Energy-scaling approach to phase-ordering growth laws

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We present a simple, unified approach to determining the growth law for the characteristic length scale $L(t)$ in the phase-ordering kinetics of a system quenched from a disordered phase to within an ordered phase. This approach, based on a scaling assumption for pair correlations, determines $L(t)$ self-consistently for purely dissipative dynamics by computing the time dependence of the energy in two ways. We derive growth laws for conserved and nonconserved O(n) models, including two-dimensional XY models and systems with textures. We demonstrate that the growth laws for other systems, such as liquid crystals and Potts models, are determined by the type of topological defect in the order parameter field that dominates the energy. We also obtain generalized Porod laws for systems with topological textures.

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I. INTRODUCTION

The quench of a system from a disordered phase to an ordered phase is a nonequilibrium process in which energy is dissipated and topological defects, if present, are eliminated. Typically, the system develops a scaling structure with a single length scale that evolves in time as the various broken-symmetry phases compete to select the ordered phase [1–3]. In the thermodynamic limit, this length scale will grow without bound, and the scaling structure will hold at late times. When scaling holds, any theoretical or experimental analysis of a system is simplified. It is then natural to explore the growth law of the single length scale $L(t)$. That is the aim of this paper, aspects of which have appeared elsewhere [4,5].

Previous investigations of growth laws have been carried out on a case-by-case basis, and many of the predictions have been controversial. The approach presented here is simple yet powerful, dealing with all phase-ordering systems within a common framework. This approach is based on the role played by topological defects in fixing the large-momentum behavior of two-point correlation functions. The only restriction is that the dynamics be purely dissipative, the major assumption that the scaling hypothesis is valid. Even this assumption may be relaxed: in systems with more than one characteristic scale, our approach sets one relationship between the length scales.

Systems with scalar order parameters, such as binary alloys and Ising models, have been well studied [1–3,6–20]. In such systems domains of both phases grow, and the intervening domain walls (the characteristic topological defects of scalar systems) decrease in total area and hence dissipate energy. Recently there has been a growing interest in systems with vector and more complex order parameters [3]. A series of experiments [21–23] and simulations [24–40] have explored such systems. A characteristic aspect of many of these systems is the existence of stable topological defects with singular cores. These defects include domain walls, vortex lines, and points [41]. If a system has a scaling structure, then the single length scale $L(t)$ will also characterize the curvature of any domain walls and vortex lines, and the separation of any point defects. One can use the nature of the defects to determine the short-distance correlations in the system. This connection between the topological defects of the system and its correlations can be exploited. The scaling assumption and the knowledge of the defect structure are together sufficient to determine the growth law.

We calculate the growth law of the characteristic length scale, *if scaling exists*, for quenched systems with either scalar or vector fields, with either short- or long-range attractive interactions, and with or without conservation laws. We do this by considering the time dependence of the energy density as the system relaxes towards a ground state. First we calculate the energy density of the system. Then we equate its time derivative to the rate of energy-density dissipation independently calculated from the local evolution of the order parameter. From this we self-consistently determine the growth law, $L(t)$. Our approach is independent of the details of the system and of the initial conditions, and hence reflects the observed universality of growth laws among physical systems and simulations.

The symmetries of the system are reflected in the topological defects seen in the system. The defect structure determines the asymptotic behavior of correlations, so that we do not need to use any approximation schemes in our approach. For short-range interactions we find three regimes (shown in Fig. 1): defect dominated scalar and XY [O(2)] systems both with and without relevant conservation laws, and spin-wave dominated [42] systems

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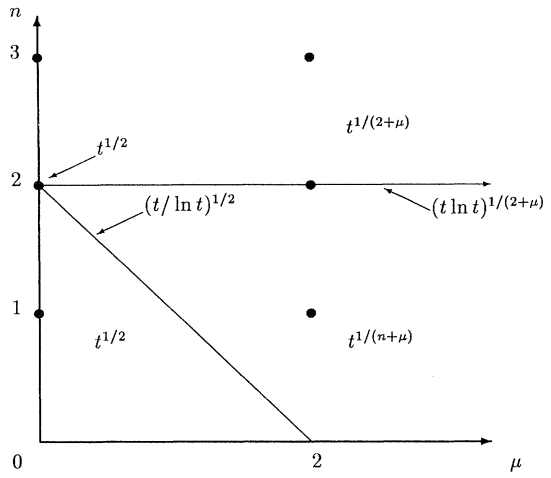

FIG. 1. The growth law $L(t)$ of the length scale for phase-ordering systems which scale. The growth laws are shown for various numbers of components n, where $n \leq d$, and conservation laws μ. The black circles correspond to possible physical systems. For $n = d = 1$ and $n = d = 2$ the growth laws in Table I apply. Scaling systems with no topological defects ($n > d + 1$), or with textures ($n = d + 1$), will have $L \sim t^{1/(2+\mu)}$.

in which conservation laws are always relevant. In the defect dominated regime topological defects provide short-scale structure that determines the energy density and the growth laws. In the spin-wave dominated regime, topological defects, if present, do not dominate the energetics or determine the growth law. For all regimes, we find power-law growth laws which are independent of the spatial dimension of the system, apart from special cases of one-dimensional (1D) scalar systems and two-dimensional (2D) XY models. For the marginal cases separating regimes we find logarithmic factors in the growth laws.

It is informative to contrast our approach with renormalization-group (RG) work by Bray [43,44] (see also [3,45]). In common is the universality with respect to the details of the phase-ordering system, and the assumption of dynamical scaling. There are three significant differences. The first is that the RG approach only determines the dynamical exponent z, i.e., the power law $1/z$ of the growth, but does not determine any logarithmic factors. The second one is that the RG approach implicitly assumes the existence of a simple fixed point in which *all* correlation functions scale with a single length, while our approach only requires the scaling of the two-time two-point correlation function. The third difference is that the RG approach is limited to systems with a relevant conservation law, while our approach also applies with nonconserved dynamics. In contrast to the RG methods, our "energy-scaling" approach places the growth laws of both conserved and nonconserved scaling systems on equivalent theoretical footing, and explicitly predicts any logarithmic factors. In principle, the results of this work could be reached by the RG approach using an explicit RG structure for nonequilibrium phase-ordering problems, but, to our knowledge, that approach has not yet been developed.

In Sec. II we introduce the model. In Sec. III we obtain growth laws for a variety of O(n) systems. We introduce the scaling assumptions and then calculate the energy density and the energy-density dissipation. We also discuss systems with long-range attractive interactions. These results have mainly been presented before [4], though in a brief form (see also [5]). In Sec. IV we discuss systems without topological defects, then consider the collapse of isolated defect structures such as spherical domains, 2D XY point defect pairs, and nonsingular textures. We use these results to obtain growth laws for 2D XY models, and also asymptotic correlations and growth laws for systems with topological textures. These have mostly *not* been presented before and provide a detailed perspective on scalar, XY, and textured phase-ordering systems. In Sec. V we review relevant work on phase-ordering growth laws. We then discuss our results in Sec. VI and summarize them in Sec. VII.

II. THE MODEL

A generic energy functional for a phase-ordering system with an n-component order parameter $\vec{\phi}(\mathbf{x})$ and short-range interactions is

$$H[\vec{\phi}] = \int d^d x \left[(\nabla\vec{\phi})^2 + V(\vec{\phi}) \right] , \tag{1}$$

where d is the spatial dimension and $V(\vec{\phi})$ is an isotropic, sombrero-shaped potential [46] such as

$$V(\vec{\phi}) = V_0(\vec{\phi}^2 - 1)^2. \tag{2}$$

After a temperature quench into the ordered phase, the equation of motion for the ordering kinetics of the Fourier components $\vec{\phi}_{\mathbf{k}}$ [47] is

$$\partial_t \vec{\phi}_{\mathbf{k}} = -k^\mu \, (\delta H/\delta\vec{\phi}_{-\mathbf{k}}), \tag{3}$$

where we only consider systems with purely dissipative dynamics. (Note that bold face denotes spatial vectors, while overarrows denote vectors in the order-parameter space.) We work at temperature $T = 0$, with no thermal noise, since T is an "irrelevant variable" within the ordered phase [43,48]. The conventional nonconserved model-A and conserved model-B dynamics are $\mu = 0$ and $\mu = 2$, respectively. However, any $\mu > 0$ enforces a global conservation law for $\vec{\phi}$ [6].

Shortly after the quench, the magnitude of the order parameter nearly saturates and evolution takes place by the motion of topological defects and, for vector systems, by the relaxation of the director field of the order parameter. The role of the potential $V(\vec{\phi})$ in these later stages of evolution is different between scalar and vector systems, though in neither case does it dominate the gradient term in (1). Near defect cores, where the or-

der parameter vanishes in order to reduce the gradient energy, both the gradient and potential terms contribute to the energy density in proportion to the defect-core density. For vector systems, away from defect cores, the field is close to saturation and varies over scales of order $L(t)$, the characteristic length. Consider a small region, away from defect cores, with a uniform gradient in the unit magnitude director field $\hat{\vec{\phi}}$ and a uniform magnitude $|\vec{\phi}| = 1 - \delta$. The local energy density $\mathcal{H} \equiv (\nabla\vec{\phi})^2 + V(\vec{\phi})$ will be of the same form for any potential $V(\vec{\phi})$ with a quadratic minimum:

$$\mathcal{H}[\vec{\phi}] \simeq (1-\delta)^2(\nabla\hat{\vec{\phi}})^2 + 4V_0\delta^2. \tag{4}$$

The local energy density is then minimized if

$$\delta \simeq (\nabla\hat{\vec{\phi}})^2/(4V_0). \tag{5}$$

Since $\nabla\hat{\vec{\phi}} \sim 1/L(t)$, the contribution to ϵ from the potential term, of order $\delta^2 \sim L^{-4}$, is clearly subdominant for large $L(t)$. Including nonuniform gradients in the director field, and corresponding gradients of δ, only contributes more subdominant terms. For vector systems, the energy due to the magnitude variations of the order parameter balance between the potential and gradient terms, but both are dominated by the gradient term of the director field. For scalar systems, the director field is uniform away from defects so that Eq. (4) involves $(\nabla\delta)^2$. Minimizing the local energy shows that δ decays exponentially away from the interface, with a decay length $\xi \sim V_0^{-1/2}$ [49]. As a result, the energy density is dominated by the domain wall energy, which is balanced between the potential and gradient terms.

