Anomalous diffusion and the correspondence principle

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We study the quantum behavior of the standard map in the so-called accelerator state, which, within the theoretical framework of classical physics, would result in anomalous diffusion. In agreement with the behavior of the systems, which classically exhibit full chaos and are consequently characterized by positive Lyapunov coefficients λ, quantum uncertainty increases very quickly and leads to the breakdown of the overwhelming majority of the classical trajectories in a time $t_\chi = (1/\lambda)\ln(1/\hbar)$. In the case of normal diffusion, the diffusion process is unaffected by this rapid transition from classical to quantum physics. However, in the case of anomalous diffusion, we find the existence of a new breakdown process, corresponding to a statistical departure of quantum from classical dynamics. We argue that this new kind of breakdown, which does not have anything to do with the well known phenomenon of localization, takes place on a time scale $t_B = (1/\lambda_f)\ln(1/\hbar)$ larger than t_χ and with Lyapunov coefficient λ_f determined essentially by the stochastic trajectories moving on the border between the stochastic sea and the accelerator islands. If our arguments are confirmed, they would lead to the possibility of observing the breakdown of the correspondence principle in the statistical sense in times compatible with experimental observation.

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I. INTRODUCTION

One of the most striking aspects of the dynamics of quantum systems that would be classically chaotic is the discovery of possible new channels for the macroscopic manifestation of quantum mechanics. The majority of authors [1–4], with but a few exceptions [5], agree on this issue with different arguments leading, however, to equivalent conclusions. We think that probably the most intuitive way to reach this conclusion is given by the picture recently used by Zurek and Paz [4]. Following them [4], let us consider a quantum system in the physical condition where, according to a traditional wisdom, quantum dynamics is supposed to be virtually coincident with classical mechanics. Let us consider, for instance, a driven one-dimensional system, namely, a particle moving in a time-dependent potential $V(q,t)$. Even though we provide a direct numerical integration of the quantum dynamics in the text, in this section, for pedagogical reasons, we discuss the qualitative nature of the quantum system using a phase-space equation of evolution. We adopt the Wigner formalism [6] leading to the equation of motion for the quasiprobability $\rho_W(q,p;t)$,

$$\frac{\partial}{\partial t}\rho_W(q,p;t) = [L_{\rm class} + L_Q]\rho_W(q,p;t) \ , \tag{1}$$

where $L_{\rm class}$ coincides with the classical Poisson bracket and L_Q is the quantum contribution reading

$$L_Q = \sum_{r=3}^{\infty}\left(\frac{\partial^r}{\partial q^r}V(q,t)\right)\left(\frac{\hbar}{2i}\right)^{r-1}\frac{1}{r!}\frac{\partial^r}{\partial p^r} \ . \tag{2}$$

Let us consider the case where the Planck constant $\hbar$ is extremely small compared to the typical macroscopic action ΔI, namely, the volume of the phase space available to the system within the classical representation. Under this condition, where the classical picture is expected to hold, we adopt an initial distribution given by a smooth Liouville density $\rho_L(q,p;0)$. The classical evolution of the system is given in terms of the Liouville density

$$\rho_L(q,p;t) = \exp(L_{\rm class}t)\rho_L(q,p;0) \ . \tag{3}$$

The quantum evolution of the system is given in terms of the Wigner density

$$\rho_W(q,p;t) = \exp[(L_{\rm class} + L_Q)t]\rho_L(q,p;0) \ . \tag{4}$$

The validity of the classical picture is broken when $\rho_W(q,p;t)$ significantly departs from $\rho_L(q,p;t)$. In the case of ordinary classical motion the time required for this breakdown to take place is inversely proportional to the Planck constant to a power of the order of unity and consequently a time so astronomically large as to result in a complete fulfillment of the expectations of the correspondence principle [1].

In the case where the classical motion is fully chaotic a completely different situation occurs. The Liouville density undergoes a rapid process of fragmentation and the originally smooth distribution quickly develops whorls and tendrils and becomes more and more finely fragmented with increasing time. This process of increasing fragmentation has the effect of enhancing the role of the operator L_Q due to the sharp gradients in the fragmented distribution. This operator would rigorously vanish in the case of linear systems, thereby making unlimited the time of validity of the correspondence principle. Even

in the nonlinear case this correction can be neglected if the Liouville density (and consequently the Wigner density) remained smooth. However, when the Wigner density, originally identical to the Liouville density, becomes extremely fragmented, the differential operator L_Q, applied to the quasidistribution, yields terms comparable in magnitude to those of the classical evolution, thereby resulting in a departure of the Liouville density from the Wigner quasidistribution. Zurek and Paz [4] argue that this departure of the classical from the quantum description takes place at a time t_χ given approximately by

$$t_\chi = \frac{1}{\lambda} \ln\left(\frac{1}{\hbar}\right) , \tag{5}$$

where λ is the largest Lyapunov exponent. The time required for the quantum-classical correspondence breakdown to take place is proportional to the logarithm of the inverse of the Planck constant. This is a relatively short time, which would allow, also for macroscopic systems, quantum effects to show up in experimentally observable times. This would raise the question of the validity of the correspondence principle in classically chaotic quantum systems.

The above argument is widely accepted and suggests that there is a time after which one cannot derive classical dynamics from quantum dynamics. It is therefore of interest to determine why the existence of such a time does not appear to concern most physicists. In part the explanation lies with the multiple definitions of the correspondence principle. The naive definition corresponds to the existence of single classical trajectories, which are thought of as the average motion of narrow quantum wave packets [definition (i)].

The postulate of the theory of measurement implies that the quantum expectation value is experimentally reproduced by making an average over a set of independent measurements or, equivalently, over a Gibbs ensemble. A proper formal definition of the correspondence principle, in line with this perspective, can be given by using the Wigner formalism [6]. According to the Wigner formalism, the mean value of a quantum-mechanical observable $\hat{A}$ can be expressed by

$$\langle\psi|\hat{A}|\psi\rangle_t = \int dqdp A_W(q,p)\rho_W(q,p;t) , \tag{6}$$

where $A_W(q,p)$ is the classical variable corresponding to the quantum observable within the Weyl picture. It is thus clear that the adoption of a Gibbs ensemble is necessary even if only one wave function (and not a statistical mixture) is considered. Within this interpretation scheme the classical behavior of our system (and consequently the correspondence principle) holds true, if in (6) $\rho_W(q,p;t)$ can be safely replaced by the Liouville density $\rho_L(q,p;t)$. This statistical definition of the correspondence principle [definition (ii)] is the one adopted in this paper.

