more details). Based on their results we can assert that for any relation between a, b, and c the modulus of $R_x + iR_y$ grows without limit with t as $\exp(\bar{\lambda}t)$. This exponential growth is described by the same exponent as the exponentially decaying tails of a localized quantum ψ function.

Our limit of a slow strain corresponds to a quasiclassical regime so that $\bar{\lambda}$ can be calculated by using a semiclassical approximation. Classically allowed and forbidden regions should be considered separately. If $U < 0$ (the region is classically allowed) then $R_x + iR_y$ is the sum of two oscillating exponents so that the value R_j of the modulus of $R_x + iR_y$ at the beginning of this interval is generally of the order of its value R_{j+1} at the end. If $U > 0$ (the region is classically forbidden) then $R_x + iR_y$ is the sum of the increasing and decreasing exponents. To estimate the ratio R_{j+1}/R_j we can neglect the decreasing exponent and find $R_{j+1}/R_j \sim \exp\left(\int \sqrt{U(t')}\, dt'\right)$; the integral here is taken between the points t_j and t_{j+1} where $U = 0$. The typical distance between these points is the correlation time τ which is assumed to be much larger than both the inverse mean strain S^{-1} (determined by a and b) and the inverse mean vorticity Ω^{-1} (determined by c). That means that the exponent determining R_{j+1}/R_j is large (it is just the reason enabling us to neglect the decaying exponent, since it is exponentially small). We conclude that with an exponential accuracy the ratio $(R_x + iR_y)(t)/(R_x + iR_y)(0)$ is determined by the regions where $U > 0$ and can be estimated as the product of R_{j+1}/R_j for these regions. That means that the rate stretching $\lambda(t) = \ln[R(t)/R(0)]$ can be written as an integral

$$\lambda(t) = \frac{1}{t}\mathrm{Re}\int_0^t \sqrt{U(t')}\, dt' \, , \tag{3.6}$$

so that we have again a central limit theorem for the statistics of $\lambda(t)$ at $t \gg \tau$: $P(\lambda) = \sqrt{t/2\pi\Delta}\, \exp[-(\lambda - \bar{\lambda})^2 t/2\Delta]$.

The expression for the Lyapunov exponent follows from (3.6):

$$\bar{\lambda} = \mathrm{Re}\left\langle \sqrt{a^2 + b^2 - c^2} \right\rangle , \tag{3.7}$$

which can be calculated for any given statistics of a, b, c. The time intervals with a negative U give no contribution to $\bar{\lambda}$ [in the main order in $(\bar{\lambda}\tau)^{-1}$] since regions with predominant vorticity do not change the modulus of $\mathbf{R}$ in a slow case.

From (3.6) we find the variance:

$$\Delta = \int \left\langle \mathrm{Re}\sqrt{U(t')}\, \mathrm{Re}\sqrt{U(0)} \right\rangle_c dt' \, ,$$

where we use standard notation $\langle AB\rangle_c \equiv \langle AB\rangle - \langle A\rangle\langle B\rangle$. We conclude that $\Delta \sim S^2\tau$ if $S \sim \Omega$. Note that $\bar{\lambda}$ is determined by simultaneous averages so it does not depend on the correlation time τ (at given values of S and Ω), while the dispersion Δ does depend on it. A rigorous treatment in Sec. V B confirms (3.7)—see also [12]. From (3.7) it follows that $\bar{\lambda}$ can be estimated as S for the case $\Omega \lesssim S$.

The case $\Omega \gg S$ deserves separate consideration since $\bar{\lambda}$ will be suppressed in this case. By calculating different-time correlation functions of $\tilde{s}$ one can see that the correlation time of $\tilde{s}$ is either $1/\Omega$ or τ depending on which value is less. For $\Omega\tau \gg 1$ we get the asymptotic law of decreasing Lyapunov exponent for the limiting case of a very strong vorticity:

$$\bar{\lambda} \sim S^2/\Omega \; . \tag{3.8}$$

To turn an estimate into a quantitative answer one should specify the statistics of the velocity field. For Gaussian statistics the answer could be found in [12] and Sec. V B for any S, Ω.

To estimate the value of the Lyapunov exponent $\bar{\lambda}$ in a simplified way, one may construct an interpolation formula for $\bar{\lambda}$ in terms of the mean strain S, mean vorticity Ω, and correlation time τ. The value of $\bar{\lambda}$ for the fast case is equal to D_s given by (3.2) which can be estimated as $S^2\tau$. Taking into account also the fact (which will be proved in Sec. V) that the asymptotics (3.7) in the slow limit is approached exponentially in $S\tau$ we come to (1.2). The concrete values of S, Ω, and τ entering this expression can be estimated from the pair correlation functions of a, b, c up to numerical factors of the order unity (depending on the statistics of a, b, c).

IV. PROBABILITY DISTRIBUTIONS: GAUSSIAN BUMP AND NON-GAUSSIAN TAILS

We postpone the general proof of a central limit theorem for the stretching rate statistics until the next section. Here, assuming this theorem to be valid, we study the probability distributions that appear for different quantities.

A. Non-Gaussian tails of the probability distributions

The central limit theorem assures us that the probability distribution function for the stretching rate λ measured during the time t that is much larger than τ is as follows:

$$P(\lambda,t)=\sqrt{t/2\pi\Delta}\,\exp[-(\lambda-\bar\lambda)^2t/2\Delta]\ . \qquad (4.1)$$

It is expressed via two parameters: Lyapunov exponent $\bar\lambda$ and variance Δ, which depend upon the statistics of the velocity field. In the limiting cases, they can be expressed via S,Ω, and τ as has been done in the preceding section (see also the next section); here we do not need their explicit form. One can generally treat t and λ as independent parameters in (4.1). $t\gg\tau$ is implied. We will consider the limit of large Péclet number Pe. In this case, for separations r_{12} taken in the convective interval (from r_{dif} to L) $\ln(L/r_{12})$ can be treated as a large value.

We are interested in a particular case when t is equal to the time of passing the distance R from r_{12} up to the external scale L. Then the values of λ and t are simply related: $\lambda t=\ln(L/r_{12})$. For such a relation, both distributions

$$P(\lambda)\propto\exp\left[-\frac{(\lambda-\bar\lambda)^2\ln(L/r_{12})}{2\Delta\lambda}\right],$$
$$P(t)\propto\exp\left[-\frac{[\ln(L/r_{12})/t-\bar\lambda]^2t}{2\Delta}\right], \qquad (4.2)$$

are generally non-Gaussian. For example, the time probability distribution (that we need for evaluating the passive scalar statistics) is close to Gaussian at $|t-\bar t|\ll\bar t=\ln(\mathrm{L}/\mathrm{r}_{12})/\bar\lambda$. At $t\gg\bar t$ that formula gives an exponential PDF $P(t)\propto\exp(-t\bar\lambda^2/2\Delta)$. Generally, λ and the transfer time t are related in a more complicated way since $\lambda(t)$ is defined as some integral over time of a fluctuating quantity. This influences preexponential factors omitted in (4.2)—see Sec. IV B and Appendix C, subsection 1. There is another source of non-Gaussianicity except for the nonlinear relation between λ and t: the PDF (4.1) is itself true only asymptotically as $t\to\infty$. At a finite t PDF $P(\lambda,t)$ has non-Gaussian corrections that depend on the statistics of $\hat\sigma$. In Appendix B, we show that the account of those corrections can add only a numerical factor $c_2\simeq1$ in the exponent so that at $t\gg\bar t$

$$P(t)\propto\exp(-tc_2\bar\lambda^2/2\Delta)\ . \qquad (4.3)$$

Note that this tail is not generally of the form $\exp(-t/\bar t)$. For a δ-correlated case, $\Delta=\bar\lambda$ and $P(t)\propto\exp(-\bar\lambda t/2)$. For a long-correlated case with $S\simeq\Omega$, one has $\Delta\sim\bar\lambda^2\tau$ so that $P(t)\propto\exp(-ct/\tau)$ with the dimensionless coefficient c that depends on the statistics of $\hat\sigma$. For small $t\ll\bar t$ the probability sharply decreases: $P(t)\propto\exp[-\ln^2(L/r_{12})/2t\Delta]$.

B. Statistics of the passive scalar

The consideration of the correlation functions of the passive scalar in the locally comoving reference frame (briefly presented before in [4]) will be based upon the representation (2.5) giving the formal solution for θ in terms of the pumping "force" ϕ. To find a correlation function of θ we should first average over the statistics of ϕ and second over the statistics of $\hat\sigma$. The result of the first averaging can be expressed in terms of the correlation functions of ϕ; an example is given by (2.8). From that formula and (4.2) one immediately gets the exponential factor in the distribution function for the simultaneous product $\langle\theta(\mathbf r_1)\theta(\mathbf r_2)\rangle_\phi=P_2t(r_{12})\equiv Q$ averaged over ϕ only:

$$\mathcal P(Q)\propto\exp\left[-\frac{[\ln(L/r_{12})P_2/Q-\bar\lambda]^2Q}{2P_2\Delta}\right], \qquad (4.4)$$

which is Gaussian near the maximum (within a variance interval). The exponential behavior far from the maximum given by (4.4) is exactly correct only for a δ-correlated $\hat\sigma$ [see (C11) and (C12)] otherwise a numerical factor of order unity appears as well as in (4.3). To calculate the preexponential factors in (4.2) and (4.4) it would be wrong to substitute $\lambda t=\ln(L/r_{12})$ into (4.1)—such a substitution would give a probability that erroneously counts trajectories that reach $R=L$ at earlier times as well. The contribution of the trajectories with nonmonotonous $R(t)$ to the preexponential factor is substantial. To illustrate this, we calculate the whole $\mathcal P(Q)$ for the δ-correlated case—see (C11) and (C12) in Appendix C, subsection 1.