In both scalar and vector cases, the gradient term is proportional to the local energy density. Thus the scaling behavior of the average energy density of the system, $\epsilon \equiv \langle\mathcal{H}\rangle$, is captured by

$$\epsilon \sim \langle(\nabla\vec{\phi})^2\rangle \sim \int_{\mathbf{k}} k^2 \left\langle \vec{\phi}_{\mathbf{k}} \cdot \vec{\phi}_{-\mathbf{k}} \right\rangle, \tag{6}$$

where $\int_{\mathbf{k}}$ is the momentum integral $\int d^dk/(2\pi)^d$, and the angular brackets represent an average over initial conditions, or, equivalently, a spatial average in the thermodynamic limit. We independently calculate the rate of energy density dissipation, $\dot{\epsilon} \equiv \partial_t\epsilon$, by integrating the contribution from each Fourier mode:

$$\begin{aligned}\dot{\epsilon} &= \int_{\mathbf{k}} \left\langle (\delta H/\delta\vec{\phi}_{\mathbf{k}}) \cdot \partial_t\vec{\phi}_{\mathbf{k}} \right\rangle, \\ &= -\int_{\mathbf{k}} k^{-\mu} \left\langle \partial_t\vec{\phi}_{\mathbf{k}} \cdot \partial_t\vec{\phi}_{-\mathbf{k}} \right\rangle,\end{aligned} \tag{7}$$

where we use the equation of motion (3) to obtain the second line. We see the special role of the two-point equal-time correlation function,

$$S(\mathbf{k},t) = \left\langle \vec{\phi}_{\mathbf{k}} \cdot \vec{\phi}_{-\mathbf{k}} \right\rangle, \tag{8}$$

which determines the energy density, and of the two-point time-derivative correlation function,

$$T(\mathbf{k},t) \equiv \left\langle \partial_t\vec{\phi}_{\mathbf{k}} \cdot \partial_t\vec{\phi}_{-\mathbf{k}} \right\rangle, \tag{9}$$

which determines the rate of energy-density dissipation. These correlation functions will be the basis of our discussion of growth laws for systems that satisfy dynamical scaling in the following section and subsequently of isolated defects and growth laws in scalar, XY, and textured systems. We will equate the time derivative of ϵ from (6) to $\dot{\epsilon}$ from (7), and self-consistently determine the growth laws.

Before proceeding, we digress to discuss what can be learned from dimensional analysis of the dynamics in Eq. (3). From the linear part, associated with the gradient term in H, of the equation of motion (3) we have the dimensional relation $[L] = [t^{1/(2+\mu)}]$. What other length scales are there in the problem? The nonlinear part of (3), associated with the potential term in H, gives $[L] = [\xi]$ where ξ, given by $V_0^{-1/2}$ for the potential (2), is the core size of any topological defects present, e.g., the width of a domain wall. Finally, there is the length scale ξ_0 associated with any short-range correlations in the ensemble of initial conditions. Thus dimensional analysis gives $[L] = [t^{1/(2+\mu)}] = [\xi] = [\xi_0]$, implying

$$L(t) = t^{1/(2+\mu)} F(\xi^{2+\mu}/t,\ \xi_0^{2+\mu}/t). \tag{10}$$

The growth law can only be changed from the naive (without ξ or ξ_0) dimensional result $L \sim t^{1/(2+\mu)}$ by nontrivial behavior of F at late times, when its arguments approach zero. However, ξ_0 cannot enter into the growth law unless scaling is violated (see, e.g., [50,70]). If ξ_0 did enter into the asymptotic growth law, then changing the initial condition from the state at $t = 0$ to one at $t_1 > 0$ (with a larger ξ_0) will lead to a multiplicative change in the amplitude of the growth law, rather than merely a shift of origin of the time coordinate. This is unphysical, so any ξ_0 dependence implies a lack of scaling. Conversely, systems which break scaling need the initial conditions to set, through the initial correlation length, the relative amplitudes of the multiple growing length scales. (The earlier argument against ξ_0 dependence does not apply to systems which break scaling, since changing ξ_0 in such systems is no longer simply equivalent to an offset in the time coordinate, but also changes the relative amplitudes of the growing length scales.) It is possible, but unlikely, that multiple length scales will have relative amplitudes set only through the core scale ξ. So we expect scaling violations if and only if we observe ξ_0 dependence in the amplitude of the growth law. For scaling systems, therefore, we may set the second argument of the function F in (10) to zero. Then, in cases where no topological defects are present, or the defects do not dominate the dynamics, $L(t)$ will not depend on ξ either, and (10) gives $L(t) \sim t^{1/(2+\mu)}$. This is the generic result, covering the region $n > 2$ in Fig. 1. For $n \leq 2$, the defects *do* dominate the dynamics (in that they dominate the energy density, as we shall see in Sec. III B), and $L(t)$ can depend on the core scale ξ. The remainder of the paper is mainly devoted to understanding these cases.

III. GROWTH LAWS

A. Scaling assumption

Many phase-ordering systems are empirically found to scale at late times from a quench, after initial transients have decayed. Accordingly, the correlation function of the order parameter, $C(\mathbf{r},t) = \langle \vec{\phi}(\mathbf{x},t) \cdot \vec{\phi}(\mathbf{x}+\mathbf{r},t) \rangle$, exhibits the scaling form $C(\mathbf{r},t) = f(r/L(t))$, with a single characteristic length scale $L(t)$ and a time-independent scaling function $f(x)$. This is the dynamic scaling hypothesis [11]. Fourier transforming the scaling form of the correlation function, we obtain the scaling form of the structure factor,

$$S(\mathbf{k},t) = L(t)^d \, g(kL(t)), \tag{11}$$

where d is the spatial dimension and $g(y) = \int d^d x e^{i\mathbf{x}\cdot\mathbf{y}} f(x)$. This scaling hypothesis can be generalized to two-time correlations by dimensional analysis,

$$\begin{aligned} S(\mathbf{k},t,t') &\equiv \left\langle \vec{\phi}_{\mathbf{k}}(t) \cdot \vec{\phi}_{-\mathbf{k}}(t') \right\rangle , \\ &= k^{-d} \tilde{g}(kL(t), kL(t'), t/t'), \end{aligned} \tag{12}$$

so that $g(y) \equiv y^{-d}\tilde{g}(y,y,1)$. From this we obtain the scaling form of $T(\mathbf{k},t)$,

$$\begin{aligned} \left\langle \partial_t \vec{\phi}_{\mathbf{k}} \cdot \partial_t \vec{\phi}_{-\mathbf{k}} \right\rangle &= \left. \frac{\partial^2}{\partial t \partial t'} \right|_{t=t'} \left\langle \vec{\phi}_{\mathbf{k}}(t) \cdot \vec{\phi}_{-\mathbf{k}}(t') \right\rangle , \\ &= \dot{L}^2 k^{2-d} h(kL), \end{aligned} \tag{13}$$

where $h(x)$ is a new scaling function [51].

Our energy-scaling approach is based on the scaling behavior in Eqs. (11) and (13). Since the structure of the system determines the energy, we can obtain the energy as a function of L. The time dependence of the energy will come solely through $L(t)$, since the structure is invariant up to rescaling. However, the rate of energy dissipation is directly determined by $T(\mathbf{k},t)$ and can be calculated as a function of L and $\dot{L}$, with no other time dependence. By equating the time derivative of ϵ from (6) with $\dot{\epsilon}$ from (7), we self-consistently obtain $\dot{L}$ as a function of L. From this we solve for the growth law of the length scale, $L(t)$. We first consider systems where $n \leq d$, which have singular topological defects [41]. We will discuss systems without topological defects ($n > d+1$) in Sec. IV A and systems with nonsingular topological textures ($n = d+1$) in Sec. IV C.

B. Energy density

To calculate the energy density we use the scaling form of the structure function (11). If the thermodynamic limit of the phase-ordering system exists, then the infrared (IR) limit of the integral (6) is well behaved. Hence either the momentum integral converges in the ultraviolet (UV) and $\epsilon \sim L^{-2}$ is extracted using the scaling form and a change of variables, or momenta on the order of the UV cutoff dominate the integral.

When structure in the UV limit dominates the integral, we need to know the behavior of the correlation function in that limit, with $kL \gg 1$. This small-scale structure will only come from topological defects, since small-scale nondefect structures relax quickly via the dissipative dynamics. As a result, the structure factor is proportional to the density of the defect core, $S(\mathbf{k},t) \propto \rho_{\text{def}}$ for $kL \gg 1$. Since in an n-component model in d dimensions the defect core will have dimension $d-n$, it follows from scaling that the core density $\rho_{\text{def}} \sim L^{d-n}/L^d \sim L^{-n}$ [52]. Hence $S(\mathbf{k},t) \sim L^{-n}$, and the scaling form (11) implies [52–55]

$$S(\mathbf{k},t) \sim L^{-n} \, k^{-(d+n)} , \quad kL \gg 1. \tag{14}$$

The form of the Porod tail is purely geometrical in origin and does not depend on the overall defect structure [53], or on the details of the defect core (this holds even for defects with asymmetric time-independent core structure [56]). This generalized Porod's law is valid when $n \leq d$, so that singular topological defects exist.

Using the asymptotic expression (14) as needed in the integral for the energy density (6), and imposing a UV cutoff at $k \sim 1/\xi$, we obtain [55]

$$\epsilon \sim \begin{cases} L^{-n} \, \xi^{n-2} , & n < 2 \\ L^{-2} \ln(L/\xi) , & n = 2 \\ L^{-2} , & n > 2 . \end{cases} \tag{15}$$

The integral (6) is UV divergent for $n \leq 2$ and convergent for $n > 2$. For $n = 2$, when the integral is logarithmically divergent, we impose an effective lower cutoff at $k \sim 1/L$, which is the length scale at which (14) breaks down and which reflects the relative lack of structure at scales longer than $L(t)$. We see that the energy is dominated by the defect core density ρ_{def} for $n < 2$, by the defect field at all length scales for $n = 2$, and by variations of the order parameter at scale $L(t)$ for $n > 2$.

C. Energy-density dissipation

We calculate the scaling behavior of the rate of energy-density dissipation, $\dot{\epsilon}$, as a function of $L(t)$ in an analogous manner. If the energy-dissipation integral (7) converges we use the scaling form (13) and change variables to obtain $\dot{\epsilon} \sim L^{\mu-2}\dot{L}^2$. For the cases when the momentum integral diverges in the UV, we must evaluate the time-derivative correlation function in the $kL \gg 1$ limit.