One might be tempted to believe that definitions (i) and (ii) are equivalent. Actually it is not so. We remark that in the fully chaotic case the departure of $\rho_W(q,p;t)$ from the Liouville density $\rho_L(q,p;t)$ takes place first on a scale of the phase space that is of the order of $\hbar$ [1]. Thus, if on the same scale the observable $A(q,p)$ is a "smooth" function of the phase-space coordinates, the quantum and classical averages can be essentially equivalent. As a consequence, definition (ii) can lead to a virtual equivalence between quantum and classical mechanics well beyond the crossover time t_χ. The quantum standard map [7] is an especially illuminating example of this property since the conflict with the correspondence principle defined in the statistical sense above takes place only when localization occurs and this is known to occur at times $1/\hbar^\gamma$, where γ is of the order of unity [7], and consequently at astronomically large times if the system under study is macroscopic.

Similar investigations have been recently carried out by Heller and co-workers [8]. These authors focused their attention on the time over which the semiclassical approximation to the quantum propagator is valid. According to a well known argument by Berry [1], for a classical chaotic system, the semiclassical approximation should break down after a time proportional to the logarithm of the inverse of the Planck constant. Note that this estimate is substantially equivalent to the above heuristic calculation by Zurek. Contrary to this expectation, Heller and co-workers showed that the semiclassical treatment of the quantum propagator leads to a surprising agreement with the exact quantum-mechanical prediction well beyond the time predicted by Berry. However, we want to point out that the successful application of the semiclassical method, which is essentially a genuinely quantum-mechanical theory, taking quantum interference into account, does not necessarily imply that a similar range of validity holds also for the correspondence principle as given by definition (ii).

The search for a statistical manifestation of the breakdown of the correspondence principle occurring at logarithmic times is the current subject of the research work of Berman and co-workers [9]. These authors prove that a real physical system of N paramagnetic atoms in a resonant cavity interacting with a constant magnetic field and with a resonant external magnetic field can be prepared in a chaotic semiclassical condition, leading to a crossover time t_χ logarithmically dependent on the number of atoms involved and consequently to a breakdown process compatible with experimental observation. From their current results, however, it is not yet clear why the breakdown of single trajectories and the rapid birth of strong quantum correlation should have such an effect on the quantum-mechanical mean values as to make them distinctly different from the corresponding classical averages. In the case of the standard map in the fully chaotic state, for instance, it is known [10] that the quantum correlation function of the momentum departs from the corresponding classical correlation function at logarithmic times; on the other hand, on the same time scale no effect is observed on the evolution of the energy, which keeps increasing linearly in time well after the crossover time t_χ.

We believe that the breakdown of the correspondence principle according to definition (ii) can only occur when observing statistical phenomena, whose existence is

strongly determined by the existence of single trajectories. To check the validity of this conjecture, we focus our attention here on anomalous rather than normal diffusion. The dynamical approach to anomalous diffusion in classical physics is the current object of an intensive search by some groups [11–14]. To the best of our knowledge, this is the first paper where the same problem is examined in a quantum-mechanical context.

The plan of the paper is as follows. Section II is devoted to reviewing the key theoretical arguments for deriving anomalous diffusion from classical deterministic dynamics. Section III illustrates the numerical results concerning the corresponding quantum dynamics. Section IV is devoted to a theoretical estimate of the time at which the mean quantum-mechanical value departs from the corresponding classical prediction. In Sec. V we summarize our conclusions: the quantum dynamics of systems that, within the theoretical framework of classical physics would lead to anomalous diffusion, yield the breakdown of the correspondence principle at logarithmic times even if the orthodox definition (ii) is adopted.

II. ANOMALOUS DIFFUSION IN CLASSICAL PHYSICS

We apply our investigation to a prototype of classical chaos [15]. This is the standard map, which reads

$$\begin{aligned} p_{n+1} &= p_n + K \sin\theta_n \ , \\ \theta_{n+1} &= \theta_n + p_{n+1} \ , \qquad \mathrm{mod}(2\pi) \ . \end{aligned} \tag{7}$$

This is an area-preserving map that describes the discrete evolution of a classical rotator kicked at regular intervals of time by a momentum proportional to $K \sin\theta$. The standard map has been widely studied as a prototype for deterministic diffusion [16]. Focusing our attention on the diffusive regime, we study the case of very large n and we adopt a continuous time picture. For the sake of notational simplicity we identify the momentum p_n with the continuous variable x (to be compared with a diffusing spacelike variable) and the quantity $K \sin\theta_n$ with the corresponding "velocity," denoted by ξ. We thus obtain

$$\dot{x} = \xi \ . \tag{8}$$

Under the stationarity assumption the solution to (8) leads to the second moment

$$\langle x^2(t)\rangle = 2\langle \xi^2\rangle_{\mathrm{eq}} \int_0^t dt' \int_0^{t'} dt'' \Phi_\xi(t'') + \langle x^2(0)\rangle \ , \tag{9}$$

where

$$\Phi_\xi(t) \equiv \frac{\langle \xi(0)\xi(t)\rangle_{\mathrm{eq}}}{\langle \xi^2\rangle_{\mathrm{eq}}} \tag{10}$$

is the stationary correlation function of the velocity ξ and the subscript eq on the brackets an average in the equilibrum state. Ordinary diffusion is obtained when the correlation function of the velocity undergoes so rapid a relaxation process as to result in a finite value for the correlation time τ, defined by

$$\tau = \int_0^\infty dt \Phi_\xi(t) \ . \tag{11}$$

There are, however, special parameter values for the underlying dynamical process resulting in a decay of $\Phi_\xi(t)$ with no time scale. For certain values of K deterministic islands appear in the phase space of the standard map [11], called accelerator islands. These are islands created around stable periodic orbits $\theta(t)$ of period Q that satisfy

$$\begin{aligned} p_{n+Q} - p_n &= 2\pi\ell \ , \\ K \sum_{i=1}^{Q} \sin\theta_{n+i} &= 2\pi\ell \ , \end{aligned} \tag{12}$$

where ℓ is a nonzero integer. These islands appear in symmetrical pairs distributed around $\theta = 0$, depending on the sign of ℓ. If the particle is located inside an accelerator island, at any step the particle changes its momentum by a quantity $2\pi\ell/Q$, jumping inside the corresponding island located in the next or the preceding cell of the phase space, according to whether ℓ is positive or negative. If the particle is located on the border between the chaotic sea and the small accelerator island, at any step the particle makes jumps into the next or the preceding cell, in a position still on the border between the chaotic sea and the corresponding island. In other words, for an extended period of time the trajectory remains in the neighborhood of the accelerator islands. Due to the fractal nature of this boundary region, the resulting distribution of sojourn times $\psi(t)$ in the long-time limit has the inverse power-law form [12]