In considering general correlation functions of θ we assume for simplicity that the correlation functions of odd products of ϕ are zero and therefore the statistics of ϕ is characterized by the set of irreducible correlation functions Ξ_2, Ξ_4, ... of ϕ. For example,

$$\begin{aligned}&\langle\phi(\mathbf q_1)\phi(\mathbf q_2)\phi(\mathbf q_3)\phi(\mathbf q_4)\rangle\\&\quad=\Xi_4(\mathbf q_1;\mathbf q_2;\mathbf q_3;\mathbf q_4)+\Xi_2(\mathbf q_1;\mathbf q_2)\Xi_2(\mathbf q_3;\mathbf q_4)\\&\qquad+\Xi_2(\mathbf q_1;\mathbf q_3)\Xi_2(\mathbf q_2;\mathbf q_4)+\Xi_2(\mathbf q_1;\mathbf q_4)\Xi_2(\mathbf q_2;\mathbf q_3).\end{aligned} \qquad (4.5)$$

Here we designate $\mathbf q_i=(t_i,\mathbf r_i)$. As we have explained previously, in the comoving reference frame one can treat the field ϕ as δ-correlated in time:

$$\begin{aligned}\Xi_2(\mathbf q_1,\mathbf q_2)&\to\delta(t_1-t_2)P_2\xi_2(\mathbf r_1-\mathbf r_2),\\ \Xi_4(\mathbf q_1;\mathbf q_2;\mathbf q_3;\mathbf q_4)&\to\delta(t_1-t_2)\delta(t_1-t_3)\delta(t_1-t_4)\\&\quad\times P_4\xi_4(\mathbf r_1,\mathbf r_2,\mathbf r_3,\mathbf r_4),\end{aligned} \qquad (4.6)$$

etc. The quantity P_4 (introduced by analogy with P_2) is the production rate of θ^4 at the scale L. Using those notations we can express the different-time pair correlation function as follows:

$$\begin{aligned}&\langle\theta(t_1,\mathbf r_1)\theta(t_2,\mathbf r_2)\rangle\\&\quad=P_2\int_{-\infty}^{t}dt'\left\langle\xi_2[\hat W(t_1,t')\mathbf r_1-\hat W(t_2,t')\mathbf r_2]\right\rangle_\sigma,\end{aligned} \qquad (4.7)$$

where $t=\min(t_1,t_2)$. If $t_1=t_2$ then (4.7) reduces to (2.8). For sufficiently small r_1 and r_2, the integration time in (4.7) is large so that the absolute values of $\hat{W}(t_1,t')\mathbf{r}_1$ and $\hat{W}(t_2,t')\mathbf{r}_2$ can be approximated by $\exp[\bar\lambda(t_1-t')]r_1$ and $\exp[\bar\lambda(t_2-t')]r_2$, respectively. If $r_1\sim r_2$ then the argument of ξ_2 in (4.7) will be determined by the largest of two times t_1 and t_2, say, t_1 (then $t=t_2$). In this case the integration over t' in (4.7) is within the interval $t_1-\bar\lambda^{-1}\ln(L/r_1)<t'<t_2$. Therefore with logarithmic accuracy

$$\langle\theta(t_1,\mathbf{r}_1)\theta(t_2,\mathbf{r}_2)\rangle=P_2\left[\bar\lambda^{-1}\ln(L/r_1)-(t_1-t_2)\right], \quad (4.8)$$

which implies $t_1>t_2$, $r_1\sim r_2$. The expression (4.8) is correct if $\bar\lambda^{-1}\ln(L/r_1)>(t_1-t_2)>\bar\lambda^{-1}$. Note that the correlation function (4.8) does not depend only on the difference r_{12} since we lost the homogeneity at passing to the comoving reference frame. We see that the correlation time of the scalar is logarithmically large in this frame independently of the correlation time of σ. This is a manifestation of the fact that θ is a Lagrangian invariant of the dynamics.

Consider now the higher-order correlation functions of θ. From (2.5), (4.5), and (4.6) it follows that

$$\langle\theta_1\theta_2\theta_3\theta_4\rangle=P_4\int_{-\infty}^{t}dt'\langle\xi_4(\hat{W}(t_1,t')\mathbf{r}_1,\hat{W}(t_2,t')\mathbf{r}_2,\hat{W}(t_3,t')\mathbf{r}_3,\hat{W}(t_4,t')\mathbf{r}_4)\rangle_\sigma$$
$$+P_2^2\left\langle\int_{-\infty}^{t_{12}}dt'\xi_2(\hat{W}(t_1,t')\mathbf{r}_1-\hat{W}(t_2,t')\mathbf{r}_2)\int_{-\infty}^{t_{34}}dt''\xi_2(\hat{W}(t_3,t'')\mathbf{r}_3-\hat{W}(t_4,t'')\mathbf{r}_4)\right\rangle_\sigma+\cdots, \quad (4.9)$$

where the dots designate two additional terms originating from $\Xi_2\Xi_2$ products in (4.5), $t=\min(t_1,t_2,t_3,t_4)$, $t_{12}=\min(t_1,t_2)$, etc. The same arguments as before show us that with logarithmic accuracy the first term in the right-hand side of (4.9) can be estimated as $P_4\bar\lambda^{-1}\ln(L/r)$ and the second term in the right-hand side of (4.9) is reduced to the product of the two pair correlation functions (4.8). This product contains the second power of the large logarithm and therefore the first term in the right-hand side of (4.9) is negligible in comparison with the second one. Thus we conclude that the main contribution to the fourth-order correlation function $\langle\theta_1\theta_2\theta_3\theta_4\rangle$ is determined by its reducible part which is the sum $\langle\theta_1\theta_2\rangle\langle\theta_3\theta_4\rangle+\langle\theta_1\theta_3\rangle\langle\theta_2\theta_4\rangle+\langle\theta_1\theta_4\rangle\langle\theta_2\theta_3\rangle$. The same assertion is obviously true for higher-order correlation functions of θ until some order n (see below). Therefore the statistics of θ is Gaussian with logarithmic accuracy, which means that we calculated exactly only factors at the terms with the largest power of the logarithms while neglecting additive constants and terms with smaller powers of the logarithms.

Now we discuss the deviations from Gaussian statistics due to finiteness of the Péclet number. Those deviations appear at sufficiently large θ (or for sufficiently high orders of the correlators). We study them for the simultaneous correlation functions. Averaging over the statistics of $\hat\sigma$ can be replaced by averaging over λ:

$$F_2=\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\rangle=P_2\int d\lambda P(t,\lambda)\lambda^{-1}\ln(L/r_{12})\,. \quad (4.10)$$

Here $P(t,\lambda)$ is the PDF of λ on the time interval t and $t\lambda=\ln(L/r_{12})$. Since $P(t,\lambda)$ has a sharp maximum at $\lambda=\bar\lambda$ the integration over λ in (4.10) gives (2.8). Expressions analogous to (4.10) can be deduced for higher-order simultaneous correlation functions of θ. Consider, e.g., the fourth-order correlation function. If all separations r_{12}, r_{34},... are of the same order then with logarithmic accuracy

$$F_4=\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\theta(\mathbf{r}_3)\theta(\mathbf{r}_4)\rangle$$
$$=P_2^2\int d\lambda P(t,\lambda)\lambda^{-2}[\ln(L/r_{12})\ln(L/r_{34})$$
$$+\ln(L/r_{13})\ln(L/r_{24})+\ln(L/r_{14})\ln(L/r_{23})]. \quad (4.11)$$

The term $\int d\lambda P(t,\lambda)\lambda^{-2}$ in (4.11) can be substituted by $\bar\lambda^{-2}$ which gives the sum of the products of the pair correlation functions. In the $(2n)$th-order correlation function the term

$$\int d\lambda P(t,\lambda)\lambda^{-n} \quad (4.12)$$

will arise. It can be substituted by $\bar\lambda^{-n}$ if the number n is not very large. The largest value of n allowing for this substitution can be found by using (4.1): $n\lesssim(\bar\lambda/\Delta)\ln(L/r)$. We conclude that the $(2n)$th-order correlation function of θ is reduced to the product of the pair correlation function up to the number $n\sim(\bar\lambda/\Delta)\ln(L/r)$ which is large due to the suggested large value of $\ln(L/r)$. For higher n the Wick theorem (that is, the Gaussianicity) is violated.

The crossover number $n\sim(\bar\lambda/\Delta)\ln(L/r)$ can be readily appreciated as the ratio of the transfer time $\bar\lambda^{-1}\ln(L/r)$ to the correlation time of the stretching rate fluctuation. The latter is $\tau_\lambda\simeq\min\{\tau,\lambda^{-1}\}$ according to Appendix C, subsection 2, while $\Delta\simeq\max\{\lambda,\lambda^2\tau\}$. For the Gaussianicity of the nth correlation function, the time of mutual correlations $n\tau_\lambda$ should be less than the transfer time.

To determine the value of F_{2n} for $n\gg(\bar\lambda/\Delta)\ln(L/r)$ it is worthwhile to pass to the integration over t using $t\lambda=\ln(L/r)$. Then

$$F_{2n}\simeq P_2^n(2n-1)!!\ln(L/r)\int dt\,t^{n-2}P(t,\lambda)\,, \quad (4.13)$$

where all separations are assumed to be of the same order. For large values of n this integral is determined by large t. Substituting $P(t)\propto\exp(-tc_2\bar\lambda^2/2\Delta)$ into (4.13) we find $F_{2n}\propto(2n)!(P_2/2)^n\ln(L/r)\left(2\Delta/c_2\bar\lambda^2\right)^{n-1}$. This behavior can be described in terms of the probability distribution function

$$P(\theta) \propto \ln(L/r) \exp(-\sqrt{\bar{\lambda}^2 c_2/P_2\Delta} \mid \theta \mid) . \qquad (4.14)$$

We see that the exponent here does not depend on $\ln(L/r)$; it enters (4.14) only as a factor. Therefore the function (4.14) can be used to characterize the large-θ tail of the single-point PDF $P(\theta)$; the only difference is that instead of $\ln(L/r)$ the factor $\ln(L/r_{dif}) = \ln \mathrm{Pe}$ should be substituted into (4.14)—see Appendix A. This expression is valid at $\theta^2 \gg \langle\theta^2\rangle = (P_2/\bar{\lambda}) \ln \mathrm{Pe}$. For the most physically interesting case $S \simeq \Omega \simeq \tau^{-1}$ one has $P(\theta) \propto \exp(-c_1\sqrt{\bar{\lambda}/P_2} \mid \theta \mid)$ with some dimensionless coefficient c_1 depending on the statistics of the velocity field. The basic statement on the exponential tail of $P(\theta)$ agrees with that of Shraiman and Siggia [5], who examined the particular case of δ-correlated strain.

If the third-order correlation function of the pumping is nonzero then odd correlation functions of the scalar are also nonzero. However, they are logarithmically suppressed compared to even ones. That could be established by a procedure similar to that which gave (4.8) and (4.9): $\langle\theta_1^n\theta_2^{n+1}\rangle \propto \ln^n(L/r_{12})$. The powers of the logarithm here are less than one would expect from the scaling $\theta \sim \ln^{1/2}$ prompted by the expressions for the even correlation functions.