In the UV limit, structure comes from defect cores. The field sufficiently close to a defect core comoves with the core. If the core has a local velocity $\mathbf{v}$, then

$$\partial_t \vec{\phi} \simeq R_\omega \vec{\phi} - \mathbf{v} \cdot \nabla \vec{\phi} \tag{16}$$

close to the core (i.e., at distances small compared to L), where R_ω is a rotation matrix. In the momentum representation, the second term of (16) probes the structure of the defect and scales as $k\dot{L}$, while the rotation term does not and only scales as $\dot{L}/L$. Hence the rotation term is negligible when $kL \gg 1$. From the comoving assump-

tion, $\mathbf{v}$ is spatially uniform on length scales much smaller than the local radius of curvature (or defect separation for point defects), and is only defined near the defect core. This leads to $\partial_t \vec{\phi}_{\mathbf{k}} \sim kv\vec{\phi}_{\mathbf{k}}$ for $kL \gg 1$, and

$$\begin{aligned} \left\langle \partial_t \vec{\phi}_{\mathbf{k}} \cdot \partial_t \vec{\phi}_{-\mathbf{k}} \right\rangle &\sim \langle v^2 \rangle_k \, k^2 \left\langle \vec{\phi}_{\mathbf{k}} \cdot \vec{\phi}_{-\mathbf{k}} \right\rangle, \qquad kL \gg 1 \\ &\sim \langle v^2 \rangle_k \, k^{2-d-n} L^{-n}, \qquad kL \gg 1, \end{aligned} \tag{17}$$

where $\langle v^2 \rangle_k$ is the square velocity of the defect core, averaged over the core elements that contribute to structure at momentum k. To obtain the second line we have used the generalized Porod's law (14) and have implicitly assumed that all lengths are large with respect to the core size: $k^{-1}, L \gg \xi$. We emphasize that (16) is valid sufficiently close to a slowly moving defect core. Since $\partial_t \vec{\phi}$ is purely dissipative from (3), the defect core in (16) must be moving "downhill" in the energy landscape of the local order-parameter field and, for extended defects, under the influence of the surface or line tensions of the defect core itself.

The characteristic value of the defect velocity v is $\dot{L}$. If the energy dissipation is dominated by the evolution of large defects at the characteristic size L, then using Eq. (17) and replacing $\langle v^2 \rangle_k$ with $\dot{L}^2$ within the energy-dissipation integral (7) is appropriate. (We show below that this is the case for $n < d$ or for $n > 2$.) If the integral converges in the UV this is equivalent to using the scaling form (13). If the integral diverges in the UV, this implies that the energy dissipation is due to the slow evolution at characteristic scales (such as shrinking domains or vortex loops), leading to a reduction of the core volume. The actual asymptotics of (17) is in general *not* given by using $\langle v^2 \rangle_k \sim \dot{L}^2$, as we will see explicitly in Sec. IV B, so we must directly check the appropriateness of the substitution in calculating $\dot{\epsilon}$.

For $n \leq 2$, when the energetics is dominated by the defects [following Eq. (15)], the rate of energy dissipation can be written as the rate of change of the energy of all the defect features [4]:

$$\begin{aligned} \dot{\epsilon} &\sim \frac{\partial}{\partial t} \int_{\xi}^{\infty} dl \, n(l,t) \, E(l) \\ &= -\int_{\xi}^{\infty} dl \frac{\partial j(l,t)}{\partial l} E(l) \\ &= j(\xi) E(\xi) + \int_{\xi}^{\infty} dl \, j(l,t) \frac{\partial E(l)}{\partial l}, \end{aligned} \tag{18}$$

where $n(l,t)$ is the number density of defect features of scale l [57], $E(l)$ is the average energy of a defect feature of scale l, and $j(l)$ is the number flux of defect features. The defect size l corresponds to the local radius of curvature for $n < d$ or to the separation of point defects for $n = d$. We use the continuity equation, $\partial n/\partial t + \partial j/\partial l = 0$, to obtain the second line in (18). The total number density of defect features, N, scales as the inverse scale volume, $N \sim 1/L^d$, and hence $\dot{N}$ is slowly varying for times of order $L/\dot{L}$. Since defects only vanish at the core scale ξ, we have $\dot{N} \sim j(\xi)$. This implies that $j(l)$ is independent of l for $l \ll L$, in order to provide a steady rate of defect extinction. We would like to know whether the integral in Eq. (18) is dominated by small scales, with $l \ll L$, and so we need to know $E(l)$ in that limit. We assume here that small defect features have an average energy related to the scaling form (15) for the energy density:

$$E(l) \sim \begin{cases} l^{d-n}, & n < 2 \\ l^{d-2} \ln(l/\xi), & n = 2 \\ l^{d-2}, & n > 2. \end{cases} \tag{19}$$

This is explicitly confirmed for isolated symmetric defects in Sec. IV B. Using (19), we see that for $d > n$ or $n > 2$ the integral in (18) is well behaved at $l \ll L$ and dominates the $j(\xi) E(\xi)$ term. [For $n > 2$, Eq. (18) describes the dissipation due to small defect features.] For these cases, structures with scales and separations $l \sim L(t)$ always dominate the energy dissipation, so that $\langle v^2 \rangle_k \sim \dot{L}^2$ can be used in evaluating $\dot{\epsilon}$ with Eq. (17). For $d = n \leq 2$ the integral in (18) diverges for $l \ll L$. For $d = n < 2$ the dissipation is dominated by defect pairs annihilating at $l \sim \xi$. We treat the case $d = n = 1$ below. For $d = n = 2$ the integral is logarithmically divergent, and $\dot{\epsilon}$ has contributions from defect pairs at all scales. We treat this case in Sec. IV B by determining $\langle v^2 \rangle_k$ directly.

For systems with $n < d$ or $n > 2$ we use $\langle v^2 \rangle_k \sim \dot{L}^2$ in (17), *within* the energy-dissipation integral (7). This gives

$$\dot{\epsilon} \sim \begin{cases} L^{-n} \xi^{n+\mu-2} \dot{L}^2, & n+\mu < 2 \\ L^{-n} \ln(L/\xi) \dot{L}^2, & n+\mu = 2 \\ L^{\mu-2} \dot{L}^2, & n+\mu > 2, \end{cases} \tag{20}$$

apart from $n = d \leq 2$. For $n + \mu < 2$ the integral is UV divergent and we have used the $kL \gg 1$ form (17) between its lower limit of applicability, $k \gtrsim L^{-1}$, and the UV cutoff at $k \sim 1/\xi$. For $n + \mu = 2$ the integral is logarithmically divergent. Again we use the $kL \gg 1$ asymptotics in its region of applicability. [The scaling contribution from $kL \sim 1$ merely changes the effective ξ in the logarithm of (20).] For $n + \mu > 2$ the integral is convergent and we use the scaling form (13). Equating $\dot{\epsilon}$ from (20) to the time derivative of ϵ from (15), we obtain the growth laws shown in Fig. 1.

As an example, consider nonconserved scalar fields. For this case, with $\mu = 0$ and $n = 1$, Eq. (15) gives $\epsilon \sim 1/L\xi$, implying $\dot{\epsilon} \sim -\dot{L}/L^2\xi$, while (20) gives $\dot{\epsilon} \sim -\dot{L}^2/L\xi$. Equating these two results for $\dot{\epsilon}$ yields $\dot{L} \sim 1/L$, implying $L \sim t^{1/2}$. The other results displayed in Fig. 1 were obtained in the same way.

For 1D scalar systems ($n = d = 1$), equal-time correlations scale [5,7] but $T(\mathbf{k}, t)$ *breaks* scaling due to rapidly annihilating domain wall pairs [5]. We can still apply the energy-scaling argument to the exponentially suppressed interaction energy between domain wall pairs, expressed as an energy density, $\epsilon_{\text{int}} \sim e^{-L/\xi}/L$. We equate the time derivative of this to the energy dissipation from Eq. (20) (since the interaction energy changes from the slow $\langle v^2 \rangle \sim \dot{L}^2$ evolution of the defects). This leads to logarithmic growth $L(t) \sim \ln t$ for both nonconserved

and conserved scalar models in one dimension, as shown in Table I.

The growth laws shown in Fig. 1 and Table I are only the leading time dependence of the properly integrated equations of motion for the length scale. Further corrections come from subdominant parts of the energy integrals, and from subdominant corrections to the scaling forms of the correlation functions. Corrections to integrals dominated by the UV limit are given by contributions from $kL \sim 1$, and by corrections to the generalized Porod's law (14). The latter are not generally known, except for three-dimensional scalar systems where the leading correction to Porod's law is $L^{-3}k^{-6}$ [58]. Corrections to convergent or logarithmically divergent integrals can come from the UV cutoff. For instance, for nonconserved XY systems the $\ln L$ factors in (15) and (20) will in general have different effective cutoffs, of order of the core size. This leads to an additive correction to the growth law, $O(L(t)/\ln t)$. It is important to be aware of such corrections for each particular system.

D. Long-range attractive interactions

For systems with long-range *attractive* interactions [5,12,20,44] the rate of energy-density dissipation will still be given by (7), but the energy density has a new contribution

$$\epsilon_{\rm LR} \sim \int_{\mathbf{k}} k^{\sigma} \left\langle \vec{\phi}_{\mathbf{k}} \cdot \vec{\phi}_{-\mathbf{k}} \right\rangle , \tag{21}$$

with $0 < \sigma \leq 2$. This reduces to the short-range case for $\sigma = 2$, and is subdominant to the short-range interactions for $\sigma > 2$. This interaction can be motivated by a term in the energy-functional:

$$H_{\rm LR} = \int d^d x \int d^d r \frac{\left[\vec{\phi}(\mathbf{x}+\mathbf{r}) - \vec{\phi}(\mathbf{x})\right]^2}{r^{d+\sigma}}, \tag{22}$$

where we take $\sigma > 0$ for a well defined thermodynamic limit. As in the short-range case, the local potential $V(\vec{\phi})$ does not dominate the gradient, and can be neglected. We see this for vector systems with $0 < \sigma \leq 2$ by generalizing Eq. (4) to balance a long-range energy $(1-\delta)^2 L^{-\sigma}$ with the potential term $4A\delta^2$. The local energy density is minimized if $\delta \sim 1/L^{\sigma}$, where $|\vec{\phi}| = 1 - \delta$. This corresponds to a $1/r^{\sigma}$ tail in the profile of an isolated defect, a generalization of the $1/r^2$ tail for vector systems with only short-range interactions. This tail does not change the generalized Porod's law (14) for either short- or long-range interactions, because the local structure of the defect does not change as the length scale $L(t)$ grows. As before, the structure factor for $kL \gg 1$ is simply proportional to the density of the defect core. Using Porod's law in (21), we find

$$\epsilon_{\rm LR} \sim \begin{cases} L^{-n}\,\xi^{n-\sigma}\,, & n < \sigma \\ L^{-\sigma}\,\ln(L/\xi)\,, & n = \sigma \\ L^{-\sigma}\,, & n > \sigma. \end{cases} \tag{23}$$

Comparing with the energy density from the short-range interactions (15), we see that $\epsilon_{\rm LR}$ always describes the scaling of the energy density at late times for $0 < \sigma \leq 2$.