$$\psi(t) = \frac{1}{t^\mu} \ , \tag{13}$$

with [11]

$$2 < \mu < 3 \ . \tag{14}$$

When the waiting time distribution is of the form (13) the trajectory spends most of its time in the neighborhood of the accelerator islands. In this case we can safely neglect the time spent in the fully chaotic region and the trajectory appears to switch at random from the border of one accelerator island to the border of the other, corresponding to the opposite value of ℓ. In this approximation the expression for the correlation function $\Phi_\xi(t)$ holds true [17]

$$\Phi_\xi(t) = \frac{1}{\langle t\rangle \langle \xi^2\rangle_{\mathrm{eq}}} \int_t^\infty dt'(t'-t)\psi(t') \ , \tag{15}$$

where $\langle t\rangle$ is the mean sojourn time in the accelerated condition. Thus, when the distribution of sojourn times (13) is applicable the correlation function (15) inserted into (9) yields, by making time tend to infinity,

$$\langle x^2(t)\rangle = \ \mathrm{const} \times t^{4-\mu} \ . \tag{16}$$

Therefore when the inverse power-law index is in the in-

terval (14) diffusion ranges from normal ($\mu = 3$) to ballistic ($\mu = 2$). The case $2 < \mu < 3$ is referred to as *anomalous diffusion.*

For the property (16) to hold true, with a distinct difference from the normal case, it is necessary for a single trajectory to sojourn in the region at the border between accelerator islands and the chaotic sea for an arbitrarily long time. Any process resulting in a random perturbation of the single trajectory would provoke the particle to escape from the trapping region, thereby producing a truncation of the inverse power law of (13). In this case we should replace (13) with

$$\psi(t) = \frac{1}{t^{\mu}} F(t, \epsilon) \ , \tag{17}$$

where the function $F(t, \epsilon)$ depends on the strength of perturbation ϵ. Typically, for a small perturbation the effect of $F(t, \epsilon)$ is important for times longer than $1/\epsilon$, while in the short-time period the value of $F(t, \epsilon)$ will be close to unity. For instance, in the case of thermal fluctuations it was shown [14] that a plausible form for $F(t, \epsilon)$ is given by

$$F(t, \epsilon) = \exp(-\epsilon t) \ . \tag{18}$$

At times longer than $1/\epsilon$ anomalous diffusion is lost and normal diffusion with an anomalously large diffusion coefficient appears, which tends to diverge with decreasing the intensity of the environmental fluctuations [14]. Thus we see that the effect of a weak fluctuation on anomalous diffusion is much more drastic than it is in the case of normal diffusion, where the effect of a fluctuation is easily proved to yield only a slight change in the diffusion coefficient [18]. If we make the reasonable assumption that quantum fluctuations produce effects similar to those of classical environmental fluctuations, then we have an intuitive explanation of why quantum fluctuation cannot make the standard diffusion process depart from the classical prediction (before the occurrence of the localization process, which, at small $\hbar$, would take place at extremely long times [10]). We also have an intuitive explanation of why, on the contrary, in the case of anomalous diffusion, quantum fluctuations might provoke a significant departure of the diffusion process from the classical prediction. Although we do not expect quantum fluctuations to be equivalent to classical fluctuations, due to the fact that they imply long-range space and time correlations, we do believe that anomalous diffusion is as sensitive to quantum as it is to classical fluctuations. For this reason we think that the crossover time beyond which the classical trajectories are lost might correspond to statistical effects making quantum predictions depart significantly from the classical one.

In summary, while standard diffusion is not affected by the smearing of the trajectory, anomalous diffusion seems to be strongly dependent on the existence of single trajectories and the smearing of them is proven to result in a strong departure from the anomalous behavior that the system would exhibit in the absence of fluctuations, either quantum or thermal.

III. NUMERICAL RESULTS

The theoretical expectation of the previous sections, on the sensitivity of anomalous diffusion to quantum-mechanical fluctuations, is confirmed in this section through numerical calculations. The analysis of quantum dynamics implies the evaluation of the time evolution of the wave function. This is given by applying the Floquet operator F corresponding to the map (7) to the wave function

$$|\psi^{n+1}\rangle = F|\psi^{n}\rangle \ , F = e^{-\frac{i}{2\hbar}p^2} e^{-\frac{i}{\hbar}k\cos q} \ . \tag{19}$$

It is convenient to express the operator F in the basis of the momentum eigenstates $|m\rangle$, defined by $p|m\rangle = \hbar m|m\rangle$, thereby resulting in the mapping for the expansion coefficients $c_m^n \equiv \langle m|\psi^n\rangle$,

$$c_m^{n+1} = \sum_{m'} (-i)^{m'} J_{m'}(K/\hbar) c_{m+m'}^{n} \ , \tag{20}$$

where J_m denotes a Bessel function of order m. This means that at each temporal step n we have to define a new vector. This vector in principle should be infinite dimensional, with m' ranging from $-\infty$ to $+\infty$. For practical purposes we are obliged to truncate these vectors. This truncation must be made with caution because the dimension of the vectors must be sufficiently large as to not conflict with the nature of anomalous diffusion, which implies a rapid energy increase and consequently a larger basis set of eigenstates of the momentum than in the standard diffusion case. The requirement of very large dimensions is determined by the fact that the correspondence principle refers to the case of very small $\hbar$, as well as by the nature of the diffusion under study. In fact, to consider small $\hbar$ we are forced to adopt extremely large dimensions for two different reasons. The first is that the breakdown of anomalous diffusion takes place at longer and longer times upon decreasing the value of $\hbar$. This means that the mean value of the square of the classical momentum becomes exceptionally large. The second compelling reason is given by the analytical form of (20). The truncation of the expansion basis set can be safely made when the index m in (20) is two or three times larger than the argument itself of the Bessel function. This argument becomes larger upon decrease of the Planck constant $\hbar$, thereby increasing the dimension necessary for a fair numerical treatment of the problem under discussion.

If N is the size of the vectors used, the iterating map (20) implies a computational time proportional to N^2. This computational time can be drastically reduced by using a numerical procedure based on the fast Fourier transform algorithm [19]. When we apply the operator $e^{-\frac{i}{\hbar}k\cos q}$, using the Fourier transform, we switch from the momentum representation to the coordinate representation, thereby making the operator diagonal. Then before applying the second operator $e^{-\frac{i}{2\hbar}p^2}$, by an inverse Fourier transform we return to the momentum representation, thereby also keeping this operator diagonal. It is straightforward to show that the adoption of this calculation strategy reduces the computational time from a

quantity proportional to N^2 to a value proportional to $N \ln N$. When we use extremely large values of N (for the smallest value of $\hbar$ we choose $N = 2^{21}$), this implies a significant reduction of computer time.