C. Fluxes of the integrals of motion

The initial equation (2.1) without pumping and diffusion conserves an infinite sequence of the integrals $\int \theta^{2n}(\mathbf{r})\, d\mathbf{r}$. The way of pumping θ^2 radically differs from that of pumping high-order ($n > 1$) integrals [4,11]. For the steady flux of θ^2 in the convective interval of scales $L \gg r_{12} \gg r_{dif}$, one gets directly from (2.1)

$$\langle[(\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1) + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2)]\theta_1\theta_2\rangle$$
$$= \langle\phi_1\theta_2 + \phi_2\theta_1\rangle + \kappa\langle\theta_1\Delta\theta_2 + \theta_2\Delta\theta_1\rangle \ . \quad (4.15)$$

The first term on the right-hand side is constant at $r_{12} \ll L$ and it is equal to P_2 which is the pumping rate of θ^2, while the second term is negligible for $r_{12} \gg r_{dif}$. That means that the flux of θ^2 is constant in the convective interval. For θ^4 one gets similarly

$$\langle[(\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1) + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2)]\theta_1^2\theta_2^2\rangle$$
$$= \langle\phi_1\theta_2^2\theta_1 + \phi_2\theta_1^2\theta_2 + \kappa\theta_1\theta_2^2\Delta\theta_1 + \kappa\theta_2\theta_1^2\Delta\theta_2\rangle \ .$$

Besides the irreducible part that is constant in the convective interval, the correlator on the right-hand side necessarily contains the reducible parts. The main contributions due to one-point means $2\langle\theta^2\rangle\big(\langle\phi\theta\rangle + \langle\theta\Delta\theta\rangle\big)$ are canceled because of the conservation of θ^2 which requires $2\langle\phi\theta\rangle = -2\kappa\langle\theta\Delta\theta\rangle = P_2$. In the different-point part we can neglect $\kappa\langle\theta_1\Delta\theta_2\rangle$ comparing to $\langle\phi_1\theta_2\rangle$ as we did in considering (4.15). We thus get $\langle\phi_1\theta_2\rangle\langle\theta_1\theta_2\rangle = P_2\langle\theta_1\theta_2\rangle$, which changes with r_{12} as the pair correlator. Since our pair correlation function is logarithmic, it grows as r_{12} decreases. This means that for sufficiently small r_{12}/L one can neglect the constant irreducible contribution determined by P_4 in comparison with $P_2^2\bar{\lambda}^{-1}\ln(L/r_{12})$. That is why all the fluxes for $1 < n < \ln \mathrm{Pe}$

$$\langle[(\mathbf{v}_1 \cdot \boldsymbol{\nabla}_1) + (\mathbf{v}_2 \cdot \boldsymbol{\nabla}_2)]\theta_1^n\theta_2^n\rangle \propto P_2^n\bar{\lambda}^{1-n}\ln^{n-1}(L/r_{12})$$

are expressed in terms of P_2 and are nonconstant in the convective interval. It is worth emphasizing that the reason for this is the presence of the external action at any scale, i.e., the absence of the convective intervals for higher integrals. The flux change has nothing to do with nonconservation.

This simple consideration shows how an asymptotic Gaussianicity appears for logarithmic correlation functions and how the set of correlation functions appears to be independent of the influxes of higher integrals of motion. To conclude this section we would like to repeat that if the flow is nonergodic and there are separate space regions with different values of the pumping then on averaging of the correlation functions over space the Gaussianicity is lost (see also [13]) while the logarithmic dependencies of the correlation functions remain the same.

V. ANSATZ FOR T EXPONENT

In Sec. III we have established the asymptotic Gaussianicity of the statistics of the stretching rate and found the Lyapunov exponent at the limiting cases of rapid and slow strain. Although velocity fields that produce such strain can exist, the most interesting and widespread cases certainly correspond to a strain correlation time that is of the order of the turnover time. Starting from this section we shall manage to prove a central limit theorem and to find a way to calculate $\bar{\lambda}$ for an arbitrary τ. For pursuing the first aim we should find such a representation for $\ln|\hat{W}(T,t)\mathbf{r}|$ at large T that can be presented as an integral of a scalar function with a finite correlation time.

It is possible to extract from (2.6) and (2.10) $\hat{W}_s(t)$ only as the antichronological time-ordering exponent (2.7), not as some regular function of $\hat{\sigma}$. To calculate averages over $\hat{\sigma}$ we can use the formalism of a path integral

$$\langle f(\hat{W})\rangle = \int \mathcal{D}\hat{\sigma}\, \exp(-S\{\hat{\sigma}\}) f(\hat{W}) \qquad (5.1)$$

with an appropriate action $S\{\hat{\sigma}\}$ that determines the statistics of $\hat{\sigma}$. Such a formalism enables one to pass from the variables $\hat{\sigma}$ to other variables that give $\hat{W}$ as a regular function. A similar problem—transformation of the time-ordered exponent of some linear combination of spin SU(2) operators—has been solved by Kolokolov [8] for an exact functional representation of the partition function of a quantum Heisenberg ferromagnet. The main idea of the ansatz is as follows: by using the commutation relations of the spin algebra, find out new integration variables in the functional integral, such that $\tilde{T}\exp$ becomes some regular function. Here, we suggest a modification of this ansatz. Besides the possibility of establishing the statistics of λ for an arbitrary τ, our representation will allow us to generate consistent perturbation theory in cases of both a rapid and a slow strain.

A. Central limit theorem for an arbitrary correlation time

First, we expand the $\hat{s}$ matrix (which is the symmetric part of $\hat{\sigma}$) over the spin 2×2 matrices $\hat{s} = a\hat{\sigma}_z + b\hat{\sigma}_x$. Then we introduce a new basis of the spin algebra $\hat{\sigma}_y$, $\hat{\sigma}_\pm$ with

$$\hat{\sigma}_\pm = \hat{\sigma}_z \pm i\hat{\sigma}_x = \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \tag{5.2}$$

that corresponds to the rotation of the quantization axis from the usual position (parallel to the z axis) to the new one—parallel to the y axis. We choose, instead of $a(t), b(t)$ fields, the new ones $\varphi^\pm \equiv (a \pm ib)/2$, representing $\hat{s}$ in a more compact form $\hat{s} = \varphi^-\hat{\sigma}_+ + \varphi^+\hat{\sigma}_-$.

Let us consider the matrix function given in the explicit form

$$\begin{aligned}\hat{A}(T,t) &= \exp\left[-\hat{\sigma}_-\Phi^+(t)\right]\exp\left[-\hat{\sigma}_+\Phi^-(T)\right] \\ &\quad\times \exp\left[\hat{\sigma}_y\Phi^y(T)\right]\exp\left[\hat{\sigma}_-\Phi^+(T)\right],\end{aligned} \tag{5.3}$$

where $\Phi^\pm, \Phi^y$ are the functionals of the new dynamical fields $\psi^\pm, \rho$:

$$\Phi^+(T) = \psi^+(T)\ ,\ \Phi^-(T) = \int_t^T \psi^- e^{2\int_t^{\tilde{t}} \rho dt'} d\tilde{t}\ ,$$
$$\Phi^y(T) = \int_t^T \rho d\tilde{t}. \tag{5.4}$$

We recall that the field c characterizes the antisymmetric part of $\hat{\sigma}$. Using the commutation relations of spin operators (5.2) one can show that the matrix function $\hat{A}$ obeys the differential equation

$$\begin{aligned}\partial_T\hat{A} = \hat{A}\big\{&-\hat{\sigma}_+\psi^- + \hat{\sigma}_-\left[4\psi^-(\psi^+)^2 - 2\rho\psi^+ + \dot{\psi}^+\right] \\ &+\hat{\sigma}_y(-4\psi^-\psi^+ + \rho)\big\},\end{aligned} \tag{5.5}$$

and the first factor in (5.3) ensures the boundary condition $\hat{A}(t,t) = 1$. Comparing (5.5) with (2.10) we find that the substitution $\varphi^- = \psi^-$ and

$$\varphi^+ = -\dot{\psi}^+ - 2ic\psi^+ + 4\psi^-(\psi^+)^2, \tag{5.6}$$
$$\rho = 4\psi^-\psi^+ - ic, \tag{5.7}$$

guarantees the coincidence of $\hat{W}_s$ and $\hat{A}$. It allows us to obtain an explicit functional integral for any averages written in terms of $\hat{W}_s$ by means of changing the variables ($\varphi^\pm \to \psi^\pm$). The transformation (5.6) contains the derivative of the field ψ^+ with respect to time on its right-hand side. Therefore it should be supplied with some initial conditions. Another point is that in the course of calculation it is necessary to average some functions of the operator $\hat{A}(T,t)$ at fixed time moment T over the velocity statistics. For a given T it is convenient to fix the final value of the field ψ^+ (see also [9,14]): $\psi^+(T) = -1/2$. The Jacobian of the map (5.7) is the determinant of a triangle matrix and depends on the choice of the regularization. We choose here the same variant of discretization of the map (5.6) as was used before in the papers [9,14] [$\varphi_n^\pm = \varphi^\pm(t_n); n = 1, ..., M; h = \frac{T-t}{M} \to 0; t_n = t + hn; M \to \infty$],

$$\begin{aligned}\varphi_n^- &= \psi_n^-\ ,\\ \varphi_n^+ &= -\frac{1}{h}(\psi_{n+1}^+ - \psi_n^+) - ic_n(\psi_{n+1}^+ + \psi_n^+) \\ &\quad+\psi_n^-(\psi_n^+ + \psi_{n+1}^+)^2,\end{aligned} \tag{5.8}$$

which gives the following expression for the Jacobian:

$$\mathcal{D}\varphi^\pm = \mathcal{J}[\psi^\pm]\mathcal{D}\psi^\pm,$$
$$\mathcal{J} = \text{const}\times\exp\left(4\int_t^T \psi^+\psi^- dt' - i\int_t^T c dt'\right).$$

The matrix $\hat{A}(T,t)$ being multiplied on the initial vector

$$\mathbf{R}(O) = \begin{pmatrix} r \\ 0 \end{pmatrix}$$

produces the following simple expressions for the squared vector (3.1):

$$\begin{aligned}R^2(T-t) &= [\hat{A}(T,t)\mathbf{R}(O)]^2 \\ &= -2\psi^+(t)\exp\left[8\int_t^T (\psi^+\psi^- - ic/4)dt'\right]r^2.\end{aligned} \tag{5.9}$$

Here we exploited the isotropy condition and chose $\mathbf{R}(O)$ without any loss of generality. The formula (5.9) immediately gives a desired asymptotic (at large T) expression for the stretching rate where only the real exponents contribute:

$$\lambda(T) = \frac{4}{T}\int_0^T \psi^+(t)\psi^-(t)dt\ . \tag{5.10}$$

We thus succeeded in representing the stretching rate as an integral of a scalar function. Expression (5.10) allows us to prove the positivity of $\bar{\lambda} = \lim_{T\to\infty}\langle\psi^+(T)\psi^-(T)\rangle$ and the central limit theorem for the statistics of $\lambda(t)$. At first glance, the substitution (5.6) and (5.7) makes ψ^+ not be complex conjugated to ψ^- so that $\psi^+\psi^-$ may be negative and even complex. It can be shown that in calculating averages one can deform the integration contour in the plane $\{\psi^+, \psi^-\}$ into the contour $(\psi^+)^* = \psi^-$ without encountering singularity [12]. We thus can conclude that $\bar{\lambda} > 0$. This could also be obtained from generalization of the classical results of Furstenberg [10] to the case of finite correlation time. To establish a central limit theorem one should prove that the random process $\psi^+(t)\psi^-(t)$ has a finite correlation time for a finite-correlated $\hat{\sigma}$. This will be shown in Appendix C, Subsection 2. As time t is getting much larger than the correlation time of $\psi^+(t)\psi^-(t)$ then the statistics of $\lambda(t)$ approaches a Gaussian one. Let us recall that t is bounded from above by $\tau_* = \bar{\lambda}^{-1}\ln \mathrm{Pe}$ so that the Gaussian statistics is an asymptotic property of a high Pe regime.