The asymptotics of $T(\mathbf{k},t)$ are still given by (17). Applying Eq. (18) to the long-range case shows that we can use $\langle v^2 \rangle_k \sim \dot{L}^2$ *within* the energy-dissipation integral (7) for all cases except $n = d \leq \sigma$ [4]. Comparing the rate of energy dissipation, still given by (20), but excluding $n = d \leq \sigma$, and the time derivative of (23), we find $\dot{L}$ and hence $L(t)$. The results are summarized in Fig. 2. We treat the cases $n = d = 1 \leq \sigma < 2$, where $\langle v^2 \rangle_k$ is dominated by small defect features, elsewhere and find $L(t) \sim t^{1/(1+\sigma)}$ [5].

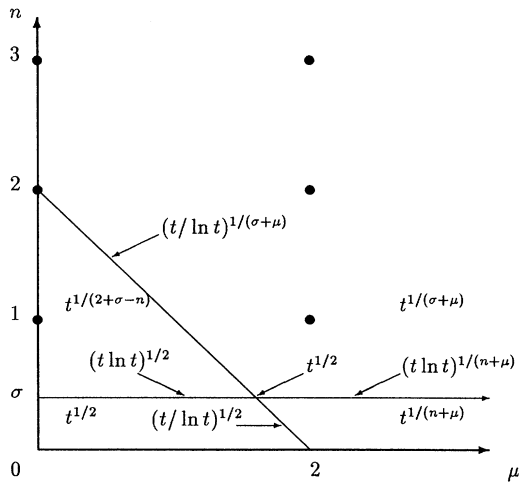

FIG. 2. The growth law of the length scale $L(t)$ for attractive long-range interactions with a fixed σ, $0 < \sigma \leq 2$, for systems which satisfy our scaling assumptions. The growth laws are shown for various numbers of components n and conservation laws μ. For $n = d = 1 \leq \sigma < 2$, $L(t) \sim t^{1/(1+\sigma)}$ [5]. For $\sigma = 2$ and $n = d \leq 2$, see Table I. Scaling systems with no topological defects ($n > d+1$), or with textures ($n = d+1$), will have $L \sim t^{1/(\sigma+\mu)}$.

TABLE I. Growth laws $L(t)$ for short-range systems with $n = d \leq 2$.

	$n = d = 1$	$n = d = 2$
$\mu = 0$	$\ln t$	$(t/\ln t)^{1/2}$
$\mu > 0$	$\ln t$	$t^{1/(2+\mu)}$

IV. SPECIAL CASES

A. Systems without topological defects

Systems with no topological defects [such as O(n) models with $n > d+1$] will have no power-law UV structure, and hence will have convergent energy integrals, giving $L(t) \sim t^{1/(\sigma+\mu)}$, where $\sigma = 2$ for short-range interactions. This result for $L(t)$ also follows directly from the dimen-

sional analysis discussed at the end of Sec. II, if the system scales. Any power-law UV structure would indicate a scaling density of singular structure — which we do not expect if locally stable defects are absent. For instance, Rao and Chakrabarti find stretched exponential tails to the asymptotic structure factor [27] for 2D $n = 4$ systems. Of course, this result applies to any system with initial conditions that evolve into a scaling structure without topological defects, and not just to $n > d + 1$. For example, if the 2D XY model is quenched *from* below T_{KT} then the scaling state has no free topological defects. In the nonconserved case scaling is obeyed and the growth law is $L \sim t^{1/2}$ [59], as expected. In the conserved 2D XY model quenched from below T_{KT}, $t^{1/4}$ growth (or, more generally, $t^{1/(\sigma+\mu)}$) will obtain, provided scaling holds. Of course, scaling can be violated even without topological defects — as seen in the conserved spherical model [60].

B. Isolated defects

In this section we obtain the size $l(\tilde{t})$, as a function of time to collapse $\tilde{t}$, of an isolated symmetric defect. We will then use this, for scalar and 2D XY systems, to obtain the proper asymptotics of $T(k,t)$, i.e., an expression for $\langle v^2 \rangle_k$ in (17). For scalar systems this will confirm the growth laws we have obtained. For 2D XY systems, we will be able to obtain the scaling growth laws (see Table I) where previously we were prevented by our crude knowledge of $\langle v^2 \rangle_k$. (In the following section we will apply a similar approach to systems with nonsingular topological textures.)

Applying Eqs. (6) and (7) to isolated defects, in a "microscopic" approach, is appealing but not trivial. For scaling systems, with many defects, the disordered initial conditions will cut off any momentum integral in the infrared (IR). Equivalently, there is no structure at scales much greater than the inverse of the characteristic length, L^{-1}, which thus provides a natural IR cutoff. On the other hand, for solitary defects (such as a single spherical domain), an IR cutoff is provided by the structure and evolution of the single defect itself. We will apply the quasiadiabatic assumption to the defect structure: that the order-parameter field is locally equilibrated to the defect-core configuration. The breakdown of this assumption away from the core, due to a finite core velocity, will provide a natural IR cutoff — cutting off the structure at small k, corresponding to large spatial scales. We must apply the largest appropriate IR cutoff, and we will focus on small defects within larger phase-ordering systems, where both cutoffs to the IR behavior are appropriate. Whenever possible, we check our approach with established results. We restrict our attention to the collapse of a spherical scalar domain of radius l with $d > 1$, and the annihilation of a point-defect–antidefect pair of separation l in a 2D XY model ($n = d = 2$). We calculate both the structure factor $S(k,l)$ and the time-derivative correlations $T(k,l)$, use these in the independent energy equations (6) and (7), and self-consistently solve for $l(\tilde{t})$ [62].

We first consider generic systems with an isolated, symmetric, singular topological defect structure ($n \leq d$) [41] characterized by a single length scale much larger than the core size, $l \gg \xi$: e.g., spherical domain walls, or a point-defect–antidefect pair. The defect size l corresponds to the local radius of curvature for $n < d$ or to the separation of point defects for $n = d$. The structure factor at small scales, $kl \gg 1$, is proportional to the core volume of the defect, l^{d-n}, and has the spherically averaged asymptotic momentum dependence $k^{-(d+n)}$ [53] of a "flat" or stable defect structure. (This asymptotic momentum dependence leads to a generalized Porod's law for scaling systems [52–55].) For $kl \gg 1$ the field around an element of the defect core will comove with it (without a rotation term because of the symmetry of the defect): $\partial_t \vec{\phi} = (dl/dt)(\nabla \vec{\phi})_{\text{radial}}$, where we have taken the radial component of the gradient. We then have, for the isolated defect,

$$S(k,l) = l^{d-n}\, k^{-(d+n)}\, f_S(kl)\ ,$$
$$T(k,l) = (dl/dt)^2\, l^{d-n}\, k^{-(d+n-2)}\, f_T(kl)\ . \tag{24}$$

The scaling functions $f_S(x)$ and $f_T(x)$ are positive definite and tend to constants for large x, if we ignore oscillations due to the finite size of the defect (these oscillations never dominate the energy integrals). We use the spherically averaged correlations in this section as they are all that are required in the energetics. From (6), the energy of the isolated defect of scale $l \gg \xi$ is given by

$$E(l) \sim \int_{\mathbf{k}} k^2 S(k,l),$$
$$\sim \int_0^{\xi^{-1}} d^d k\, l^{d-n}\, k^{-(d+n-2)}\, f_S(kl)\ , \tag{25}$$

where we cut off the integral above at the inverse core size. The energy integral (25) always converges at small k, as will be checked on a case-by-case basis. Evaluating the integral gives

$$E(l) \sim \begin{cases} l^{d-1}\xi^{-1}, & n = 1 \\ l^{d-2} \ln(l/\xi), & n = 2 \\ l^{d-2}, & n > 2. \end{cases} \tag{26}$$

(We include $n > 2$ for completeness. Isolated defects with $n > 2$ will generically be asymmetric [56,61], and not described by a single scale l.) The results for $n = 1, 2$ are, of course, familiar: for $n = 1$ the energy is just proportional to the surface area l^{d-1} of the domain, while the factor ξ^{-1} is an estimate of the surface tension; for $n = 2$, the energy is given by the defect-core volume l^{d-2} multiplied by a logarithmic factor from the "far field," with a short-distance cutoff at the core scale ξ and a large-distance cutoff at the size l of the defect, reflecting the self-screening of the defect (a vortex loop for $d = 3$, or vortex-antivortex pair for $d = 2$) at large distances.

Next we evaluate the analogue of Eq. (7), the energy dissipation rate:

$$\dot{E}(l) = -\int_0^{\xi^{-1}} d^d k\, k^{-\mu}\, T(k,l)$$
$$\simeq -\int_{k_{\rm cut}}^{l^{-1}} d^d k\, k^{-\mu}\, T(k,l) - \int_{l^{-1}}^{\xi^{-1}} d^d k\, k^{-\mu}\, T(k,l)$$
$$\equiv \dot{E}_{\rm IR}(l) + \dot{E}_{\rm UV}(l)\ , \qquad (27)$$

where $\dot{E}_{\rm IR}$ and $\dot{E}_{\rm UV}$ correspond to the momentum integrals over $k < l^{-1}$ and $k > l^{-1}$, respectively. The $kl \gg 1$ behavior, given by Eq. (24) with $f_T(kl)$ =const, can be used to calculate $\dot{E}_{\rm UV}$. However, we will see that, unlike in the calculation of $E(l)$, the IR behavior (through $\dot{E}_{\rm IR}$) sometimes dominates $\dot{E}(l)$, and so we need to determine both the IR structure of $T(k,l)$ and the effective IR cutoff, $k_{\rm cut}$. For scalar systems, $\dot{E}_{\rm IR}$ is only significant for conservation laws with $\mu \geq d$, and $k_{\rm cut}$ depends on whether or not the isolated domain is in a larger phase-ordering system or not. For 2D XY systems, $\dot{E}_{\rm IR}$ is always significant, and $k_{\rm cut}$ is provided by the breakdown of the quasiadiabatic approximation at large distances due to the nonzero evolution velocity of the defect core.