We devote special attention to the adoption of initial conditions that do not make the quantum distribution overlap with the accelerator islands. These important initial conditions are determined by setting the system on an eigenstate of the momentum with an eigenvalue distinctly different from the values of p at which, within the classical approximation, the accelerator islands appear, i.e., we choose $\langle \psi^0 | p | \psi^0 \rangle = 0.5$. In the classical treatment of the same problem, we adopt an equivalent condition, namely, a distribution of phase space points with $p = 0.5$, randomly distributed over θ. This means that these points are located within the chaotic sea: the birth of anomalous diffusion depends on the fact that, sooner or later, due to the diffusion process, these phase space points stick to the border between the chaotic sea and an accelerator island.

For each numerical treatment of the quantum case we also carry out the much easier numerical treatment of the classical case, namely, the numerical evaluation of the map (7) and then we compare the two results. Figures 1 and 2 provide a qualitative but quite transparent illustration of the fact that anomalous diffusion implies a special sensitivity to quantum fluctuations. These two figures refer to two slightly different values of K: $K = 7.1$ in the case of Fig. 1 and $K = 6.9115$ in the case of Fig. 2. Although these two values of K are close to one another, the latter corresponds to a value that is known to result in anomalous diffusion [11], whereas the former results in a standard diffusive process. We see that in the latter case the classical prediction, denoted by the full line, leads to a dependence of the second moment of p on time that is distinctly larger than the linear dependence of standard diffusion [11,12]. Upon increasing the value of $\hbar$ the quantum calculation leads to a rapid departure of the quantum from the classical prediction, which turns out to be much more rapid for anomalous than for standard diffusion (note that the time scale of Fig. 2 is much shorter than that of Fig. 1). We see from Fig. 1 that the classical prediction, denoted again by a full line, is a straight line, implying as it must a linear dependence of $\langle p_n^2 \rangle$ on time. To point out the strong difference between the two cases, let us consider the smallest value of $\hbar$ used, $\hbar = 0.005$. We see that, while in the anomalous case the quantum result significantly departs from the classical prediction after relatively short times (Fig. 2), in the standard case we could not find any appreciable difference between the two in the time range considered (Fig. 1).

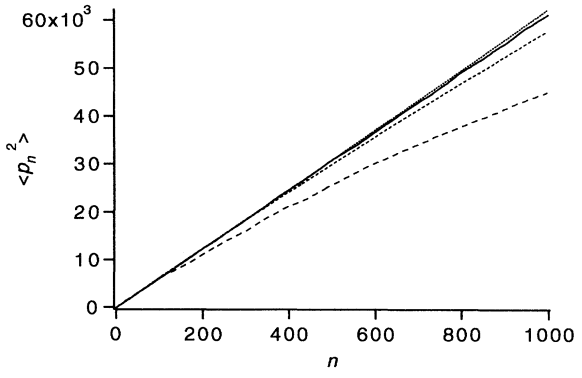

FIG. 1. Time evolution of $\langle p_n^2 \rangle$ (twice the mean energy) in the standard diffusion case. The solid line denotes the classical calculation while the dashed lines denote the quantum results. The values of the parameters are $K = 7.1$, $\hbar = 0.1$ (– – –), 0.01 (- - -), and 0.005 (· · ·).

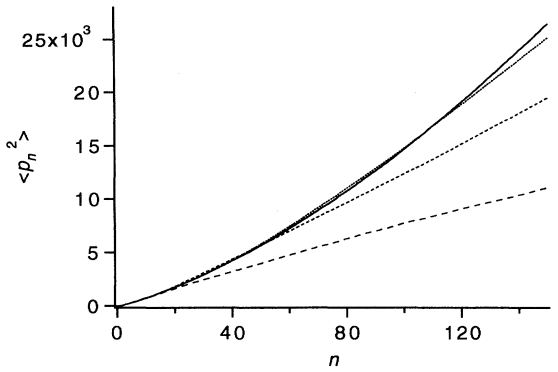

FIG. 2. Time evolution of $\langle p_n^2 \rangle$ (twice the mean energy) in the anomalous diffusion case. The solid line denotes the classical calculation while the dashed lines denote the quantum results. The values of the parameters are $K = 6.9115\hbar = 0.1$ (– – –), 0.01 (- - -), and 0.005 (· · ·).

To set these results on a more quantitative basis, we evaluated the mean value of energy increase per kick $R_n \equiv \langle E_{n+1} \rangle - \langle E_n \rangle$. It is evident that standard diffusion in the classical case implies that R_n remains constant, whereas anomalous diffusion implies that R_n steadily increases with n. This is confirmed by the results of Fig. 3, which show that after a very sharp increase in R_n at very short times it fluctuates around a fixed mean value. It is interesting to notice that the quantum result departs almost immediately from the classical prediction. However, since both the quantum and the classical pre-

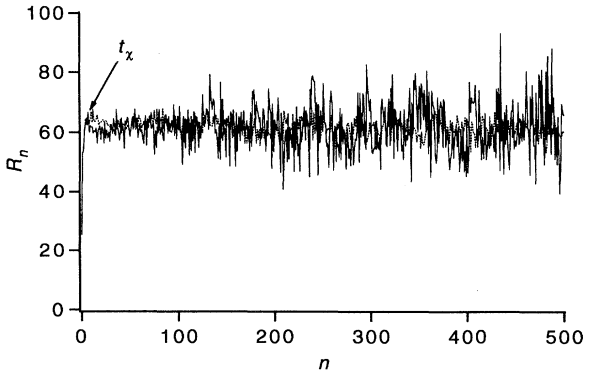

FIG. 3. Mean energy increase per kick R_n in the standard diffusion case. The solid line is the classical result and the dashed line is the quantum result. The value of the parameters are $K = 7.1$ and $\hbar = 0.005$.

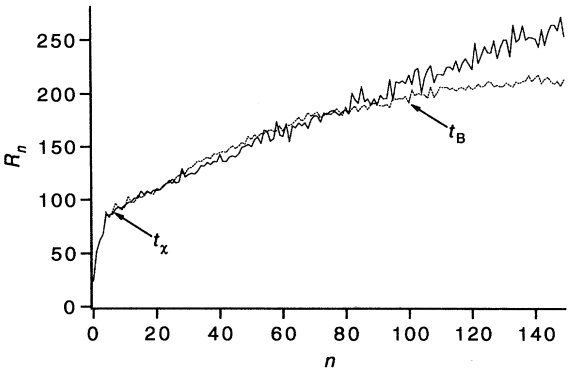

FIG. 4. Mean energy increase per kick R_n in the anomalous diffusion case. The solid line is the classical result and the dashed line is the quantum result. The value of the parameters are $K = 6.9115$ and $\hbar = 0.005$.