The representation (5.10) for the stretching rate enables us to develop a consistent perturbation theory in limits of both slow and fast strain. To this end, one should look for the value $\bar{\lambda} = \lim_{T\to\infty} 4\psi^+\psi^-$ substituting as ψ^+ the solution of Eq. (5.6) and averaging the result. Extracting from Eqs. (5.6) ψ^-, one obtains

$$\varphi^+ + 2ic\psi^+ = -\dot{\psi}^+ + 4\varphi^-(\psi^+)^2 \,. \tag{5.11}$$

The right-hand side of Eq. (5.11) consists of two terms. The first term is leading in the fast case while the second one is leading in the slow case. Then we can look for the corrections to those leading terms that will produce two different asymptotic series. In this way we obtain a set of recursion relations enabling us to formulate asymptotic expansions for $\bar{\lambda}$ for the cases of the slow and fast fields. Such an iteration procedure can be found in [12]; it confirms the simple approach of Sec. III.

B. Gaussian strain with an arbitrary correlation time

Let us emphasize that up to now in this section we have not specified the statistics of the velocity field. All the above statements have universal character and do not depend on a detailed structure of velocity statistics. Still, to get a precise quantitative description in the case of an arbitrary correlation time one should specify the statistics. Let us choose for further investigation the case of Gaussian statistics as the simplest (yet nontrivial) example. Namely, we will consider the particular case of the following Gaussian statistics of $\hat{\sigma}$:

$$\mathcal{D}\hat{\sigma}(t)\exp(-S_0) = \mathcal{D}a\mathcal{D}b\mathcal{D}c\exp(-S_0),$$
$$S_0 = \frac{1}{2D_s}\int [a^2 + b^2 + \tau^2(\dot{a}^2 + \dot{b}^2)]dt + \frac{1}{2D_a}\int [c^2 + \tau^2\dot{c}^2]dt. \tag{5.12}$$

The expression of σ via the fields a, b, c is given by (2.3). In (5.12) we introduced two different values, D_s and D_a, characterizing, respectively, the amplitudes of the strain (described by the fields a and b) and of the vorticity (described by the field c). The mean values S and Ω of the strain and the vorticity can now be defined as

$$S^2 = D_s/\tau, \quad \Omega^2 = D_a/\tau \,. \tag{5.13}$$

To determine $\bar{\lambda}$ (or another averaged quantity) one should calculate the functional integral with the measure that is obtained by substituting (5.6) into (5.12). Before doing it let us rewrite the measure by introducing auxiliary fields ξ^+ and ξ^-:

$$\mathcal{D}\hat{\sigma}(t)\exp(-S_0\{a,b,c\}) \Rightarrow \mathcal{D}\varphi^{\pm}\mathcal{D}\xi^{\pm}\mathcal{D}c\exp(-S_1\{\varphi,\xi,c\}),$$
$$S_1 = \frac{2}{D_s}\int \left[\varphi^+\varphi^- + \xi^+\xi^- + \varphi^+\xi^- + \tau^2\ddot{\varphi}^-\xi^+\right] dt + \frac{1}{2D_a}\int \left[c^2 + \tau_a^2(\dot{c})^2\right] dt. \tag{5.14}$$

Let us perform the substitution (5.6) and the following linear change of variables:

$$\psi^- = D_s\pi^- \,, \quad \psi^+ = \eta^+, \quad \xi^- = -D_s\pi^- + \frac{D_s}{2}\eta^- \,,$$
$$\xi^+ = -D_s\pi^+ \,, \quad c = \sqrt{D_sD_a}z \,. \tag{5.15}$$

The Lyapunov exponent is thus the average of $4D_s\lim_{T\to\infty}\pi^-(T)\eta^+(T)$ with respect to the new measure $N\mathcal{D}\pi^{\pm}\mathcal{D}\eta^{\pm}\mathcal{D}z\exp(-S_3\{\pi,\eta,z\})$

$$S_3 = \int \left[2(D_s\tau)^2\dot{\pi}^+\dot{\pi}^- - \dot{\eta}^+\eta^- + 2\pi^+\pi^- - 4\pi^-\eta^+ - \pi^+\eta^- + 4\pi^-(\eta^+)^2\eta^-\right] dt'$$
$$-i\sqrt{\frac{D_a}{D_s}}\int(2\eta^+\eta^- - 1)zdt' + \frac{1}{2}\int \left[z^2 + (\tau D_s)^2(\dot{z})^2\right] dt', \tag{5.16}$$

where $t' = tD_s$. The action (5.16) is of the Feynman-Kac type [16]. Our path integral is the matrix element of the quantum-mechanical evolution operator $\exp(-\hat{H}_1T)$ where the Hamiltonian $\hat{H}_1$ corresponds to a system with three degrees of freedom:

$$\hat{H}_1 = -\frac{\Delta_r}{2(\tau D_s)^2} + \frac{r^2}{2} - \frac{r}{2}[4e^{-i\vartheta}\hat{d}^-(1-\hat{d}^+\hat{d}^-) + e^{i\vartheta}\hat{d}^+]$$
$$-\frac{1}{2\tau^2D_s^2}\frac{\partial^2}{\partial z^2} + \frac{z^2}{2} - 2i\sqrt{\frac{D_a}{D_s}}\hat{d}^+\hat{d}^-z \,. \tag{5.17}$$

Here $(r\cos\vartheta, r\sin\vartheta) = (\pi^+ + \pi^-, i\pi^- - i\pi^+)$, so that Δ_r is the Laplacian operator in two-dimensional r space, and $\hat{d}^+, \hat{d}^-$ are some creation and annihilation operators with the usual commutation relation $\{\hat{d}^-, \hat{d}^+\} = 1$, corresponding to η^- and $-\eta^+$ fields in the path integral over the measure (5.16) accordingly. Let us note that the essential difference between "quantization" procedures for π and η fields stems from different structures of the "kinetic" terms in the action (5.16) (of Weyl and Wick types [17,18], respectively). In general terms Wick's quantization procedure requires one to fix an ordering of creation and annihilation operators in the Hamiltonian. In our case, it is possible to show straightforwardly that the regularization of the map (5.8) and corresponding regularization of all the expressions which we used fixes the ordering of operators in the Hamiltonian (5.17) unambiguously. However, there exists a simpler way to check this statement. Indeed, the energy of the ground state

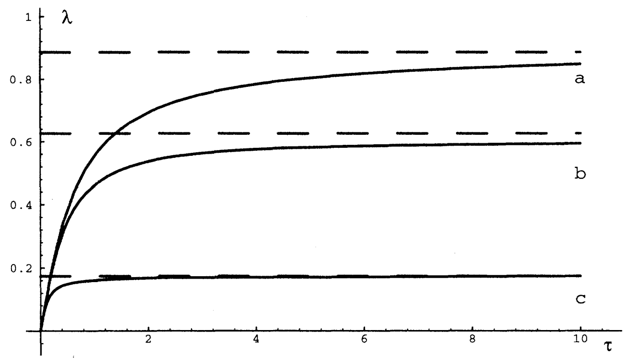

FIG. 1. Lyapunov exponent $\bar{\lambda}$ as a function of τ at $D_s = \tau$. The three plots correspond to different values of D_a/D_s: (a) $D_a = 0$; (b) $D_a = D_s$; (c) $D_a = 25D_s$, the dashed lines marking the asymptotics that have been obtained analytically.

of the Hamiltonian (5.17) must coincide with the corresponding one of the harmonic oscillator. But the only ordering satisfying this requirement is the one chosen in (5.17).

All the local in time averages with respect to the measure (5.16) are the respective averages over the ground state of the Hamiltonian (5.17). So, to calculate the Lyapunov exponent one has to average $4D_s\pi^-\hat{d}^-$ over the ground state of the Hamiltonian (5.17). In [12] this well-defined quantum mechanics is described in detail. The wave function of the ground state is found as an expansion into the series over some polynomials, where the expansion coefficients connect with each other linearly. The system of linear equations for the coefficients can be solved with any precision required. The contribution of high-order polynomials is negligible; the number of terms giving a substantial contribution grows with τ and is finite for a finite τ. We computed the Lyapunov exponent $\bar{\lambda}$ up to large enough correlation times ($\tau \approx 10D_s^{-1}$). Figures 1 and 2 summarize our numerical evidence for the Lyapunov exponent as a function of the correlation time τ and the ratio D_a/D_s. They show a good qualitative agreement with the interpolation formula (1.2). It is worth noting that the Lyapunov exponent $\bar{\lambda}$ is a

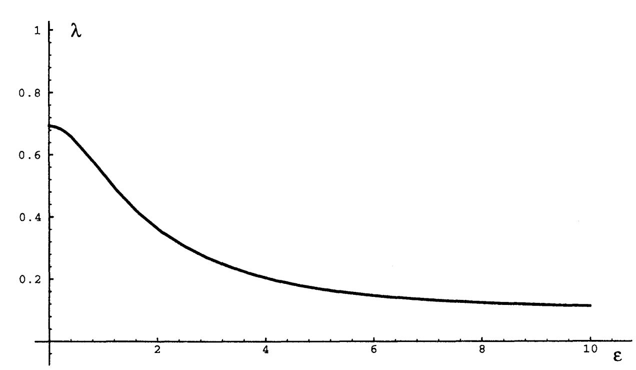

FIG. 2. Lyapunov exponent $\bar{\lambda}$ as a function of $\varepsilon = (D_a/D_s)^{1/2}$ at $D_s = \tau = 2$.

monotonically growing function of the correlation time τ at fixed mean values S and Ω of the strain and vorticity introduced by (5.13) and $\bar{\lambda}$ is a monotonically decreasing function of Ω.