1. *Scalar systems*

In scalar systems with $d > 1$, we consider a spherical domain with radius l. Under the quasiadiabatic approximation, the fields inside and outside the domain are taken to minimize the energy subject to the configuration of the domain wall. In the IR limit, with $kl \ll 1$, the field is dominated by the saturated value inside and outside the domain walls and so $\phi_k \sim l^d$. For conserved fields we know that $\phi_{k=0}$ does *not* evolve with l and so the quasiadiabatic approximation must break down as $k \to 0$. The finite evolution rate of the domain will determine the effective $k_{\rm cut}$. Using $\phi_k \sim l^d$, we easily obtain the IR asymptotics:

$$\begin{aligned} S(k,l) &\sim l^{2d}, && k_{\rm cut} < k \ll l^{-1}, \\ T(k,l) &\sim (dl/dt)^2 l^{2d-2}, && k_{\rm cut} < k \ll l^{-1}, \end{aligned} \qquad (28)$$

where we have not yet determined $k_{\rm cut}$. This IR limit, combined with the UV behavior from (24) and the appropriate $k_{\rm cut}$, will be sufficient to determine the evolution of an isolated domain. Note that the small kl forms (28) are consistent with the general scaling forms (24). In the case of spherical domains the complete scaling functions f_S and f_T can easily be calculated. For a domain of $+1$ in a background of -1, we find that for $k > 0$ $\phi(k,l) = 2\int_0^l d^d r \exp(i\mathbf{k}\cdot\mathbf{r}) = 2(2\pi)^{d/2}(l/k)^{d/2} \mathrm{J}_{d/2}(kl)$. For $kl \gg 1$, for example, this leads to $S(k,l) \simeq 2^{d+2}\pi^{d-1} l^{d-1} k^{-(d+1)}\{1 + \cos[2kl - (d-1)\pi/2]\}$. [The oscillatory term adds $O(E(l)\xi/l)$ corrections to the energy $E(l)$ and can be ignored.] In practice, however, we do not need the precise structure for $kl \sim 1$, as its scaling properties with l are embodied in the IR and UV limits.

For nonconserved systems, the IR asymptotics imply that the energy-dissipation integral (27) converges in the IR and is dominated by $\dot{E}_{\rm UV}$, leading to

$$\dot{E} \sim -(dl/dt)^2 l^{d-1} \xi^{-1}. \qquad (29)$$

Comparing this with the time derivative of (26) recovers the standard result [8] $dl/dt \sim -l^{-1}$ for $d > 1$, which gives

$$l(\tilde{t}) \sim \tilde{t}^{1/2}, \qquad (30)$$

where $\tilde{t} = t_{\rm max} - t$ is the time left before the defect annihilates.

We can also extract a scale-dependent effective mobility for nonconserved systems [63], $\eta(l)$, by taking

$$\frac{dl}{dt} \sim -\eta \frac{\partial E/\partial l}{l^{d-n}}, \qquad (31)$$

so that $\eta(l)$ is the proportionality factor between the driving force per core volume and the evolution speed. This gives $\eta \sim \xi$ for $n = 1 < d$. While the mobility of a scalar system depends on the details of the potential through ξ, i.e., on the surface tension, the evolution velocity dl/dt is independent of those details [for free energy functionals of the form (1)]; this was first observed by Allen and Cahn [8].

For conserved systems with $\mu < d$, the IR limit still does not dominate $\dot{E}$ and the dependence on l is still captured by $\dot{E}_{\rm UV}$. The results are summarized in Table II: for $\mu < 1$ the UV limit dominates the energy-dissipation integral; for $\mu = 1$ the integral is logarithmically divergent and is cut off by the detailed structure at $kl \sim 1$; for $\mu > 1$ the integral of $\dot{E}$ converges in the UV, and structure at $kl \sim 1$ dominates. The scaling behavior of this contribution is given by power counting on the second line of (27).

For strong-enough conservation laws, $\mu \geq d$, the energy-dissipation integral no longer converges at small k. It is dominated by the IR behavior of $\dot{E}_{\rm IR}$ and we need to know the appropriate cutoff $k_{\rm cut}$. First we consider the breakdown of the quasiadiabatic approximation far from the defect. This provides the only cutoff for a solitary domain in an infinite system. The cutoff is provided by the behavior of the field ϕ far from the defect, where we linearize the evolution equation around the ordered state $|\vec{\phi}| = 1$ and can neglect all but the lowest derivatives. From Eq. (3) we have

$$\partial_t(\delta\phi_{\mathbf{k}}) = -8Ak^{\mu}\delta\phi_{\mathbf{k}}, \qquad kl \ll 1. \qquad (32)$$

Any breakdown of quasiadiabaticity (i.e., the field not following the static configuration given by the configuration of the domain wall) is due to the finite evolution speed dl/dt. Dimensionally from Eq. (32), $dl/dt \sim Ak_{\rm vel}^{\mu-1}$, which determines a velocity-dependent IR cutoff $k_{\rm vel} \sim (dl/dt)^{1/(\mu-1)}$. This cutoff can be used for $k_{\rm cut}$ in Eq. (27) to determine the evolution of an isolated domain (with $l \gg \xi$) in an infinite system. However, within the context of a larger phase-ordering system, other evolving domains provide a natural cutoff $k_{\rm cut} \sim L^{-1}$, where L is the characteristic length scale of the system. This cutoff can be used to determine the l dependence of dl/dt (i.e., the "scaling" behavior of the domain), again with Eq. (27). The two results can be compared and we find that the velocity cutoff is only larger than L^{-1} for domains with $l \lesssim L^{1-1/d}$. This is a vanishing proportion of do-

TABLE II. Results of the energy dissipation and the domain size dynamics of an isolated scalar domain of scale $l \ll L$ with $d > 1$. The time left before domain annihilation is $\tilde{t}$.

	$\partial_t E(l)$	dl/dt	$l(\tilde{t})$
$\mu < 1$	$-(dl/dt)^2 l^{d-1} \xi^{\mu-1}$	$-l^{-1}$	$\tilde{t}^{1/2}$
$\mu = 1$	$-(dl/dt)^2 l^{d-1} \ln(l/\xi)$	$-[l \ln(l/\xi)]^{-1}$	$(\tilde{t}/\ln \tilde{t})^{1/2}$
$1 < \mu < d$	$-(dl/dt)^2 l^{d+\mu-2}$	$-l^{-\mu}$	$\tilde{t}^{1/(1+\mu)}$
$1 < \mu = d$	$-(dl/dt)^2 l^{2d-2} \ln(L/l)$	$-l^{-d}/\ln(L/l)$	$[\tilde{t}/\ln(L^{1+d}/\tilde{t})]^{1/(d+1)}$
$\mu > d > 1$	$-(dl/dt)^2 l^{2d-2} L^{\mu-d}$	$-l^{-d} L^{d-\mu}$	$\tilde{t}^{1/(d+1)} L^{(d-\mu)/(d+1)}$

mains in the scaling limit, so we have used $k_{\rm cut} \sim L^{-1}$ in Table II. (The difference between the cutoffs does imply that for scalar systems with $\mu \geq d$ a solitary domain will evolve qualitatively differently than a typical domain in a phase-ordering system.) Comparing the dominant energy dissipation ($\dot{E}_{\rm IR}$ in this case) with the time derivative of (26), we find Table II for $d > 1$. We obtain the standard $l \sim \tilde{t}^{1/3}$ for $\mu = 2 < d$ [2], and recover the $\mu = d = 2$ result of Rogers and Desai [9] for small domains: $dl/dt \sim -1/[l^2 \ln(L/l)]$, where, unlike Ref. [9], we have retained the defect scale in the logarithm.

We can estimate $\langle v^2 \rangle_k$ for scalar systems with our results in Table II. This will enable us to determine the UV asymptotics of $T(k,t)$ directly through (17), and with that to check our previous results and the consistency of our approach. We assume that small defect features, with $l \ll L$, evolve as described in Table II with the local rms velocity of the defect core given by $v(l) \sim dl/dt$. We further assume that small defect features will evolve, at least in their l dependence, like segments of spherical domains, with a local radius of curvature decreasing as dl/dt, so that the number flux is given by $j(l) = n(l)v(l)$. Then

$$\langle v^2 \rangle_k \simeq \frac{\int_{l_{\rm min}}^{L} dl\, v^2(l) n(l) l^{d-n}}{\int dl\, n(l) l^{d-n}},$$

$$\sim \dot{L} L^{n-d-1} \int_{l_{\rm min}}^{L} dl\, |v(l)| l^{d-n}, \quad kL \gg 1. \qquad (33)$$

If small features ($l \ll L$) dominate we use $j(l) \sim j(\xi) \sim -\dot{L} L^{-(d+1)}$ in (33), while if small features do not dominate then the same approximation will give the expected $\langle v^2 \rangle_k \sim \dot{L}^2$. To evaluate the integral, with $n = 1$ for scalar systems, we take $v(l) \sim dl/dt$ for $l \ll L$ and use Table II. Only features large enough to contribute to the asymptotic structure of $T(k,t)$ are counted in $\langle v^2 \rangle_k$, and this sets $l_{\rm min}$ in (33). Comparing Eqs. (17) and (24) we see that $l_{\rm min} = k^{-1}$. We use this in Eq. (33) and summarize our results in Table III. These results, used in Eq. (17), give the true $kL \gg 1$ limit of $T(k,t)$. These asymptotics are *not* given by simply approximating $\langle v^2 \rangle_k \sim \dot{L}^2$, at least for $\mu \geq d$. However, the true asymptotics, used in (7), give the same growth laws as obtained earlier with our less detailed approach in Sec. III (which has the advantage of requiring fewer assumptions about the dynamics of small defect features within the phase-ordering system). For $\mu \leq 1$, where $\dot{\epsilon}$ is dominated by the UV asymptotics of $T(k,t)$, then the proper asymptotics show that $\langle v^2 \rangle_k \sim \dot{L}^2$. Conversely, for $\mu \geq d$, where $\langle v^2 \rangle_k$ is *not* given by $\dot{L}^2$, we confirm that $\dot{\epsilon}$ is not controlled by the UV asymptotics of $T(k,t)$. Hence the details of our treatment in Sec. III, as well as the growth laws, are consistent with this more microscopic treatment.

2. 2D XY systems

We can apply the same quasiadiabatic assumption to the 2D XY model: that the order-parameter field is given by the static configuration with boundary conditions imposed by the defect configuration. For the 2D XY model, the hard-spin ($V_0 \to \infty$) energy is optimized by XY phases which satisfy $\nabla^2 \theta = 0$ away from the vortices. For an isolated-defect–antidefect pair separated by l, this leads to

$$\theta(\mathbf{r}) = \tan^{-1}\left[\frac{y}{x - l/2}\right] - \tan^{-1}\left[\frac{y}{x + l/2}\right], \qquad (34)$$

where $\mathbf{r} = (x,y)$ and the order parameter $\vec{\phi}(\mathbf{r}) = (\cos\theta, \sin\theta)$ [64]. For quasiadiabatic evolution, the time dependence is only through l, and the IR behavior of the spin-spin correlations is straightforward to determine from (34):

$$\begin{aligned} S(k,l) &\sim l^2/k^2, & kl \ll 1, \\ T(k,l) &\sim (dl/dt)^2/k^2, & kl \ll 1. \end{aligned} \qquad (35)$$

Note once more that these results are consistent with the general scaling forms (24). Using (35), the energy integral converges in the IR and is given by (26). For nonconserved systems, both $\dot{E}_{\rm IR}$ and $\dot{E}_{\rm UV}$ (27) are logarithmically divergent. For conserved systems $\dot{E}_{\rm IR}$ dominates and is infrared divergent. For both cases we must determine the appropriate IR cutoff $k_{\rm cut}$.