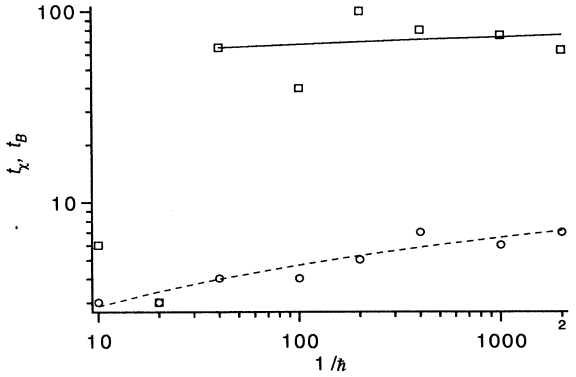

FIG. 5. Dependence of the two breakdown times on $1/\hbar$. The small circles denote the logarithmic time t_χ while the dashed line is the theoretical prediction of Eq. (5). The small squares denote the second breakdown time t_B and the solid line is a logarithmic fit $t_B = \frac{1}{\lambda_f} \ln \frac{1}{\hbar} + c$ with the additive constant $c = 56$ and $\lambda_f = 0.4$. A power-law fitting of the same data $t_B = \kappa(\hbar)^{-\eta}$ would result in $\kappa = 58.7$ and $\eta = 0.03$ and in the scale of this figure in a curve almost indistinguishable from the logarithmic one. The values of the nonlinear parameter is $K = 6.9115$, corresponding to the anomalous diffusion case.

diction of R_n fluctuate around the same constant value, they will result in the same time evolution for the mean energy $\langle E_n \rangle$.

We believe that this behavior of R_n is the result of the insensitivity of the statistical predictions to whether the distribution is classical or quantum, mentioned in Sec. I. This is why, if we adopt the definition (ii) of the correspondence principle, we can safely conclude that the correspondence principle is confirmed by the field of quantum chaos. The reasons for the insensitivity of standard diffusion to whether the distribution is classical or quantum are in turn those stressed in Sec. II: standard diffusion is not affected by weak fluctuation producing a broadening of its trajectories.

We see from Fig. 4 that similar but distinct properties are exhibited in the case of anomalous diffusion. Even in this case the departure of the classical from the quantum evolution of R_n is extremely rapid. However, in this case the quantum and the classical curve fluctuate around two steadily increasing curves, very close to one another. Then, the quantum prediction departs from the classical one. In conclusion, we see that there are two significant time scales here. The first coincides with the time t_χ given by the heuristic formula (5) and it has to do with the departure of the Liouville density from the Wigner quasiprobability. According to the arguments of Sec. I as well as several others proposed by various authors [1–4], this breakdown time depends logarithmically on the inverse of the Planck constant. In addition to this we also find a second breakdown time, which has a much greater influence on R_n and is related to the statistical departure of the quantum from the classical prediction in the sense of definition (ii). We call this second breakdown time t_B. In the case of standard diffusion shown in Fig. 3, this second breakdown time is not exhibited in the time region depicted in the figure. It is expected to take place on a much longer time scale and must be related to the well known phenomenon of quantum localization [10,20]. Quantum localization implies that R_n will regress to a vanishing value, thereby implying that even in the case of standard diffusion there is an eventual statistical departure of quantum from classical dynamics. There is wide agreement in the literature on the fact that the localization process takes place on a time scale proportional to $1/\hbar^\gamma$ with γ of the order of the unity. Therefore, in the case of anomalous diffusion we might also expect t_B to have the same power dependence on the Planck constant. Our numerical results prove that it is not so and that t_B has a quite different dependence on $1/\hbar$.

We are now in a position to set a more quantitative basis for our prediction on the special sensitivity of anomalous diffusion to quantum fluctuation. This is illustrated by Fig. 5, where we plot both breakdown times as a function of the inverse of $\hbar$. The small circles illustrate the dependence of t_χ on $1/\hbar$ and the comparison with the predicted logarithmic dependence (dashed line) confirms the accuracy of the heuristic prediction. Note that the dashed line is derived from (5) using as the Lyapunov coefficient the value $\lambda = \ln(K/2)$ [21] equal to $\lambda = \ln(6.9115/2) \approx 1.24$ in the case considered here. Note also that Eq. (5) in the original form of Zurek and Paz [4] is supplemented by an additive constant that is here adopted as a fitting parameter.

We think that the numerical results depicted in Fig. 5 clearly show that the time t_B depends logarithmically on $1/\hbar$. Let us discuss this key point in some detail. The first two data points, corresponding to the largest values of $\hbar$ considered in our calculations, seem to correspond to a condition where quantum uncertainty is so large as to make the system lose any dependence on the structure of the classical phase space. We found a similar property in a recent work on the influence of chaos on quantum tunneling [22]. Since we are interested in the behavior of t_B for small $\hbar$, in the following analysis we shall disregard these first two points.

We fit the numerical result on t_B with both an inverse power-law expression $t_B = \kappa(\hbar)^{-\eta}$ and a logarithmic dependence on $1/\hbar$. We find that the numerical results are accounted for by adopting an extremely small value for

the fitting parameter η, namely, $\eta \sim 3 \times 10^{-2}$. It must be stressed that there are technical difficulties in making $\hbar$ very small. During the course of our calculations, as a result of technical improvements we have been able to make calculations with smaller and smaller values of $\hbar$. We noticed that decreasing the value of $\hbar$ turned out to result in a general tendency towards smaller values of η without any convergent behavior. We have therefore concluded that a further decrease of $\hbar$ would result in a further decrease of the fitting parameter η. Furthermore, we must say that we are not aware of any theoretical approach resulting in so weak a value for the power η. For these reasons we are confident that the logarithmic dependence of t_B on $1/\hbar$ has much firmer theoretical ground than the power-law dependence, as we are now going to discuss.