VI. CONCLUSION

We developed a theory for quite arbitrary temporal characteristics of the turbulent velocity field. As far as the spatial requirement of velocity being large scale is concerned, it is not very restrictive as well. If the energy spectrum $E(k)$ of the velocity field decays at $kL > 1$ fast enough to produce the main strain by scales of the order L, then the above theory is valid. This is so, in particular, in a viscous-convective range at large Prandtl number (viscosity to diffusivity ratio) [19] where the energy spectrum $E(k)$ decays exponentially. What if we consider the convective interval for the velocity as well so that the spectrum is powerlike $E(k) \propto (kL)^{-x}$? For distributions with $x > 3$ [21–23] the strain is large scale and our theory is applicable. Besides, for steady turbulence one can prove that $x \geq 3$ (see, e.g., [20]). As far as turbulent vorticity cascade is concerned, it corresponds to $x = 3$. This case does not satisfy the applicability conditions of the above theory and the calculation of the scalar PDF is still ahead of us. The r dependencies of the scalar correlation functions can be found nevertheless. If $E(k)$ were exactly k^{-3}, then all correlators would be proportional to integer powers of the logarithm as above; for instance, $\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\rangle \propto \ln(L/r_{12})$ [19]. We know, however, that the velocity spectrum is logarithmically renormalized $E(k) \propto (kL)^{-3}\ln^{-1/3}(kL)$ [24,4] so that both the effective strain and vorticity depend on the scale and grow with k to provide $\bar{\lambda}(k) \propto \ln^{1/3}(kL)$. In this case, the correlation functions of the passive scalar coincide with those of the vorticity: $\langle\theta^n(\mathbf{r}_1)\theta^n(\mathbf{r}_2)\rangle \propto \ln^{2n/3}(L/r_{12})$ [4].

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APPENDIX A: DIFFUSION AND CORRELATION FUNCTIONS AT SMALL DISTANCES

As long as one considers correlation functions at sufficiently small distances, in particular one-point statistics, taking account of diffusion or other dissipative mechanisms is unavoidable. This causes no substantial difficulties, yet some formulas do get bulky. Here we show how a formalism similar to (2.5) and (2.8) could be applied to the complete equation (2.4) and prove that the one-point statistics of θ is the same as the statistics of the different-point products in the convective interval: first $n < \ln \mathrm{Pe}$ moments are Gaussian, i.e., are determined by the value $\langle\theta^2\rangle$.

Let us look for a solution of (2.4) in the following form:

$$\theta(t,\mathbf{r}) = f(t,\mathbf{R}) \equiv f(t,\hat{W}(t,t_0)\mathbf{r}), \tag{A1}$$

where the evolution of the matrix $\hat{W}(t,t')$ is given by (2.6) and (2.7) and t_0 is a fixed moment of time. Then, the function $f(t,\mathbf{R})$ should satisfy the following equation:

$$\dot{f}(t,\mathbf{R}) = \phi(t,\hat{W}^{-1}\mathbf{R}) + \kappa W_{\alpha\gamma}W_{\beta\gamma}\frac{\partial}{\partial R_\alpha}\frac{\partial}{\partial R_\beta}f(t,\mathbf{R}). \tag{A2}$$

This linear equation may be rewritten in the Fourier representation with respect to $\mathbf{R}$,

$$\dot{f}_{\mathbf{k}}(t) = \tilde{\phi}_{\mathbf{k}}(t) - \kappa(\hat{W}\hat{W}^T)_{\alpha\beta}k_\alpha k_\beta f_{\mathbf{k}}(t), \tag{A3}$$

$$f_{\mathbf{k}}(t) \equiv \int d\mathbf{R}e^{i(\mathbf{k}\cdot\mathbf{R})}f(t,\mathbf{R}),$$
$$\tilde{\phi}_{\mathbf{k}}(t) \equiv \int d\mathbf{R}e^{i(\mathbf{k}\cdot\mathbf{R})}\phi(t,\hat{W}^{-1}\mathbf{R}). \tag{A4}$$

The dynamical equation (A3) is readily solved

$$f_{\mathbf{k}}(t) = \int_0^\infty dt'\tilde{\phi}_{\mathbf{k}}(t-t')e^{-\kappa\mathbf{k}\hat{\Lambda}(t,t-t';t_0)\mathbf{k}}, \tag{A5}$$

$$\hat{\Lambda}(t_1,t_2;t_0) \equiv \int_{t_2}^{t_1}\hat{W}(\tau,t_0)\hat{W}^T(\tau,t_0)d\tau. \tag{A6}$$

Performing an inverse Fourier transform of (A5), one gets

$$f(t,\mathbf{R}) = \int_0^\infty \frac{dt'}{4\pi\kappa\sqrt{det\hat{\Lambda}(t,t-t';t_0)}} \times\int d\mathbf{R}'\phi\Big(t-t',\hat{W}^{-1}(t-t',t_0)\mathbf{R}'\Big) \times\exp\left[-\frac{(\mathbf{R}-\mathbf{R}')\hat{\Lambda}^{-1}(t,t-t';t_0)(\mathbf{R}-\mathbf{R}')}{4\kappa}\right]. \tag{A7}$$

The auxiliary moment of time t_0 may be removed from the solution after a change of variables $d\mathbf{R}' \to d\mathbf{r}'$, $\mathbf{r}' = \hat{W}(t-t',t_0)(\mathbf{R}'-\mathbf{R})$:

$$\theta(t,\mathbf{r}) = \int_0^\infty \frac{dt'}{4\pi\kappa\sqrt{\det\hat{M}(t,t-t')}} \times\int d\mathbf{r}'\phi\Big(t-t',\mathbf{r}'+\hat{W}(t,t-t')\mathbf{r}\Big) \times\exp\left[-\frac{\mathbf{r}'\hat{M}^{-1}(t,t-t')\mathbf{r}'}{4\kappa}\right], \tag{A8}$$

where

$$\hat{M}(t_1,t_2) \equiv \hat{\Lambda}(t_1,t_2;t_2) = \int_{t_2}^{t_1}\hat{W}(\tau,t_2)\hat{W}^T(\tau,t_2)d\tau, \tag{A9}$$

and we used the following features of $\hat{W}$ as a time-ordered exponent of a traceless matrix: $\det[\hat{W}] = 1$, $\hat{W}^T\Lambda^{-1}\hat{W} = [\hat{W}\hat{\Lambda}(\hat{W}^T)^{-1}]^{-1}$. The expression (A8) is the formal solution of (2.4) and is the direct generalization of the diffusionless expression (2.5).

Similarly to what has been done in Sec. II for the diffusionless case, the correlation functions of the passive scalar could now be rewritten in terms of the known correlation functions of the pumping. Performing integrations with respect to inner temporal (realizing the source δ function) and spatial (convolution of two diffusive Green functions) variables we arrive at the following general expression:

$$\langle\theta(\mathbf{r}_1)\theta(\mathbf{r}_2)\cdots\theta(\mathbf{r}_{2n})\rangle = P_2^n\left\langle\sum_{\{l\}}\prod_{i=1}^{2n}\sqrt{t(\mathbf{r}_{il_i};\kappa)}\right\rangle_\sigma, \tag{A10}$$

$$t(\mathbf{r};\kappa) = \int_0^\infty\frac{dt}{8\pi\kappa\sqrt{\det\hat{M}(0,-t)}} \times\int d\mathbf{r}'\xi_2\Big(\mid\mathbf{r}'+\hat{W}(0,-t)\mathbf{r}\mid\Big) \times\exp\left[-\frac{\mathbf{r}'\hat{M}^{-1}(0,-t)\mathbf{r}'}{8\kappa}\right], \tag{A11}$$

where $\{l\} \equiv \{l_1,\ldots,l_{2n}\}$ is a reordering set of numbers from 1 to n. The term with the maximal number of integrations (each giving the large logarithmic parameter) is kept in (A11) while the terms with P_n for $n > 2$ are omitted with logarithmic accuracy as has been done in (4.9) and (4.11). The expression (A11) is thus valid when all the distances between points r_{il} are much less that L yet maybe, however, small. Let us show that if all the distances are large compared to r_{dif} then a further simplification is possible: the term $\mathbf{r}'$ can be neglected in the argument of ξ_2 so that the integration over $d\mathbf{r}'$ gives unity and we come back to the formalism of Secs. II and IV. To see that, we should compare typical fluctuations of two terms under the argument of ξ_2. For the largest eigenvalue of $\hat{M}(0,-t)$ [the smallest eigenvalue of $\hat{M}^{-1}(0,-t)$], the eigenvector $\boldsymbol{\rho}_0$ increases exponentially $\hat{M}(0,-t)\boldsymbol{\rho}_0 \sim (\boldsymbol{\rho}_0/\lambda)\exp(2\lambda t)$, which gives the following estimation for a characteristic fluctuation of r': $r' \sim \sqrt{\kappa/\lambda}e^{\lambda t}$. The second term under the argument of ξ_2 grows with the same exponent $\mid\hat{W}(0,-t)\mathbf{r}\mid\sim re^{\lambda t}$ according to Sec. III. Therefore, the terms under the argument of ξ_2 differ in prefactors only. Comparison of the prefactors shows the following.

(i) If $r \gg \sqrt{\kappa/\lambda}$ one can neglect diffusion coming from (A11) back to $t(r;0)$ which has been considered in Secs. II and IV.

(ii) If $r \ll \sqrt{\kappa/\lambda}$ we can drop the $\mathbf{r}$ dependence entirely to obtain the major contribution. If all the distances between points are less than $\sqrt{\kappa/\lambda}$ the correlation functions are getting independent of the distances and could be considered at one point. The rest of this appendix is devoted to the second case.