The field far from the vortices can only follow their motion for infinitesimal dl/dt. For nonzero evolution speeds, the failure of the quasiadiabatic approximation far from

TABLE III. The average square velocity $\langle v^2 \rangle_k$ of defect cores which contribute to $T(\mathbf{k},t)$ at $kL \gg 1$ for scalar and 2D XY systems. The overdots indicate time derivatives.

	$n = 1 < d$	$n = d = 2$
$\mu = 0$	$\dot{L}^2$	$\dot{L} L^{-1} \ln(kL)/\ln(L/\xi)$
$0 < \mu < d$	$\dot{L}^2$	$\dot{L} L^{-1} k^{\mu}$
$\mu = d$	$\dot{L} L^{-\mu} \ln\ln(kL)$	$\dot{L} L^{-1} k^{\mu}$
$\mu > d$	$\dot{L} L^{-\mu} \ln(kL)$	$\dot{L} L^{-1} k^{\mu}$

the vortices provides an infrared cutoff. Far from the vortices, the component of $\partial_t\vec{\phi}$ parallel to the field is still dominated by Eq. (32), but the transverse component is given by the gradient term: $(\partial_t\vec{\phi})_\perp \sim -(-\nabla^2)^{\mu/2}\nabla^2\vec{\phi}$. Dimensionally from this equation, the transverse component provides a larger IR cutoff than Eq. (32), so that $dl/dt \sim k_{\rm cut}^{\mu+1}$. This breakdown of quasiadiabaticity implies an IR cutoff $k_{\rm cut} \sim (dl/dt)^{1/(1+\mu)}$. This cutoff can be used with the IR asymptotics (35) in the energy-dissipation integrals of Eq. (27) to determine the evolution of a solitary vortex pair. We summarize the results in Table IV. For closely separated vortex pairs within a phase-ordering system, other defect pairs will provide an effective cutoff at L^{-1}. The largest cutoff will apply in Eq. (27). We find that for nonconserved systems the breakdown of quasiadiabaticity provides the dominant cutoff for $l \lesssim L/\ln(L/\xi)$, and for conserved systems for $l \lesssim L$. Since in the nonconserved case $k_{\rm cut}$ only enters logarithmically, for conserved and nonconserved systems any defect with $l \ll L$ will have an effective IR cutoff provided by the breakdown of quasiadiabaticity. For nonconserved systems, since the cutoff only enters logarithmically, the velocity screening is effectively the same as the length-scale screening used by Yurke *et al.* [24], though from (35) we see that $T(k,l)$ for the defect-antidefect is *not* screened in the IR through the static configuration. The effective mobility from Eq. (31) is $\eta \sim -[\ln|\xi dl/dt|]^{-1}$ for a small defect pair, which agrees with Ryskin and Kremenetsky [65]. We see that for nonzero velocities the mobility does not depend on the system size, if the system is large enough.

We can estimate $\langle v^2\rangle_k$ for 2D XY systems with our results in Table IV. This will enable us to determine the UV asymptotics of $T(k,t)$ directly through (17), and with that to determine the growth laws of 2D XY systems which scale — which we were unable to do in Sec. III. We assume that vortex-antivortex pairs with separation $l \ll L$ evolve as described in Table IV, with a rms speed of annihilation given by $v \sim dl/dt$. Then the number flux of defect pairs is $j(l) = n(l)v(l)$. If small features dominate then for $l \ll L$ we use $j(l) \sim j(\xi) \sim -\dot{L}L^{-(d+1)}$ in (33), using $n = 2$, while if small features do not dominate then the same approximation will give the appropriate $\langle v^2\rangle_k \sim \dot{L}^2$. To evaluate the integral we take $v(l) \sim dl/dt$ for $l \ll L$ and use Table IV. For 2D XY systems, all vortex separations contribute to the energy dissipation since $T(k,l)$ has the same form for large and small kl. However, we require the quasiadiabatic approximation to apply for the smallest separation, so we need $k > k_{\rm cut}(l_{\rm min})$. This leads to $l_{\rm min} \sim k^{-1}$. We use $l_{\rm min}$ in Eq. (33) and summarize our results in Table III. These results can be used in Eq. (17) to get the true $kL \gg 1$ limit of $T(k,t)$. We use these true asymptotics of $T(k,t)$ in the energy-dissipation integral (7). The energy-dissipation integral is logarithmically divergent with $\dot{\epsilon} \sim -\dot{L}L^{-3}\ln(L/\xi)$. Equating this to the time derivative of ϵ from (15) we find that, although scaling is consistent, the growth law is not determined. However, imposing the scaling form (13) on the asymptotics (17) of $T(k,t)$, using Table III, determines the growth law $L \sim (t/\ln t)^{1/2}$ for nonconserved systems and $L \sim t^{1/(2+\mu)}$ for conserved systems. These growth laws differ by logarithmic factors in comparison to XY models in $d > 2$ (see Fig. 1), so that the 2D XY model is a special case of both our treatment and the growth laws.

TABLE IV. Evolution of a pair of defects of separation $l \ll L$ for $n = d = 2$.

	$\partial_t E(l)$	dl/dt	$l(\tilde{t})$
$\mu = 0$	$(dl/dt)^2\ln\lvert\xi dl/dt\rvert$	$[l\ln\lvert\xi dl/dt\rvert]^{-1}$	$(\tilde{t}/\ln\tilde{t})^{1/2}$
$\mu > 0$	$-(dl/dt)^{(2+\mu)/(1+\mu)}$	$-l^{-(1+\mu)}$	$\tilde{t}^{1/(2+\mu)}$

C. Topological textures

Systems with $n > d$ cannot have stable topological defects with singular cores, but systems with $n = d+1$ can support a nonsingular topological texture [41]. We construct an isolated texture of scale X by stereographically projecting the field configuration from a d-sphere of radius $R = X/2$ surrounding an n-dimensional point defect in n-dimensional space onto the d-dimensional space of the physical system. For $d = 1$ we rest a circle (1-sphere) on the 1D system. For $d = 2$ we rest a sphere (2-sphere) on the planar system. We set $\vec{\phi}(\mathbf{x})$ parallel to the radius vector of the sphere at the point of intersection with the line joining $\mathbf{x}$ to the top of the sphere (p_+), as shown in Fig. 3. This constructs an n-component texture that winds once over the d-dimensional system. The texture is topologically stable and can only vanish when $X \to 0$. We obtain

$$\vec{\phi}(\mathbf{x}) = (p_{\mathbf{x}}\hat{x}, n_{\mathbf{x}}), \tag{36}$$

where the d-sphere touches the system at $x = 0$. From Fig. 3 we have

$$\begin{aligned} p_{\mathbf{x}} &= \sin 2\alpha, \\ &= 2xX/(X^2 + x^2), \end{aligned} \tag{37}$$

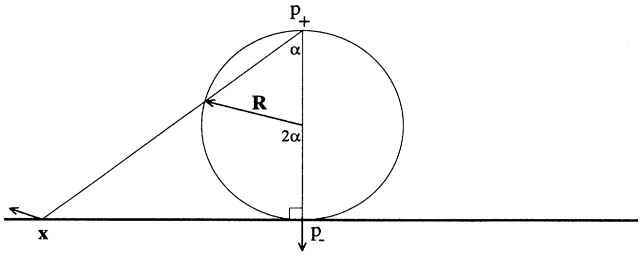

FIG. 3. The construction of a $(d+1)$-component texture in a d-dimensional system by stereographically projecting a d-sphere onto the system (shown is a plane containing $\mathbf{x}$ and the two poles $p_\pm$ of the d-sphere). The order parameter $\vec{\phi}(\mathbf{x}) = \hat{\vec{R}}$, where the radial vector $\mathbf{R}$ meets the d-sphere at the intersection with a line joining $\mathbf{x}$ with the pole p_+ of the sphere. The scale of the constructed texture, X, is twice the radius of the d-sphere.

and

$$n_{\mathbf{x}} = -\cos 2\alpha, \\ = (x^2 - X^2)/(X^2 + x^2). \tag{38}$$

The scale X of the nonsingular texture is determined by $n_X = 0$. We find

$$\vec{\phi}(\mathbf{x}) \cdot \vec{\phi}(\mathbf{x}+\mathbf{r}) = 1 - \frac{2r^2X^2}{(X^2+x^2)\left[X^2+(\mathbf{x}+\mathbf{r})^2\right]}, \tag{39}$$

and

$$S(\mathbf{k},X) = \rho_{\rm def}\int d^dx \int d^dr e^{i\mathbf{k}\cdot\mathbf{r}}\vec{\phi}(\mathbf{x})\cdot\vec{\phi}(\mathbf{x}+\mathbf{r}), \\ = (2\pi)^d\delta(\mathbf{k}) + 2\rho_{\rm def}X^2\nabla_{\mathbf{k}}^2\left[\int d^dx \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{X^2+x^2}\right]^2, \\ \simeq 2^{d+2}\pi^{d+1}\rho_{\rm def}\frac{X^{d+1}}{k^{d-1}}e^{-2kX}, \qquad kX \gg 1, \tag{40}$$

where $\rho_{\rm def}$ is the number density of defects (in the case of a single texture it is the inverse volume of the system) and we use the asymptotics of

$$\int d^dx \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{X^2+x^2} = (2\pi)^{d/2}X^{d-2}(kX)^{1-d/2}K_{d/2-1}(kX), \tag{41}$$

where $K_\nu(x)$ is a Bessel function of imaginary argument [66]. For $d = 2$ this construction generates the minimal energy texture [67] and has the same structure as unpublished results of Bray and Puri [68]. In general dimension, this stereoscopic projection does not give the minimal energy, and we expect that a different convex surface (other than the d-sphere) is needed. However, we expect that the form of the structure factor for $kX \gg 1$ will be unchanged and that the scaling form for a *single* texture will be

$$S(\mathbf{k},X) = X^{2d}g_T(kX), \tag{42}$$

which follows directly from the second line of Eq. (40). The time-derivative structure for $kX \gg 1$ is, using Eqs. (36)–(38),

$$T(\mathbf{k},X) = \rho_{\rm def}\dot{X}^2\frac{\partial^2}{\partial X_1\partial X_2}\bigg|_{X_1=X_2=X}\int d^dx \\ \times \int d^dr\left[e^{i\mathbf{k}\cdot\mathbf{r}}\vec{\phi}(\mathbf{x},X_1)\cdot\vec{\phi}(\mathbf{x}+\mathbf{r},X_2)\right], \\ \simeq 2^{d+2}\pi^{d+1}\frac{X^{d+1}}{k^{d-3}}\dot{X}^2\rho_{\rm def}e^{-2kX}, \quad kX \gg 1. \tag{43}$$