IV. THEORETICAL ESTIMATE OF t_B

Anomalous diffusion within a quantum-mechanical picture, such as standard diffusion, is characterized by two critical times. However, while the former is the same for both processes, standard and anomalous, the latter has a physical origin in the anomalous case quite distinct from that of the normal case. The former time is t_χ (see [5]) and refers to quantum-mechanical fluctuations enhanced by the conventional mechanism of full chaos. This critical time still shows up in the case of anomalous diffusion, for the following reasons. In spite of the fact that anomalous diffusion, as shown in Sec. II, originates from the motion within the fractal region at the border between the accelerator islands and the chaotic sea, the motion within the chaotic sea cannot be ignored since it drives the motion from the one to the other island. The classical particle quickly diffuses in the chaotic sea and eventually sticks to the fractal border, where it spends an extremely long period of time. We think that the early rapid diffusion process is reflected by the breakdown time t_χ, which is essentially determined by the Lyapunov coefficients of the chaotic sea. This time is where the quantum energy absorbed per kick R_n departs from the classical R_n without resulting yet in a statistical departure of quantum from classical physics. So far the quantum behavior of anomalous diffusion is equivalent to that of normal diffusion.

In both cases the time t_B refers to the statistical departure of quantum from classical physics. In the case of normal diffusion t_B coincides with the well known localization time and we refer the reader to the literature on the subject [10]. In this paper we discover that in the case of anomalous diffusion the time t_B refers to a quite distinct process, which does not have anything to do with localization. After the rapid early diffusion, the motion is essentially dominated by the classical trajectories entering the fractal region. This is characterized by Lyapunov coefficients that are significantly smaller than those responsible for the former breakdown; the particle is in fact approaching a stable periodic trajectory. Consequently, even if in the fully chaotic region the concept of a classical trajectory is already lost, in the fractal region we can still imagine the quantum phase-space distribution as generated by classical trajectories. We think that when the classical-like description of the motion within the fractal region breaks down, the "quantum particle" will be forced to leave the accelerator island region so that the resulting quantum dynamics will start deviating in the sense of definition (ii) from the corresponding classical motion. As will be shown in the following heuristic calculation, such a mechanism leads to a breaking time t_B that is proportional to the logarithm of $1/\hbar$.

It seems that there is no doubt about the fact that quantum mechanics sets an upper time limit to the inverse power-law distribution of waiting times of a particle at a site. There are already calculations of the value of t_B made by Lai *et al.* [23], who estimate it to be inversely proportional to $\hbar$ to some power of the order of the unity. Here we review their arguments, while including a dynamical property of quantum chaos that leads to an estimate of t_B consistent with the results in Fig. 5. The model adopted by Lai *et al.* [23] is a Markov chain of infinite coupled states. This model has been successfully used to describe the evolution of a generic two-dimensional chaotic system in the neighborhood of a border Kol'mogorov-Arnol'd-Moser (KAM) torus, namely, the outermost invariant curve that separates the regular motion of the inner island from the chaotic dynamics of the stochastic sea [24,25]. As we already pointed out, the global behavior of the motion is dominated by those trajectories that, trapped in the neighborhood of the accelerator island, travel in the momentum direction. In general, the physical quantities of interest, such as the sojourn time or the injection and escape time, are very complicated functions of the initial conditions of the trajectory and to a large extent can be considered as random variables. In other words, to describe the dynamics of the quantum system it is more convenient to adopt a probabilistic picture than to attempt to solve the problem directly from the map equations (7).

In the probabilistic description we divide the phase space around the island into an infinite chain of coupled Markov states. The zeroth-order state $|0\rangle$ represents the particle still embedded in the open chaotic sea while the inner states $|1\rangle$, $|2\rangle$, $|3\rangle, \ldots, |\infty\rangle$ correspond to trajectories trapped at the border of the island. The division of the trapped state into an infinite number of discrete substates roughly accounts for the dynamical properties of the phase space around a border KAM torus, namely, each state represents the portion of phase space encircled by two successive cantori. These are the remnants of the destroyed invariant tori, which have the property of being minimum flux surfaces. In other words, contrary to invariant surfaces, the cantori can be crossed by the trajectories, but with a passage rate that is a local minimum. The physical picture of the evolution of the Markov model is therefore as follows: from the chaotic sea the particle crosses the outermost cantorus of the accelerator island and gets into the first inner state. Here the motion is chaotic and the particle has a given probability to cross back to the chaotic sea or to cross the successive cantorus to go deeper into the trapped state structure. Actually this scheme is highly simplified. First, the phase-space structure suggests that a

tree model would be more appropriate than the chain model to describe the structure of all the higher-order resonances [25]. Second, the particle flux is a continuous function of the phase-space position, thereby the division of the phase space into discrete states is rather a matter of mathematical convenience than a real dynamical property. Nevertheless, it has been shown that this simple model can account for all the most important qualitative properties of these systems [24]. We have now to provide the details of the probabilistic Markov model. It has been conjectured that the fractal structure of the phase space around a border island should give rise to a scaling law for the transition probability. The numerical investigations have confirmed that this ansatz is a good approximation. Following the work of previous investigators, we consider the simplest scaling relation for the transition probabilities of the chain model, i.e.,

$$\frac{P_{i,i+1}}{P_{i-1,i}} = \delta \ , \frac{P_{i,i+1}}{P_{i+1,i}} = \beta, \quad \delta < 1 \ , \tag{21}$$

where δ is the scaling exponent and β is the ratio between the transition probability to go from the state i to the next state in the chain ($P_{i,i+1}$) and the backward transition ($P_{i+1,i}$). The equation for the occupation probabilities $P_i(t)$ reads

$$\begin{aligned} \dot{P}_0(t) &= a\delta P_1(t) - cP_0(t) \ , \\ \dot{P}_i(t) &= c\delta^{i-1}P_{i-1}(t) - \delta^{i-1}P_i(t) + a\delta^{i+1}P_{i+1}(t) \ , \end{aligned} \tag{22}$$

with $a = \delta/[1+\beta\delta]$ and $c = \beta/[1+\beta\delta]$. Finally, still exploiting the fractal nature of the phase-space, we assume that the phase-space dimension of the state of the chain scales with some given exponent A ($A < 1$): namely, for n jumps towards a region of steadily decreasing dimension the resulting size decrease is given by A^n. However, it must be noticed that, since a classical trajectory can penetrate regions of arbitrarily small dimension, this parameter will play a role only in the quantum considerations. In the classical case this model has been used to obtain the long-time behavior of the waiting time distribution denoted as $P(1 \to 0; t)$, which coincides with $\psi(t)$ of Eq. (13). This is obtained from Eq. (22) by calculating the probability that a particle, initially located in the state $|1\rangle$, first reaches the open state $|0\rangle$ at time t. Using the scaling relation (21), a relatively easy calculation gives the following nonlinear equation for the Laplace transform of the waiting time distribution denoted as $P_1(z)$

$$P_1(z)\left[z + 1 - \delta c P_1\left(z/\delta\right)\right] = 1 \ , \tag{23}$$

which in the long-time limit leads to an inverse power-law decay [24]

$$P(1 \to 0; t) \approx \frac{1}{t^{\mu}}, \quad \mu = 1 + [\ln(\delta)]^{-1} \ln\left[\frac{1-\sqrt{1-4c}}{1+\sqrt{1-4c}}\right] \ . \tag{24}$$