Generally, doing calculations in the way presented above we see that $\sqrt{\kappa/\lambda}$ (which we named the diffusion scale r_{dif}) is nothing but the ultraviolet cutoff which should be put into diffusionless expressions like (2.8) when calculating simultaneous correlation functions. Indeed, the one-point version of (A10) and (A11) gives

$$\begin{aligned}\langle\theta^{2n}\rangle &= (2n-1)!!P_2^n\langle t^n(0;\kappa)\rangle\\ &= (2n-1)!!P_2^n\left\langle\left[\int_0^\infty dt\left(\det\left[\hat{1}+\hat{M}(0,-t)\frac{8k}{L^2}\right]\right)^{-1/2}\right]^n\right\rangle_\sigma ,\end{aligned}\tag{A12}$$

where we put $\xi_2(x)$ in the Gaussian form $\exp(-x^2/L^2)$ [recall that with logarithmic accuracy the final result (A13) will be the same for any function $\xi_2(x)$ if $\xi(0)=1$ and it has the characteristic scale L]. The integrand in (A12) is equal to unity until the moment of time $t=\ln[L/r_{dif}]/\lambda$, when both terms under the determinant in (A12) are of the same order; further in time the integrand decreases exponentially. That expression for the effective integration time is exact with the required logarithmic accuracy. That leads us to the final expression [compare with (4.12)]

$$\langle\theta^{2n}\rangle = (2n-1)!!\Big(P_2\ln[L/r_{dif}]\Big)^n\langle\lambda^{-n}\rangle_\sigma .\tag{A13}$$

We thus expressed the moments of the scalar via the moments of the inverse stretching rate. The knowledge of all the moments of θ (odd ones can be neglected) allows us to restore the PDF of θ as an average with respect to the statistics of λ

$$P(\theta) = \left\langle\sqrt{\frac{\lambda}{2\pi P_2\ln(L/r_{dif})}}\exp\left(-\frac{\theta^2\lambda}{2P_2\ln(L/r_{dif})}\right)\right\rangle_\sigma .\tag{A14}$$

What we learned from Secs. III–V is that (whatever be the statistics of the velocity field) the probability distribution of the stretching rate has a Gaussian core and exponential tails—see (4.1)–(4.3). Therefore the same is true for the one-point statistics of θ so that for $n<(\bar\lambda/\Delta)\ln\mathrm{Pe}$

$$\begin{aligned}\langle\theta^{2n}\rangle &= (2n-1)!!\Big(P_2\ln[L/r_{dif}]\Big)^n\bar\lambda^{-n}\\ &= (2n-1)!!\langle\theta^2\rangle^n .\end{aligned}\tag{A15}$$

That means that $P(\theta)$ also has a Gaussian core. The tails of $P(\theta)$ [determining moments with $n\gg(\bar\lambda/\Delta)\ln\mathrm{Pe}$] are exponential as well as those of $P(\lambda)$. For example, taking $P(\lambda)$ in the form (4.2) [with $r_{12}=r_{dif}$], one obtains for $\ln(L/r_{dif})\gg\Delta/\bar\lambda$

$$P(\theta)\sim\exp\left(-\frac{\bar\lambda}{\Delta}\sqrt{\ln^2(L/r_{dif})+\theta^2\Delta/P_2}\right).\tag{A16}$$

We thus conclude that *all the statements concerning the statistics of the products of the passive scalar in the convection interval are true in the diffusion interval as well.*

Besides, if one considers the statistics of the differences $\theta(\mathbf{r}_i)-\theta(\mathbf{r}_j)$ at small enough separations ($r_{ij}\ll r_{dif}$) then the major (logarithmic) contributions are canceled and only power terms remain. For example,

$$\begin{aligned}\langle[\theta(\mathbf{r}_1)-\theta(\mathbf{r}_2)]^2\rangle &= 2P_2\langle t(0;\kappa)-t(r_{12};\kappa)\rangle\\ &= P_2\left\langle\int_0^\infty\frac{dt}{4\pi\kappa\sqrt{\det\hat{M}(0,-t)}}\int d\mathbf{r}'\xi_2(r')\exp\left[-\frac{\mathbf{r}'\hat{M}^{-1}\mathbf{r}'}{8\kappa}\right]\right.\\ &\quad\left.\times\left[1-\exp\left(-\frac{\mathbf{r}\hat{W}^T\hat{M}^{-1}\hat{W}\mathbf{r}+\mathbf{r}\hat{W}^T\hat{M}^{-1}\mathbf{r}'+\mathbf{r}'\hat{M}^{-1}\hat{W}\mathbf{r}}{8\kappa}\right)\right]\right\rangle_\sigma .\end{aligned}\tag{A17}$$

The integral in the right-hand side of (A17) stems from the diffusion region at small enough times $t\sim r_{12}^2/\kappa\ll 1/\lambda$. To calculate it at small scales, we can reduce the situation to the pure diffusion one [$\hat{W}(0,-t)\to 1,\hat{M}(0,-t)\to t$]

$$\langle[\theta(\mathbf{r}_1)-\theta(\mathbf{r}_2)]^2\rangle\,|_{r_{12}\to 0}\to\frac{P_2r_{12}^2}{4\kappa} ,\tag{A18}$$

which, of course, could also be obtained by the direct integration of (4.15).

APPENDIX B: DESCRIPTION OF THE STRETCHING IN POLAR COORDINATES

This appendix is devoted to the alternative formalism based on the representation of (1.1) in polar coordinates $\mathbf{R}=(R\cos\vartheta,R\sin\vartheta)$:

$$\dot R=\alpha R , \qquad \dot\vartheta=\beta+c ,\tag{B1}$$

where $\alpha=-a\cos(2\vartheta)-b\sin(2\vartheta)$ and $\beta=a\sin(2\vartheta)-b\cos(2\vartheta)$. It is remarkable that the equation for ϑ is separated; it can be treated as a constraint enabling us to express the angle ϑ via the fields a, b. After that is done, Eq. (B1) for R becomes a scalar equation with the solution $R(t) = r\exp\left[\int_0^t dt'\,\alpha(t')\right]$. Already this expression enables us to assert that in the limit $t \to \infty$ the statistics of $\ln R$ is Gaussian since the random field $\alpha(t)$ can be shown to have a finite correlation time. Note that the representations (B1) and (5.9) could be related by using exponential substitution for ψ.

We use this representation to study the non-Gaussian tails of the PDF $P(t,\lambda)$ for the quantity $\lambda(t) = \ln[R(t,0)/r]/t$ at small λ. The PDF can be written in the following form:

$$P(t,\lambda) = t\left\langle \delta\left(\lambda t - \int_0^t dt'\,\alpha\right)\right\rangle$$
$$\equiv (t/2\pi)\int_{-\infty}^{\infty} dx\,\exp(-ix\lambda t)Y(t,x)\,, \qquad \text{(B2)}$$

where

$$Y(t,x) = Z^{-1}\int \mathcal{D}\alpha\mathcal{D}\beta\mathcal{D}c\,\exp\left[-\int_0^t dt'(\mathcal{L} - ix\alpha)\right]. \qquad \text{(B3)}$$

Here $\mathcal{L}$ is the Lagrangian density determining the action $S = \int \mathcal{L}\,dt$, and Z is the normalization constant so that $Y(t,0) = 1$. In the limit $t \gg \tau$ one gets $\ln Y \propto t$. First we calculate the contribution of small x to $P(t,\lambda)$. We can formulate the expansion of $\ln Y$ in x. The first two terms of the expansion are

$$\ln Y(t,x) \simeq ixt\bar{\lambda} - tx^2\Delta/2\,, \qquad \text{(B4)}$$

where $\bar{\lambda} = \langle\alpha\rangle$ is precisely the Lyapunov exponent and $\Delta = \int dt'\langle\alpha(t')\alpha(t'')\rangle_c$ is the variance. The angular brackets here designate the average which can be calculated as the integral over α, β, c with the weight $Z\exp(-S)$ and $\langle\cdots\rangle_c$ designates the irreducible correlation function. We expect that (B4) determines the two first terms of the regular expansion of $\ln Y$ in x with the convergence radius of the order of $\bar{\lambda}/\Delta$. The point is that the coefficients of this expansion can be expressed like $\bar{\lambda}$ and Δ via the irreducible functions of α, and the irreducible function of the nth order can be estimated as $S^n\tau^{n-1}$, which gives the convergence radius $(S\tau)^{-1} \sim \bar{\lambda}/\Delta$.

After substitution of (B4) into (B2) we obtain the Gaussian PDF

$$P(t,\lambda) = \sqrt{\frac{t}{2\pi\Delta}}\exp\left(-\frac{(\lambda-\bar{\lambda})^2}{2\Delta}t\right). \qquad \text{(B5)}$$

The higher in x terms of the expansion of $\ln Y$ (beginning from the third-order term) will produce corrections to $\ln P(t,\lambda)$ which are small in the parameter $(\lambda-\bar{\lambda})/\bar{\lambda}$ near the maximum of $P(t,\lambda)$ but are of the order of unity for small λ. Nevertheless the regular character of the expansion of $\ln Y$ in x ensures that there are no singular in λ terms originating from the region of integration $x \lesssim \bar{\lambda}/\Delta$. Thus this region produces for small λ

$$P(t,\lambda) \propto \exp\left(-\frac{f(\lambda)}{2\Delta}t\right), \qquad \text{(B6)}$$

where $f(\lambda)$ is an analytical in λ function whose value at small λ is of the order of $\bar{\lambda}^2$.

The nonanalyticity of $P(t,\lambda)$ at small λ might be expected from the region of integration $x \gg \bar{\lambda}/\Delta$ in (B2). The reason why one may worry about large values of the fields while studying the probability of anomalously slow stretching is related to the possibility of suppressing λ due to large values of the field c describing the vorticity. However, the straightforward analysis presented in [12] for arbitrary velocity statistics leads to the conclusion that the region of large x does not produce a relevant contribution to $P(t,\lambda)$ at small λ and therefore does not change an exponential behavior of the non-Gaussian tails of $P(t,\lambda)$.

APPENDIX C: GAUSSIAN VELOCITY FIELD

The exact expression for the PDF of the stretching time in the case of δ-correlated velocity field is obtained in Appendix C, subsection 1 of this appendix. In subsection 2 we use the substitution (5.6) to prove the statements necessary for the central limit theorem in the particular case of Gaussian strain and to evaluate the correlation time of the stretching rate fluctuations.

1. Statistics of the passive scalar for the δ-correlated velocity field

In this appendix we find the *exact* expression for the PDF of $Q = P_2 t(r_{12})$ in the case of the δ-correlated velocity field. According to (3.1) one can write

$$Q = P_2\int_0^T \xi_2(R(T-t))dt,$$

where $\mathbf{R} = \hat{W}(T,t)\mathbf{r}$ and T is a large value (final answers imply that we take the limit $T \to \infty$). We will look for the Laplacian transform $\mathcal{P}(s)$ of the PDF for Q: $\mathcal{P}(s) = \langle\exp(-sQ)\rangle$.