This also leads to a scaling form for the time-derivative structure of a *single* texture:

$$T(\mathbf{k},X) = \dot{X}^2X^{2d-2}h_T(kX). \tag{44}$$

We use these scaling forms to calculate the energy of an isolated texture of scale X. The energy will be, using Eqs. (6) and (42),

$$E(X) = V\int\frac{d^dk}{(2\pi)^d}k^2S(\mathbf{k},X), \\ \sim \int_0^\infty d^dk k^2X^{2d}g_T(kX), \\ \sim X^{d-2}, \tag{45}$$

where V is the system volume, so $V\rho_{\rm def} \sim 1$ for an isolated texture. The rate of energy dissipation is, using Eqs. (7) and (44),

$$\dot{E}(X) = -V\int\frac{d^dk}{(2\pi)^d}T(\mathbf{k},X), \\ \sim -\int_0^\infty d^dk\dot{X}^2X^{2d-2}h_T(kX), \\ \sim -\dot{X}^2X^{d-2}. \tag{46}$$

Comparing these results, we see that

$$\dot{X} \sim -(d-2)X^{-1}. \tag{47}$$

For $d > 2$ an isolated texture will shrink (in agreement with Derrick's theorem [69]) and vanish with $X(\tilde{t}) \sim \tilde{t}^{1/2}$ as a function of time to annihilation, in $d = 1$ isolated textures expand, while 2D textures are stable in this treatment. Indeed, in 2D Belavin and Polyakov [67] present an exact solution for a static system of textures, with $|\vec{\phi}| = 1$ everywhere and an energy independent of the scale and position of the individual textures. However, systems with mixtures of textures and antitextures will be unstable [70].

We can use these results to discuss the phase ordering of systems with many topological textures. However, we can go far with the assumption that systems with textures do not have more singular correlations than systems with point defects ($n = d$). With this, we obtain from (14) and (17) that $S(\mathbf{k},t) \lesssim k^{-2d}$, and $T(\mathbf{k},t) \lesssim k^{2-2d}$ for $kL \gg 1$. We find that the integrals (6) and (7) converge for systems with textures if $d > 2$. Hence for these textured systems $L \sim t^{1/(2+\mu)}$, with or without conservation laws. This is in accord with the dimensional argument of Sec. II.

For nonconserved systems, we can use the results for isolated textures to calculate the asymptotic $kL \gg 1$ structure factor due to a distribution of textures at small scales. Very small textures, with $X \ll L$, will collapse independently for $d > 2$. The flux of annihilating textures is $j(\xi) = -\dot{N}$, and the number density of textures scales as an inverse volume $N \sim L^{-d}$. The flux of annihilating textures at small scales is $j(X) = n_t(X)\dot{X}$, where $n_t(X)$ is the number density of textures at scale X. Using $\dot{X} \sim X^{-1}$ from (47) and $n_t(X) \sim \dot{N}/\dot{X}$ for $X \ll L$, we obtain

$$S(\mathbf{k},t) \simeq \int_0^\infty dX\, n_t(X)S(\mathbf{k},X), \\ \sim \dot{N}k^{-(2d+2)}, \qquad kL \gg 1, \quad d > 2, \tag{48}$$

and

$$T(\mathbf{k},t) \simeq \int_0^\infty dX\, n_t(X) T(\mathbf{k},X),$$

$$\sim \dot{N} k^{2-2d}, \qquad kL \gg 1, \quad d > 2, \quad (49)$$

where we have used the scaling structure of an isolated texture from Eqs. (42) and (44). These results satisfy the bounds we obtained from the assumption that the structure will not be more singular than for systems with point defects — though the time-derivative correlations saturate the bounds. The k^{-8} tail predicted for the structure factor of three-dimensional $n = 4$ systems is in rough agreement with Toyoki's simulation results, $S(k) \sim k^{-7.5\pm0.2}$, on the same system [71].

For $d = 1$ isolated textures will expand, because the energy of an isolated texture of scale X *decreases* with X (47). Hence we do not expect a singular annihilation process. This will also be true in $d = 1$ systems with conservation laws. Rather, we expect a nonsingular combination of winding and antiwinding textures. The minimal texture scale will increase with time, and the exponential factors in (40) and (43) will cause the momentum integrals to converge in the UV. This results in a *scaling* prediction of $L \sim t^{1/(2+\mu)}$. In fact, it can be shown that both nonconserved and conserved 1D XY systems do not scale [28]. The scaling violations are associated with the fact that the correlation length ξ_0 for the initial conditions enters in a nontrivial way. Extracting a length scale from the observed energy density, $\epsilon \sim L^{-2}$, the nonconserved model has $L \sim t^{1/4}$ [25], while the conserved model has $L \sim t^{1/6}$ [26–28]. These growth laws are different than the scaling results, which is consistent with the lack of scaling.

For $d = 2$, textures either will slowly shrink due to an instability caused by a soft potential [70,72], or will unwind with antitextures in a nonsingular annihilation process. In the former case, which does not seem to be the dominant process [70], a similar approach to Eqs. (48) and (49) finds bounds which lead to convergent energy and energy-dissipation integrals for any annihilation rate. In the latter case, with nonsingular processes, we also expect convergent integrals. In either case we expect $L \sim t^{1/(2+\mu)}$ when scaling is obeyed — however, different growth laws and scaling violations are observed in simulations with nonconserved dynamics [70]. Again, the scaling violations seem to be associated with a nontrivial role of the length scale ξ_0 characterizing the initial conditions [70]. Conserved 2D $n = 3$ systems have not yet been investigated.

The scaling assumption does not hold in the 1D and 2D systems with textures that have been investigated. Does it hold for $d > 2$? The scaling violations observed seem to be related to the weak interaction of the textures, which is connected to the lack of long-range order above $T = 0$ in equilibrium correlations. Systems with textures in $d > 2$ have low-temperature ordered phases, and so are qualitatively different than 1D and 2D systems. Indeed, we know of no scaling violations in systems with textures in $d > 2$.

V. COMPARISON WITH PREVIOUS RESULTS

In this section we review the current understanding of growth laws, and the evidence for our scaling assumptions, for various phase-ordering systems. Other references can be found in reviews for scalar systems [1,2,11] and in the review of Bray [3]. The numerical simulations and experiments seem to be able to check whether a particular growth law (with or without logarithmic factor) is reasonable, or to determine a growth-law exponent within about 10% without any theoretical input. The scaling of $T(k,t)$ is usually not considered. Within these constraints, existing results are consistent with the predictions of the energy-scaling approach, as we discuss in more detail below.

A. Nonconserved systems

Scalar nonconserved systems, in agreement with our results, have growth laws $L(t) \sim t^{1/2}$ as long as the growth is driven by the curvature of domain walls, i.e., for spatial dimension $d > 1$ [8,13]. This growth law has been confirmed experimentally in two and three dimensions; see, for example, the work by Mason *et al.* on twisted nematic liquid crystals [14], and the work by Shannon *et al.* on Cu_3Au [15]. In addition, simulations, such as Monte Carlo (RG) studies [16], also confirm this growth law. The scaling of the equal-time correlation function (11) also has extensive confirmation; for example, see the review by Furukawa [11]. In one dimension, scalar systems exhibit growth driven by exponentially suppressed interactions between domain walls. This case has been treated approximately by Nagai and co-workers [7] who found a logarithmic growth law, and their solution and growth law can be shown to be asymptotically correct [5].

Vector nonconserved systems have not been as well understood theoretically. We predict $L \sim t^{1/2}$ for all cases satisfying scaling, except for the 2D XY model where we expect $L \sim (t/\ln t)^{1/2}$. This has been confirmed in the limit as the number of components $n \to \infty$ [60,73]. The 1D XY model has two time-dependent length scales [28]. A modified version of the energy-scaling treatment can still be used, however, and the growth laws determined since an independent relation between the length scales can be established by physical arguments [28]. Experimentally, liquid-crystal XY-like systems seem to exhibit $L \sim t^{1/2}$ in two [21] and three [21,22] dimensions.

In 1D, simulations for systems without topological defects, with $n = 3$, 4, and 5, obtain a single length scale $L \sim t^{1/2}$, while simulations for $n = 2$ are consistent with the scaling violations mentioned above [25,28].

In 2D, careful simulations of XY models were consistent with logarithmic factors, $L \sim (t/\ln t)^{1/2}$, in the growth law [24]. These factors manifest themselves as a systematic curvature in power-law fits of earlier numerical studies [29,30]. However, tentative scaling violations in XY models have been found by Blundell and Bray [31], when the length scale was extracted from the defect density rather than directly from the correlation func-

tion. Similar inconsistencies are seen in simulations of uniaxial and biaxial 2D nematics ($n = 2$) by Zapotocky *et al.* [32]. No violations are seen when the correlations are collapsed with a length scale extracted from the correlation function itself (see, e.g., [33]). Measurement of $T(\mathbf{k}, t)$, the time-derivative correlations, may help to resolve the scaling, or absence of scaling, of these systems. For $n > 3$ both hard-spin ($|\vec{\phi}| \equiv 1$) [34] and soft-spin [33] simulations seem to find $L \sim t^{1/2}$. For $n = 3$ the hard-spin simulations find $L \sim t^{1/3}$ from the energy density ($\epsilon \sim L^{-2}$), but no scaling [34], which is confirmed by more extensive simulations of both hard- and soft-spin texture systems [70]. (Recent work by Toyoki [33], which claimed scaling and $L \sim t^{1/2}$ in this system, is hampered by early-time transients.)

In three dimensions, a growth law $L \sim t^{0.45 \pm 0.01}$ has been observed for the XY model [30,31,35,36] — less than the $t^{1/2}$ prediction. However the system seems to scale [31]. The suppressed growth law may be related to pinning effects due to the discrete dynamics — a possible explanation [31,33] for anomalously small growth exponents seen in simulations done at $T = 0$. Alternatively, corrections to scaling may be responsible. Simulations by Toyoki found $L \sim t^{1/2}$ for 3D Heisenberg ($n = 3$) systems [37]. Scaling is confirmed by Blundell and Bray [31], though again they obtained growth exponents slightly less than $1/2$.

Our results derived from the scaling assumption are consistent with the bulk of the evidence. Further work needs to be done to measure $T(\mathbf{k}, t)$ in 2D XY systems to resolve the issue of scaling, and to check the deviations from $t^{1/2}$ growth seen in $d > 2$ vector systems.