The inverse power-law behavior of distribution (24) is a direct consequence of the fractal nature of the phase space: namely, in the long-time limit the main contribution to (24) comes from those trajectories that penetrate deeply the fractal structure thereby dwelling in the inner state for an arbitrarily long time. However, to make predictions on the quantum case there must be a state of a given order, let us say of order N, beyond which the quantum-mechanical particle cannot proceed. In other words, a heuristic quantum Markov model should contain a finite number of states N, where N is related in some way to the quantum parameter $\hbar$. This is a consequence of a general feature of quantum chaotic systems, namely, the property of classical cantori of preventing quantum motion from exploring regions of the phase space that, in the long-time limit, are allowed to its classical analog [26].

Initially the two models have the same dynamical behavior (for instance, the waiting time distribution is equal), but after a time t_B the finite state model will start deviating from the prediction of the infinite Markov model. To evaluate the breakdown time t_B we divide all the trajectories that contribute to distribution (24) into two sets. In the first we consider the trajectories that reach state $|0\rangle$ without going in the state $|N+1\rangle$. The second set is the set complementary to the former one, namely, it contains those trajectories that before reaching $|0\rangle$ were at least one time in the state $|N+1\rangle$. We shall denote $P(1 \to 0, N+1; t)$ and $P(1 \to N+1, N+1 \to 0; t)$, respectively, the probability of occurrence of the former and the latter set. It is straightforward to derive

$$\begin{aligned} P(1 \to 0; t) = P(1 \to 0, N+1; t) \\ + P(1 \to N+1, N+1 \to 0; t) \ . \end{aligned} \tag{25}$$

Following Lai *et al.* [23], we define t_B as the time at which the contribution from the forbidden set $P(1 \to N+1, N+1 \to 0; t)$ is comparable to the entire probability $P(1 \to 0; t)$

$$\left.\frac{P(1 \to N+1, N+1 \to 0; t)}{P(1 \to 0; t)}\right|_{t=t_B} = \chi \ , \tag{26}$$

where χ is an arbitrary number smaller than unity. Let us now calculate the long-time behavior of $P(1 \to N+1, N+1 \to 0; t)$. First we note that this can be written as

$$\begin{aligned} &P(1 \to N+1, N+1 \to 0; t) \\ &= \int_0^t d\tau P(1 \to N+1, 0; \tau) P(N+1 \to 0; t-\tau) \ , \end{aligned} \tag{27}$$

where $P(1 \to N+1, 0; t)$ is the probability that a trajectory initially in the state $|1\rangle$ first reaches state $|N+1\rangle$ in time t, without having been in state $|0\rangle$, and $P(N+1 \to 0; t)$ is the probability that a trajectory initially in the state $|N+1\rangle$ first reaches state $|0\rangle$ in time t. The usefulness of (27) stems from the fact that in the long-time limit the distribution $P(N+1 \to 0; t)$ satisfies a scaling relation similar to that of $P(1 \to 0; t)$. In fact, it is easy to show that the Laplace transform of $P(N+1 \to 0; t)$ is written as

$$P_{N+1}(z) = \int_0^\infty d\tau e^{-z\tau} P(N+1 \to 0;\tau) \ , \tag{28}$$

$$P_{N+1}(z) = P_1(z) P_1\left(z\delta^{-1}\right) P_1\left(z\delta^{-2}\right) \times P_1\left(z\delta^{-3}\right) \cdots P_1\left(z\delta^{-N}\right) \ .$$

To study the long-time limit of $P(1 \to N+1, N+1 \to 0;t)$ we evaluate $P_{N+1}(z)$ in the limit of z that goes to zero. Using the asymptotic expansion for $P_1(z)$ and Eqs. (27) and (28) we obtain

$$\lim_{t\to\infty} P(1 \to 0;t) \approx \frac{k_1}{\Gamma(1-\mu)} \frac{1}{t^\mu} + \cdots \ ,$$

$$\lim_{t\to\infty} P(1 \to N+1, N+1 \to 0;t) \approx \frac{k_1}{\Gamma(1-\mu)} \frac{1}{t^\mu} + \frac{k_2}{\Gamma(2-2\mu)} \frac{1}{t^{2\mu-1}} + \cdots \ , \tag{29}$$

where

$$k_0 = \alpha^N \left[\frac{\alpha-1}{\alpha^{(N+1)}-1}\right], \quad \alpha = \delta^{\mu-1} \tag{30}$$

$$k_2 = k_1^2 \left\{ \frac{1}{2} \left[\frac{1-\alpha^{-(N+1)}}{1-\alpha^{-1}}\right]^2 - \left[\frac{1-\alpha^{-2(N+1)}}{1-\alpha^{-2}}\right] \right\} .$$

Condition (26), together with (29) and (30), is a very complicated function of the time and the quantum parameter N. However, if we consider the semiclassical limit, i.e., $\hbar \ll 1$ and consequently $N \gg 1$, it is possible to derive the result

$$\lim_{t\to\infty} \frac{P(1 \to N+1, N+1 \to 0;t)}{P(1 \to 0;t)} \approx 1 - k(N)\alpha^{-N} \frac{1}{t^\mu} + \cdots \ ,$$

$$\lim_{N\to\infty} k(N) \approx k(\infty) = k_1 \frac{(3\alpha+1)}{2(\alpha+1)(1-\alpha)} \frac{\Gamma(1+2\mu)}{\Gamma(1+\mu)} \frac{\sin 2\mu\pi}{\sin \mu\pi} \ . \tag{31}$$

We have now to relate the quantum parameter N, namely, the smallest state attainable by the quantum particle, to $\hbar$. It must be pointed out that the quantum particle becomes more and more embedded in the fractal region moving within a chaotic region and that the motion within the chaotic region in turn has the effect of making the size of the wave packet mimicking the classical particle increase exponentially in time [3]. Thus we find that an assumption close to reality is

$$A^N = \hbar \exp(\lambda_f t) \ , \tag{32}$$

where λ_f is the effective Lyapunov coefficient of the chaotic area within the fractal region. Equation (32) yields the breakdown time

$$t_B \approx \frac{1}{\lambda_f} \ln\left(\frac{1}{\hbar}\right) \ . \tag{33}$$

On the other hand, this logarithmic breakdown has an immediate statistical effect on the anomalous diffusion, since anomalous diffusion depends on the inverse power-law distribution of waiting times, and the breakdown of this distribution has the immediate effect of destroying the anomalous character of diffusion.