Considering the δ-correlated initial measure (5.12) we immediately obtain from (2.11) that the rotation does not effect the passive scalar statistics at all. Thus we can put $c = 0$ in the formulas (5.6)–(5.9). The weight of averaging with respect to $\psi^\pm$ gets the following form:

$$\mathcal{D}\psi^\pm\exp\left\{\frac{2}{D}\int_0^T\left[\dot{\psi}^+\psi^- - 4(\psi^+\psi^-)^2 + 2D\psi^+\psi^-\right]dt\right\}. \qquad \text{(C1)}$$

We present here a variant of the bosonization procedure that has been used in the work [9] for 1D localization. First, one makes a gauge transformation

$$\psi^\pm(t) = \chi^\pm\exp\left(\mp 8\int_t^T \chi^+\chi^- dt'\right), \qquad \text{(C2)}$$

leading to the expression $R^2(T-t) = -2r^2\chi^+(t)$. The initial condition for $\chi^+(t)$ is $\chi^+(T) = -1/2$ and the transformation Jacobian has the form: $J[\psi \to \chi] = \exp(-4\int_0^T \chi^+\chi^- dt)$. The new integration measure is

$$\mathcal{D}\chi^\pm \mathcal{D}\rho \exp\left[-\int_0^T \left(-\frac{2}{D}\dot{\chi}^+\chi^- - 8\rho\chi^+\chi^-\right)dt + 2D\rho^2\right], \tag{C3}$$

where on the first step of the bosonization procedure we introduced the new field ρ by means of the Hubbard-Stratonovich technique. The second bosonization step is again a gauge transformation

$$\chi^\pm(t) = \tilde{\chi}^\pm \exp\left(\pm 4D\int_t^T \rho dt'\right), \quad \tilde{\chi}^+(T) = -\frac{1}{2}, \tag{C4}$$

with the Jacobian $J[\chi \to \tilde{\chi}] = \exp(2D\int_0^T \rho dt)$. A regularization of the transformation (C4) as well as (C2) is assumed to provide the elimination of nonlinearities in the suitable discretized expressions (see [14]). As a result we obtain the Gaussian measure

$$\mathcal{D}\tilde{\chi}^\pm \mathcal{D}\rho \exp\left[-\int_0^T \left(-\frac{2}{D}\dot{\tilde{\chi}}^+\tilde{\chi}^- + 2D\rho^2\right)dt\right], \tag{C5}$$

and $R^2(T-t) = -2r^2\tilde{\chi}^+(t)\exp(4D\int_t^T \rho dt')$.

An average $\langle F[\tilde{\chi}^+(t)]\rangle$ of an arbitrary functional $F[\tilde{\chi}^+(t)]$ with respect to the measure (C5) is equal to $F[-1/2]$. This result is easy to get shifting $\tilde{\chi}^+ \to -1/2 + \tilde{\chi}^+$ and noting that the average of an arbitrary degree of $\tilde{\chi}^+$ is equal to zero. Thus after the integration over $\mathcal{D}\tilde{\chi}^\pm$ we arrive at the measure

$$e^{-DT/2}\mathcal{D}\rho \exp\left[2D\int_0^T \left(\rho - \rho^2\right)dt\right], \tag{C6}$$

and $R^2(T-t) = r^2\exp(4D\int_t^T \rho dt')$. In (C6), the normalization factor $\exp(-DT/2)$ provides $\langle 1\rangle = 1$. Substituting $\rho = -\dot{\varsigma}, \varsigma(T) = 0$, we conclude that the calculation of $\langle \exp(-sQ)\rangle$ becomes a quantum-mechanical problem with respect to the ς variable

$$\begin{aligned}\mathcal{P}(s) &= e^{-DT/2}\int_{\varsigma(T)=0} \mathcal{D}\varsigma \exp\left\{-\int_0^T \left[2D\dot{\varsigma}^2 \right.\right.\\ &\quad \left.\left. + P_2 s\xi_2\left(e^{2D\varsigma}\sqrt{r/L}\right) + 2D\varsigma(0)\right]dt\right\}\\ &= e^{-DT/2}\left\langle \delta(\varsigma)|e^{-\hat{H}T}|e^{2D\varsigma}\right\rangle,\end{aligned} \tag{C7}$$

with the Hamiltonian

$$\hat{H} = -\frac{1}{8D}\partial_\varsigma^2 + P_2 s\xi_2\left(r\exp(2D\varsigma)\right). \tag{C8}$$

The last average in (C7) designates a matrix element of $\exp(-\hat{H}T)$ between states described by the corresponding wave functions. Let us take for $\xi_2(x)$ the step function $\vartheta(L-x)$ which will give us the correct answer in the principal order in $\ln(L/r_{12})$. Then $\hat{H} = -\frac{1}{8D}\partial_\varsigma^2 + U(\varsigma)$, where

$$U(\varsigma) = \begin{cases} U_s = P_2 s, & \varsigma < \varsigma_0 = \frac{\ln(L/r)}{2D} \\ 0, & \varsigma > \varsigma_0. \end{cases} \tag{C9}$$

Thus $\mathcal{P}(s)$ is equal to $e^{-DT/2}\Psi(\varsigma = 0, T)$, where Ψ is defined from the following initial value problem: $\partial_T\Psi = -\hat{H}\Psi$, $\Psi(\varsigma, 0) = e^{2D\varsigma}$. At ς goes to infinity the solution growing exponentially is proportional to $e^{2D\varsigma}$ for any T. So, at $\varsigma \to +\infty, \Psi(\varsigma, T) \to e^{DT/2+2D\varsigma}$. In general terms, at $T \to \infty$ $\Psi(\varsigma)$ approaches $e^{DT/2}f(\varsigma)$, where $f(\varsigma)$ satisfies the equation $[\partial_\varsigma^2 - 8DU(\varsigma) - 4D^2]f = 0$ and has the asymptotics $f(\varsigma \to +\infty) = e^{2D\varsigma}$ and $f(\varsigma \to -\infty) < \infty$. For the potential (C9), it gives

$$f(\varsigma) = \begin{cases} e^{2D\varsigma} + Ae^{-2D\varsigma}, & \varsigma > \varsigma_0 \\ Be^{2D\sqrt{1+2U_s/D}\varsigma}, & \varsigma < \varsigma_0. \end{cases}$$

Here, constants A and B have to be defined from matching f and $\partial_\varsigma f$ at the point ς_0. We obtain, finally,

$$\begin{aligned}\mathcal{P}(s) &= f(\varsigma = 0)\\ &= B = \frac{2}{1+\sqrt{1+2P_2 s/D}}\\ &\quad \times \exp\left[\ln\frac{L}{r}\left(1 - \sqrt{1+2P_2 s/D}\right)\right].\end{aligned} \tag{C10}$$

This function has the cut along the real axis from $s^* = -\frac{D}{2P_2}$ to $-\infty$ that gives us

$$\mathcal{P}(Q) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{sQ}\mathcal{P}(s)ds = \frac{D}{2P_2}\mathcal{F}(QD/2P_2),$$

$$\mathcal{F}(y) = \frac{2}{\pi}e^{-y+\ln(L/r)}\int_{-\infty}^{+\infty} \frac{xdx}{1-ix}e^{-yx^2+ix\ln(L/r)}. \tag{C11}$$

At $y \equiv QD/2P_2 \sim \ln(L/r) \gg 1$ the integral in (C11) calculated by means of the saddle-point approximation is

$$\mathcal{F}(y) \approx \frac{2}{\sqrt{\pi y}}\frac{\ln(L/r)}{2y + \ln(L/r)}\exp\left[-\frac{1}{4y}[2y - \ln(L/r)]^2\right], \tag{C12}$$

which shows the same Gaussian bump as the formula (4.4) and non-Gaussian tails discussed in Sec. IV. The pre-exponential factor in (C12) is correct for finite deviations of y from the mean value (within many dispersion intervals) as long as $y \gg 1$. For a nonergodic flow, the averaging over the different spatial regions may change the form of the probability distribution. Let us demonstrate this by averaging the PDF $\mathcal{P}(Q)$ over some smooth pumping PDF $P(P_2)$. Integrating $P(P_2)$ over dP_2 with the function (C11) one gets in the limit of a large logarithm $\overline{\mathcal{P}(Q)} \propto P[Q\bar{\lambda}/\ln(L/r_{12})]$.

2. Finite correlation time of the fluctuations of the stretching rate

In this subsection, we show that the fluctuations of the stretching rate have a finite correlation time which generally does not coincide with τ and depends on $\bar{\lambda}$ as well. Let us suppose that the fields $\varphi^{\pm}(t)$ and $c(t)$ obey the Gaussian statistics with the correlators

$$\langle\varphi^{+}(t)\varphi^{-}(t')\rangle = K_1(t-t') \;,\; \langle c(t)c(t')\rangle = K_2(t-t'), \tag{C13}$$

where the functions $K_{1,2}(t)$ correspond to a finite correlation time τ

$$K_{1,2}(t) \le \text{const} \times e^{-|t|/\tau}. \tag{C14}$$

In this appendix we study statistical properties of the quantity $\Lambda = [\int_t^T(\Lambda_1 + 2ic)dt' + \Lambda_2(t)]/T$, where $\Lambda_1 = 4\psi^{+}\psi^{-}$, $\Lambda_2 = \ln[-2\psi^{+}(t)]$ and Eq. (5.6) defines Λ as a functional of the fields $\varphi^{\pm}$ and c. The case of zero correlation time τ and more general statistics of the local random fields has been examined exhaustively by Furstenberg [10]. The generalization of his results to the case of finite τ requires some elaboration because the direct application of a perturbation theory leads to infrared divergences. Reformulating the perturbation theory in a convergent form we prove the stability of Furstenberg's results with respect to the finite correlation time. That is, we show that with the assumptions (C13) and (C14) the following three statements are valid.

(i) The distribution functional $\mathcal{P}[\Lambda_1(t)]$ is positively defined and differs from zero only for the configurations where $\Lambda_1(t) \ge 0$ [in other words, $\Lambda_1(t)$ is a nonnegative random variable].

(ii) The two-time correlators $D_{i,j}(t_1,t)$ of fluctuations of $\Lambda_{i,j}(t)$ for $i,j = 1,2$

$$\begin{aligned} D_{i,j}(t_1,t) &= \langle\Lambda_i(t_1)\Lambda_j(t)\rangle_c \\ &\equiv \langle\Lambda_i(t_1)\Lambda_j(t)\rangle - \langle\Lambda_i(t_1)\rangle\langle\Lambda_j(t)\rangle, \end{aligned} \tag{C15}$$

go to 0 as $t_1 - t$ goes to ∞.

(iii) The asymptotic relaxation rate of $D(t_1,t) \equiv \langle\Lambda(t_1)\Lambda(t)\rangle_c$ at the conditions (C13) and (C14) is limited from below by $\min\{\bar{\lambda} = \langle\Lambda\rangle, 1/\tau\}$.

To prove the statements (ii) and (iii) we need to elaborate the nonlinearity of Eq. (5.6) defining Λ_1 in terms of the initial fields $\varphi^{\pm}, c$. It cannot be treated perturbatively since it gives spurious time correlations: the nth order of the perturbative expansion would have the effective correlation time $\sim n\tau$ going to ∞ when $n \to \infty$. The relaxation phenomenon is in this sense nonperturbative and has an essentially dynamical nature. [The above statement (iii) means that the relaxation is governed by the quantity $\langle\Lambda\rangle$ which is determined by dynamics.] To avoid a cumbersome description, we present the complete proof for the strictly finite correlation time: $K_{1,2}(t) = 0$ if $|t| > \tau$. The slight modifications required in the more physical case (C14) are given at the end of this appendix.