B. Conserved systems

We confirm that scalar conserved systems with $\mu = 2$ have the standard growth law of $L \sim t^{1/3}$ [17,18], provided the spatial dimension $d > 1$ so that the evolution of domain walls is curvature driven. This result is also obtained by RG arguments [43] and is seen experimentally, for instance, in the binary alloy $Mn_{0.67}Cu_{0.33}$ [19], as well as in simulations [16,18]. As in the nonconserved case, scaling of $S(\mathbf{k}, t)$ has been confirmed [11] but the scaling of $T(\mathbf{k}, t)$ has not been considered. Kawakatsu and Munakata [10] have considered the 1D case, in which there are no curvature effects, and found logarithmic growth in agreement with our results in Sec. III C. Scalar models with generalized conservation laws ($\mu > 0$) have also been considered, the so-called "noninteger derivative" or "long-range exchange" models [6]. For scalar systems, growth laws in agreement with our results are found, both through computer simulations and by considering the evolution of small defect features.

For vector conserved systems, with $\mu = 2$, we predict growth laws of $L \sim t^{1/4}$ — except for XY systems with $d > 2$ where we expect $L \sim (t \ln t)^{1/4}$. This agrees with RG arguments, which, assuming standard scaling, found growth exponents of $1/4$ for all $n \geq 2$ [43] (we note again that the RG treatment does not determine logarithmic factors). Coniglio and Zannetti found scaling violations for $n = \infty$ [60], so that our treatment does not apply to that case. However, Bray and Humayun [74] show how standard scaling could be recovered for any finite n.

Simulations of the conserved 1D XY model find a scaling violation that is consistent with the discussion at the end of Sec. IV C [28]. Simulations of the 2D XY model [26] obtain $L \sim t^{1/4}$, though small but systematic scaling violations seem to be indicated by the data. These "violations" could just be strong corrections to scaling, related to the corrections of order $O(1/\ln l)$ expected in the growth law [from the canceled logarithm from the energy density in (15)]. Recent simulations by Puri and Bray [40] found good scaling for both $d = 2$ and $d = 3$, with growth laws consistent with the forms $L \sim t^{1/4}$ ($d = 2$) and $L \sim (t \ln t)^{1/4}$ ($d = 3$) predicted here. For the three-dimensional XY model, earlier work by Siegert and Rao [38] also found good evidence for standard scaling. Although the growth law was originally interpreted as $L \sim t^{0.28}$ [38], the data are much better fitted by the energy-scaling prediction $L \sim (t \ln t)^{1/4}$ [39].

Again, our results are consistent with previous work, particularly the RG results for the power-law factors in the growth laws. Conserved 2D systems with textures ($n = 3$) should be investigated. In addition, the possible scaling violations in the conserved 2D XY system need to be resolved, perhaps by measuring $T(\mathbf{k}, t)$.

C. Long-range systems

For systems with attractive long-range interactions (21), Bray [44] treated the conserved case with a RG approach, and the nonconserved scalar case with physical arguments. Our results are in agreement, with additional logarithmic factors in marginal cases, for scalar and vector systems for $0 < \sigma < 2$. Numerical work by Hayakawa *et al.* [12] for nonconserved 2D scalar systems roughly agrees with our results for $\sigma = 0.5$, 1.0, and 1.5, but more sensitivity is needed to test the logarithmic factor for $\sigma = 1$. Analytical work by Hayakawa *et al.* [75] found a breaking of scaling for conserved dynamics in the spherical limit, similar to that seen in the short-range case [60]. Scaling violations in *nonconserved* 1D scalar systems for $\sigma < 1$ have been proposed by Lee and Cardy [20]. However, no scaling violations in either $S(\mathbf{k}, t)$ or $T(\mathbf{k}, t)$ are seen in more extensive simulations of 1D scalar systems [5], which find growth laws in agreement with our predictions for $\sigma = 0.5$, 1.0, and 1.5.

VI. DISCUSSION

Our approach does not depend on the initial conditions of the system [76]. If scaling is obeyed, critical ($\langle\vec{\phi}\rangle = 0$) and off-critical ($\langle\vec{\phi}\rangle \neq 0$) quenches will have identical growth laws. For nonconserved systems, off-critical quenches are found to break scaling [77,78], since the order parameter saturates, and so our growth laws do not apply. Indeed, exponentially growing length scales are seen [36]. Conserved systems, on the other hand, have

the same growth law for critical and off-critical quenches, in agreement with earlier predictions [18,43]. This has been numerically confirmed in the scalar case [79]. Initial conditions with long-range correlations (corresponding to, e.g., quenches from a critical point) will, similarly, have the same growth laws if scaling holds [80]. In general, various classes of initial conditions may affect the existence of scaling and the form of the scaling functions (11) and (13), but will not change the growth laws if scaling holds. Of course, the appropriate defect structure must be applied. For example, if the 2D XY model is quenched *from* below T_{KT} then the bound vortex pairs quickly annihilate and the scaling configurations have no topological defects free. In this case scaling is obeyed and the growth law is $L \sim t^{1/2}$ [59], as expected for a defect-free quench, without the logarithmic factor of 2D XY systems quenched from above T_{KT}.

We derive our results at zero temperature but we expect our growth laws to describe quenches to all temperatures below T_c. We do not address quenches in which thermal noise is essential, such as systems with static disorder [81], or quenches to a $T > 0$ critical point [82]. For systems with long-range order only at $T = 0$, such as XY and scalar systems in one dimension, our approach applies only at $T = 0$. For small T, such that the equilibrium correlation length $\xi(T)$ of the disordered phase is large, we expect our growth laws to apply as long as $L(t) \ll \xi(T)$. The 2D XY model quenched from high temperature to $0 < T < T_{KT}$ develops power-law order and invites further study because the low-temperature behavior is not described by a single $T = 0$ fixed point but by a line of fixed points. Numerical work for this case by Yurke *et al.* [24], at $0 < T < T_{KT}$, found a growth law in agreement with our $T = 0$ result, suggesting a similar growth law for all temperatures below T_{KT}.

Our treatment does not depend on the details of the potential $V(\vec{\phi})$ in the energy functional. It is only the symmetry properties of the potential's ground-state manifold that determine the growth law, since the ground-state manifold determines the defect structure, which in turn determines the asymptotic structure at $kL \gg 1$ through simple scaling arguments. The energy-scaling approach can be applied to systems with more complicated order parameters than n-component vectors. All we need is the existence of some short- or long-ranged "elastic" energy (σ), a conservation law (μ), and the defect structure, if any. The effective n corresponding to a defect type is the one that determines the scaling of its core volume density $\rho_{\rm core} \sim L^{-n}$. The defect type with the smallest n, and hence greatest core volume, will determine the growth law, since it will provide the dominant contribution to any UV divergences in the energy density or dissipation integrals. Systems with non-Abelian symmetries, such as cholesteric liquid crystals, are treated in the same way, since our approach is predicated on the energetics of the system rather than on the detailed nature of the dynamics. This assumes that scaling holds, so that *all* defect types scale with the same growth law.

For example, in bulk uniaxial or biaxial nematic liquid crystals the existence of string defects, with $\rho_{\rm string} \sim L^{-2}$, leads to $n = 2$. Using this in Eqs. (14) and (17) with no conservation law implies a $L \sim t^{1/2}$ growth law. This is consistent with recent experiments [21,22] and simulations [30]. We neglect the point defects, since they do not dominate the asymptotics or the energetics.

Similarly, in Potts models the existence of domain walls leads to the same growth laws as for scalar ($n = 1$) systems, consistent with the $L \sim t^{1/2}$ growth seen in nonconserved systems [83], and the $L \sim t^{1/3}$ growth suggested by numerical studies of conserved systems [84]. The existence of vertices in Potts models, or, e.g., in clock models [85], will not change the form of the asymptotic growth law. At late times the energy density and dissipation will be dominated by the change of domain wall volume, with $\rho_{\rm wall} \sim L^{-1}$, rather than by the vertices, with $\rho_{\rm vertices} \sim L^{-2}$. In general, the prefactor or time scale of the growth law will depend on the details of the model, but the exponent and any logarithmic factors of the growth law will be universal.

Our approach is restricted to systems which are governed by dissipative dynamics and a simple energy functional of the form (1) or (21). It would be interesting to speculate on the growth laws in the late stages of defect elimination in patterned systems, with *competing* short- and long-range interactions. However, while the defect types of the patterned structures (disclinations, etc.) can be clearly identified, it is not at all clear what conservation laws or long-range forces apply to the effective order parameter of the patterned structure, or even whether temperature is relevant to phase ordering [86]. This is a promising direction for further research.

For the systems where we know the "collapse" laws $l(\tilde{t})$ of the size of an isolated defect for a given time to collapse, they are of the same form as the scaling growth law $L(t)$ of the phase-ordering length scale for a given time after the quench. This agrees with our naive expectation for scaling systems that the form of the collapse law will be the same as the growth law — the intuitive picture is of collapsing defect features leaving "voids" which set the growing length scale $L(t)$.

We can say little about the growth laws in systems that break our scaling assumptions, apart from our discussion about the ξ_0 dependence at the end of Sec. II. Some systems can be explicitly shown to break scaling, such as 1D XY models [28], or conserved spherical systems [60,75]. Are there other systems that break scaling? We cannot answer this by examining only the equal-time correlation function, since the scaling of the latter does *not* imply the scaling of $T(\mathbf{k}, t)$. There are two paths to take. The first is to explicitly find scaling violations in a system. A possible example is given by the equal-time correlations in the 2D XY model, both for the nonconserved [31,32] and for the conserved (apparent in Fig. 3 of [26]) cases. The second method is to find a growth law that is in striking disagreement with our predictions, such as in the 2D $n = 3$ system [70]. Since our approach is based on the scaling assumption, any disagreement implies a scaling violation. Conversely, to demonstrate scaling the correlations must be measured directly since agreement between the observed and predicted growth laws is necessary but not sufficient to demonstrate scaling.

VII. SUMMARY

In summary, by focusing on the energetics, rather than the detailed dynamics of the system, we obtain growth laws for phase ordering (summarized in Figs. 1 and 2, as well as Table I). This leads to a powerful, unified, and physical approach to determining growth laws that rests solely on the existence of scaling. We call this the "energy-scaling" approach. It can be used for any system with purely dissipative dynamics. Any disagreement of a growth law from our predictions indicates the breaking of scaling of at least one of the two-point correlation functions. In particular, we stress the importance of the time-derivative correlation function.

In addition to the growth laws, we determine the collapse law for isolated nonconserved textures ($n = d+1$) in more than two dimensions, and use that to determine the generalized Porod's law (48) and asymptotics for $T(\mathbf{k}, t)$ (49) — these follow power laws due to annihilating textures. We also treat the dynamics of isolated defects in scalar systems and 2D XY systems. When scaling holds, we find that the growth law matches the form set by collapsing isolated defect features, of the characteristic scale L, within a larger phase-ordering system.

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