It must be pointed out that the demonstration that led us to (33) is not a rigorous proof but is essentially based on plausible heuristic arguments. We have also to recall that Lai *et al.* [23] made a different choice with

$$A^N = \hbar \ , \tag{34}$$

which would lead to the inverse power-law expression

$$t_B \approx k(\infty)^{1/\mu} h^{-\frac{\ln\delta}{\ln A}} \ . \tag{35}$$

The exponent in this equation corresponds to the fitting parameter η defined in Sec. III, and to get an estimate of it we follow Ref. [25] and consider the border torus of the accelerator island as a noble torus, i.e., a torus with a golden mean winding number. In this case it is possible to derive δ and A to obtain a value for η of the order of unity. Note that this estimate is at least one order of magnitude greater than the value provided by the numerical results of Sec. III. We think therefore that the choice (32), although made within the context of the same heuristic treatment as that of Lai *et al.* [23], reflects more satisfactorily the physical properties of the system under study.

V. CONCLUDING REMARKS

The current literature on quantum chaos affords two possible reasons for the departure of quantum from classical physics [10,27]. The first reason is related to the finite width of the Floquet states in the momentum representation. This is ruled out by the fact that it results in a power-law dependence on $1/\hbar$ with a power that is markedly larger than that stemming from our computer results, if we insist on interpreting them as compatible with a power-law prediction. The second reason is based on the scaling properties of the phase space [23,27] and is ruled out essentially for the same reason, namely, by the fact that it would provide a power index much larger than that compatible with our computational result.

Only one interpretation is not ruled out by our numerical results and this is precisely the statement that the discrepancy between quantum and classical mechanics is a manifestation of the breakdown of the classical trajectories in logarithmic time, as given by (33). Unfortunately, we are not yet in a position to explain all the aspects of our numerical results. From Fig. 5 we see that the breakdown time, as a function of the increasing value $1/\hbar$, fluctuates about a logarithmic slope. A possible reason for these fluctuations is that since the effective Lyapunov coefficient depends on the space regions explored by the trajectories, the resulting breaking time (33) can vary in a nonmonotonic way. This might be a plausible explanation because, upon increase of $1/\hbar$, the trajectories enter regions of smaller and smaller scale and the broadening of the classical trajectories, or, equivalently, the departure of the quantum distribution from the classical Liouville density, would be modulated by

the irregular dependence of the Lyapunov coefficients on the depth of the fractal region explored.

As pointed out by the analysis in Sec. II, the process of anomalous diffusion strongly rests on the existence of single trajectories entering fractal regions of the phase space of smaller and smaller size. This picture has to be properly modified if either we add thermal fluctuations or we quantize the system, that is, if we add a "quantum fluctuation" to the map (7). In the case of normal diffusion, this does not have any statistical consequence in observable times due to the insensitivity of this process to weak fluctuations of a thermal or quantum nature. On the other hand, we found that the anomalous case results in a breakdown of the correspondence principle [see definition (ii) in Sec. I] in a logarithmic time and that this behavior can be explained by the fact that, in the quantum case, the single trajectories are prevented from penetrating deeply into the fractal region, thereby invalidating the origin of anomalous diffusion.

As mentioned in the Introduction, our conclusions do not conflict with those of Ref. [8]. We must remark, first of all, that the method of the semiclassical Green function can lead to a probability distribution coincident with the quantum one and strongly departing from the classical prediction. This can be easily proved by using the prototypical case of the two-slit experiment [28] and by noticing that the semiclassical Green function incorporates the essential interference effects of quantum mechanics. Thus, in principle, there might exist physical situations where the semiclassical method leads to exact quantum-mechanical results, and to a strong disagreement with classical mechanics, and consequently with the correspondence principle even if definition (ii) is adopted. On the basis of the results of this paper we expect that the departure between the predictions of the semiclassical treatment and those of ordinary classical mechanics takes place at the time t_B, with no conflict with the agreement between the semiclassical method and the exact quantum-mechanical treatment. Our results do not exclude that there might be no upper time limit of validity to the agreement between the results of the semiclassical Green function and the exact quantum-mechanical treatment. In conclusion, we believe that the adoption of the semiclassical method, rather than conflicting with our conclusions, might help us to supplement the theoretical interpretations of our numerical results with arguments based on the interference effects among the two distinct zones of the phase space, and thus provide an alternative to those arguments of Sec. IV.

Consequently, we are convinced that the correspondence principle is violated by the quantum behavior of systems that would classically undergo anomalous diffusion. Is this an assessment on the state of deep inconsistencies for quantum mechanics as a unifying theory from which classical mechanics, in the proper limit, naturally stems? According to Ref. [4], we should also take the interaction with the environment into account. This would make the process of thermal fluctuations affect the dynamics of the single trajectories at times earlier than those corresponding to the quantum breakdown of the single trajectories, thereby ensuring the validity of the correspondence principle, if the orthodox interpretation is adopted. In the case of anomalous diffusion, the study of the transition from anomalous to ordinary diffusion is the subject of the current investigation of our group [14]. It is in principle possible to evaluate the time for the crossover from anomalous to standard diffusion, as a function of the noise intensity.

In conclusion, a satisfactory answer to whether or not the supposed generality of quantum mechanics is invalidated by the processes of anomalous diffusion, requires the settlement of the following problems.

(a) The heuristic prediction (33) should be put on a firmer basis. Probably, still more important than this, one should account for the fluctuations of the numerical results around this prediction.

(b) Using classical physics and a plausible model of the interaction between the system of interest and its environment, one should determine the time of crossover from anomalous to normal diffusion determined by the environmental fluctuations. This crossover time is made longer and longer by decreasing the temperature of the environment. The availability of temperature so low as to make this time much longer than t_B should be discussed in the light of the current experimental techniques.

Therefore we are inclined to believe that the quantum behavior of systems that classically would undergo anomalous diffusion leads to the breakdown of the correspondence principle, even if the orthodox definition (ii) is adopted. However, this is not yet a claim on a possible deep inconsistency between quantum and classical mechanics. This requires further research work concerning environmental influences. Nevertheless, this paper, which is, to the best of our knowledge, the first theoretical quantum-mechanical treatment of a process of anomalous diffusion, has certainly the merit of pointing out the importance of anomalous diffusion for the discussion of this fundamental problem. Furthermore, this paper establishes that anomalous diffusion results in a departure of quantum from classical statistical behavior over times much shorter than those corresponding to quantum localization, if any occurs.

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