(i) This statement is equivalent to the nonnegativity of all the $\Lambda_1(t)$ correlators. The averages $\langle\Lambda_1(t_1)\cdots\Lambda_1(t_n)ic(t_1')\cdots ic(t_m')\rangle$ are real due to the time-inversion invariance of the problem. The Gaussian statistics of the fields $\varphi^{\pm}(t), c(t)$ allows for the decoupling:

$$\begin{aligned} \langle\Lambda_1(t_1)\cdots\Lambda_1(t_n)\rangle & \\ = \sum_{j_1,\ldots,j_n} \int & dt_1'\cdots dt_n' K_1(t_1-t_{j_1}')\cdots K_n(t_n-t_{j_n}') \\ & \times\left[\left\langle\frac{\delta\psi^{+}(t_1)}{\delta\varphi^{+}(t_{j_1}')}\cdots\frac{\delta\psi^{+}(t_n)}{\delta\varphi^{+}(t_{j_n}')}\right\rangle + \cdots\right], \end{aligned} \tag{C16}$$

where the set $j_1,\ldots,j_n$ runs over all the permutations of numbers $1,\ldots,n$. The key equality used in this appendix follows from (5.6):

$$\frac{\delta\psi^{+}(t)}{\delta\varphi^{+}(t')} = \exp\left\{-2\int_t^{t'}[\Lambda_1(t'') + ic(t'')]dt''\right\}. \tag{C17}$$

Using (C17) we can rewrite (C16) as a linear combination (with non-negative coefficients) of averages

$$\left\langle\exp\left\{-2\sum_j\int_{t_j^*}^{t_{j+1}^*}[\Lambda_1(t') + ic(t')]dt'\right\}\right\rangle, \tag{C18}$$

with some $\{t_j^*\}$. According to the well-known Kubo cumulant formula every expectation value (C18) can be expressed as an exponential function of a series in irreducible correlators of the initial exponent. Their real values provide the positivity of the right-hand side of (C16).

(ii) First, let us prove that

$$\mathcal{G}(t) = \left\langle\exp\left[-2\int_t^T\Lambda_1(t')dt'\right]\right\rangle \to 0, \tag{C19}$$

when $T - t \to \infty$. We have already shown that $\Lambda_1(t)$ is non-negative. Thus $\mathcal{G}(t)$ is a nongrowing monotonic function of $T-t$. The only admissible asymptotic behavior in this case is $\mathcal{G}(t) \to \mathcal{G}_\infty$ where $\mathcal{G}_\infty$ is some finite constant. Let us suppose that $\mathcal{G}_\infty \ne 0$. For any $t_0 > t' > t$ the following inequality holds:

$$\begin{aligned} \mathcal{G}(t_0) \ge &\left\langle\exp\left[-2\int_{t_0}^T\Lambda_1(t'')dt'' - 2\int_t^{t'}\Lambda_1(t'')dt''\right]\right\rangle \\ \ge &\ \mathcal{G}(t). \end{aligned} \tag{C20}$$

If $T - t_0 \to \infty$ both $\mathcal{G}(t_0)$ and $\mathcal{G}(t)$ approach $\mathcal{G}(t_\infty)$. Thus the intermediate term in (C20) has the same limit independent of t'. Taking the nth derivative of (C20) with respect to t' we obtain

$$\left\langle\Lambda_1^n(t)\exp\left[-2\int_{t_0}^T\Lambda_1(t')dt'\right]\right\rangle \to 0, \tag{C21}$$

for any $n > 0$ and any $t < t_0$. We show now that the asymptotics (C21) contradicts the assumption $\mathcal{G}_\infty \ne 0$. Let $t - t_0 > 2\tau$. Then the field $\varphi^{-}(t)$ does not correlate with $\varphi^{\pm}(t_1), c(t_1)$ for $t_1 \le t_0$ and the expectation value in the left-hand side of (C21) for $n = 1$ is equal to

$$\left\langle \Lambda_1(t) \exp\left[-2\int_{t_0}^{T} \Lambda_1(t')dt'\right]\right\rangle = \int_t^T K_1(t-t')\left\langle \exp\left\{-2\int_t^{t'} [\Lambda_1(t'')+ic(t'')]dt'' -2\int_{t_0}^{T}\Lambda_1(t'')dt''\right\}\right\rangle dt'. \tag{C22}$$

Expanding the function $\exp\{-2\int_t^{t'}[\Lambda_1(t')+ic(t')]dt'\}$ in series in $\Lambda_1(t')$ and taking into account the consequence (C21) of the assumption $\mathcal{G}_\infty \neq 0$ we obtain for $t_0 \to \infty$

$$\left\langle \Lambda_1(t) \exp\left[-2\int_{t_0}^{T} \Lambda_1(t')dt'\right]\right\rangle \to \int_t^T K_1(t-t')\left\langle e^{-2i\int_t^{t'} c(t'')dt''} \times \exp\left[-2\int_{t_0}^{T}\Lambda_1(t')dt'\right]\right\rangle dt' .$$

Because of the inequalities $t_0 - t > 2\tau$, $t'' < t' < t-\tau$ the field $c(t'')$ does not correlate with $\Lambda_1(t_1)$ at $t_1 > t_0$ and the asymptotic relation takes the form

$$\left\langle \Lambda_1(t) \exp\left[-2\int_{t_0}^{T} \Lambda_1(t')dt'\right]\right\rangle \to \mathcal{G}_\infty \int_t^T K_1(t-t') \times \exp\left[-2\int_t^{t'}\int_t^{t'} K_2(t_1-t_2)dt_1dt_2\right]dt' , \tag{C23}$$

which differs from zero even in the limit $t_0 \to \infty$. It contradicts (C21), and the only possibility is that $\mathcal{G}_\infty = 0$. Turning back to the correlator $D(t_1,t)$ we note that (5.6) is equivalent to the integral equation

$$\psi^+(t) = \psi^+(t_0)\exp\left\{-\int_t^{t_0}[\Lambda_1(t')+ic(t')]dt'\right\} + \int_t^{t_0}\varphi^+(t')\exp\left[-\int_t^{t'}(\Lambda_1+ic)dt''\right]dt'. \tag{C24}$$

Thus $\psi^+(t)$ depends on the fields $\varphi^\pm(t'), c(t')$ at $t' > t_0$ via $\psi^+(t_0)$ only. For every two functionals $A[\varphi]$ and $B[\varphi]$ of random fields $\varphi(t)$ obeying Gaussian statistics with the correlator $\langle\varphi(t)\varphi(t')\rangle = K(t-t')$ (indices are assumed) the following equality holds:

$$\langle A[\varphi]B[\varphi]\rangle_c = \sum_{n=1}^{\infty}\frac{1}{n!}\int dt_1\cdots dt_n dt'_1\cdots dt'_n \left\langle\frac{\delta^{(n)}A}{\delta\varphi(t_1)\cdots\delta\varphi(t_n)}\right\rangle \times K(t_1-t'_1)\cdots K(t_n-t'_n)\left\langle\frac{\delta^{(n)}B}{\delta\varphi(t'_1)\cdots\delta\varphi(t'_n)}\right\rangle. \tag{C25}$$

Let us substitute into (C25) $\Lambda_1(t)$ and $\Lambda_1(t')$ instead of $A[\varphi]$ and $B[\varphi]$. If $t'-t_0 > 2\tau$ and $t_0 - t > 2\tau$ the resulting expressions incorporate only the functional derivatives $\delta^{(n)}\Lambda_i(t)/[\delta\varphi(t_1)\cdots\delta\varphi(t_n)]$ with $t_k > t_0, k = 1,...,n$. Then from (C24) we obtain

$$\frac{\delta^{(n)}\Lambda_1(t)}{\delta\varphi(t_1)\cdots\delta\varphi(t_n)} = \frac{\delta^{(n)}\Lambda_1(t)}{[\delta\psi^+(t_0)]^n}\frac{\delta^{(n)}\psi^+(t_0)}{\delta\varphi(t_1)\cdots\delta\varphi(t_n)}. \tag{C26}$$

Here $\varphi(t_j)$ designates $\varphi^\pm(t_j)$ or $c(t_j)$. All the derivatives of $\Lambda_1(t)$ with respect to $\psi^+(t_0)$ can be expressed explicitly in terms of $\Lambda_1(t')$ and the fields $\varphi^\pm$ and c. For example,

$$\frac{\delta\Lambda_1(t)}{\delta\psi^+(t_0)} = 4\varphi^-(t)\exp\left[-2\int_t^{t_0}(\Lambda_1+ic)dt'\right]. \tag{C27}$$

Substituting (C26) together with (C27) into (C25) we see that the irreducible correlators $D_{1,1}(t,t')$ for $t - t' \gg \tau$ can be estimated as

$$D_{1,1}(t,t_1) \le \mathcal{G}(t)F(t_0,t_1), \tag{C28}$$

where $F(t_0,t_1)$ are some finite functions of t_1 and of fixed intermediate time moment t_0. The behavior (C19) provides for the relaxation of $D_{1,1}(t,t_1)$

$$D_{1,1}(t,t_1) \to 0, \text{ at } t_1 - t \to \infty. \tag{C29}$$

The relation (C24) together with the proven applicability of the central limit theorem to the quantity $\int\Lambda_1(t)dt$ gives for the correlator $D_{2,j}(t,t_1)$, $j = 1,2$ the asymptotic inequality of the form (C28) with some other function $F(t_0,t_1)$ in the right-hand side.

(iii) The fact that Λ is self-averaging gives us the asymptotic behavior of $\mathcal{G}(t)$ at $T - t \to \infty$

$$\mathcal{G}(t) \to \text{const} \times \exp[-2(T-t)\bar\lambda] . \tag{C30}$$

From (C28) we conclude that in the case of strictly finite correlation time the relaxation rate of fluctuations of $\Lambda(t)$ is bounded from below by $2\bar\lambda$. Let us turn to the case of exponentially decaying correlators $K_{1,2}(t)$ (C14). Replacing the inequalities like $t - t_0 > 2\tau$ by $t - t_0 \gg \tau$ and neglecting then the exponentially small terms $\sim \exp[-(t-t_0)/\tau]$ we make the proofs of statements i and ii valid in this case as well. In the proof of statement iii the integration over intermediate times in (C25) goes over the whole time interval and there will be the contribution to $D(t,t')$ proportional to $\exp[-(t'-t)/\tau]$. Such contributions exist already in the zeroth order of perturbation theory and arise from the dependence of $\Lambda(t)$ on the fields $\varphi^\pm, c$ in the vicinity of the same time moment t when the dynamics is yet linear. If $(\tau)^{-1} > 2\bar\lambda$ the relaxation rate of $D(t,t')$ is again $2\bar\lambda$, otherwise $D(t,t')$ decays with the exponent $(\tau)^{-1}$.

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