Long-time behavior of correlation functions in the finite ideal gas

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We consider time-dependent correlation functions in the classical problem of a single point particle confined to a one-dimensional box with hard walls. We find that the asymptotic behavior of these correlation functions at large times depends upon the degree of differentiability of the initial ensemble and of the observable, both of which are represented as functions over the phase space. Functions that have a discontinuity in the nth derivative lead to correlations that decay as $t^{-(n+1)}$ at long times t. We conjecture that this sensitive dependence of the long-time behavior on the smoothness of ensembles is connected to the presence of trajectory instability.

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I. INTRODUCTION

The analysis of physical systems in terms of ensembles that evolve with time is central to nonequilibrium statistical mechanics. However, this mode of description is not without its problems. There is no universal agreement on the physical meaning of an ensemble. Three authors who have energetically put forth their very different views on this subject are Jaynes [1], Lebowitz [2], and Prigogine [3].

Another problem arises when we aim to derive physically meaningful conclusions from the long-time behavior of ensembles. Results on the long-time behavior of ensembles often depend on the precise mathematical definition of the space of ensembles in which calculations are performed. This is surprising for many physicists, who would expect such a choice of definition to be a mere mathematical convenience.

For example, when we work in a Hilbert space of ensembles, as in the pioneering work of Koopman [4], the time-evolution operator is unitary, so all its eigenvalues lie on the unit circle and there is no possibility of deriving an exponential decay towards equilibrium. More exotic mathematical structures, such as "rigged" Hilbert spaces, have been used to overcome this problem [5]. The operator U is extended to act on a space Φ of test functions, which is different from the Hilbert space. Eigenfunctions of U that have eigenvalues not on the unit circle may be found in the test space Φ.

In quantum mechanics, where the use of Hilbert spaces is so widespread as to be taken for granted, a similar difficulty arises when the decay of unstable states is studied. The use of rigged Hilbert spaces was first suggested by Böhm [6] in this context.

The long-time behavior of correlation functions in chaotic systems has been the subject of much recent research. A time-dependent correlation function $G_{fg}(t)$ is defined as the expectation value of a phase-space function $f(x)$ at time t, after a dynamical system has been prepared in the ensemble $g(x)$ at time zero:

$$G_{fg}(t) = \int_\Gamma dx f(T_t x) g(x), \tag{1}$$

where Γ is the phase space and T_t is the mapping that defines the dynamics of the system; if x is the state of the system at time τ, then $T_t x$ is the state at time $t+\tau$.

Ruelle [7] and others have shown that for a certain class of models, known as "axiom A" systems, these correlation functions decay exponentially at long times. However, this conclusion is restricted to the case where the phase-space functions f and g are differentiable. Recent work [8] by the present author and others has shown how, in a model system based on the baker transformation, the behavior of correlation functions at long times depends on a smoothness condition that defines the function space in which ensembles lie.

It is therefore established that in chaotic systems, the smoothness of phase-space functions f and g controls the long-time behavior of correlation functions. The purpose of the present paper is to illustrate how this is also true even for a very simple system that has no ergodic or chaotic properties apart from trajectory instability.

II. FINITE IDEAL GAS

The model system that is the subject of this paper is simply a single point particle moving classically in a one-dimensional box with perfectly reflecting walls. The particle has mass m and the length of the box is L. We denote the position of the particle, in the range $0 \leq x \leq L$, by x and the momentum by a real number p.

One might expect that the dynamics of ensembles in such a simple model would be trivial, but this is not the case. This system has been the subject of many research papers [9–13] since 1955, when Born [9] used it to illustrate some of his concerns about the nature of reality and quantum theory.

Born noted that the trajectory of a particle in the model is unstable, in the following sense. A small uncertainty δp in the value of the momentum p in an initial condition produces an uncertainty in the position x at time t, which grows as $t\delta p/m$. This uncertainty quickly becomes large compared to the system size L. He argued that for this system and others showing trajectory instability, the description of classical dynamics in terms of trajectories could not be justified and that ensembles should be regarded as having true physical meaning. This anticipates more recent authors [14] who have put forth the same conclusion in the light of the study of chaos.

Born also realized that the analysis of the system could be simplified by extending the range of possible values of x to $\pm\infty$ and imposing the conditions

$$f(x,p) = f(-x,-p) \tag{2}$$

and

$$f(x+2L,p) = f(x,p) \tag{3}$$

on ensembles $f(x,p)$. This step incorporates the effects of the reflecting walls into the definition of an ensemble, so that whenever a particle reaches the wall at $x = L$ with momentum p, another with momentum $-p$ arrives at the same point. We then have the simple Hamiltonian

$$H = \frac{p^2}{2m}, \tag{4}$$

and the evolution of an ensemble with time t is equally simple:

$$f_t(x,p) = f_0\left(x - \frac{pt}{m}, p\right). \tag{5}$$

Born examined the evolution of an ensemble that was described by a Gaussian in position and momentum and found that the decay of certain correlation functions with time was exponential. Since then, other authors [11–13] have considered the evolution of ensembles that are Gaussian in momentum and have different types of nonuniformity in position, again finding exponential decay. Lee [12] derived a diffusionlike equation that is obeyed by the reduced x distribution when the initial ensemble is described by a function that is the product of a Maxwellian momentum distribution and some function of position. Hobson [10] found that with an ensemble described by a step function in both position and momentum, the decay of correlations with time was nonexponential, going as $1/t$ at large times.

In this paper we will show that, in general, the asymptotic rate of decay of correlation functions such as Eq. (1) at long times depends upon the degree of differentiability with respect to p of the phase-space functions f and g. Functions that have a discontinuity in the nth derivative with respect to p lead to correlations that decay at long times as $t^{-(n+1)}$ or more slowly. We derive an expression for the asymptotic form of such correlation functions, valid when the function f or g has a discontinuity in the nth derivative with respect to p.

III. CORRELATION FUNCTION

We define a time-dependent correlation function $G_{fg}(t)$ for the finite ideal gas, using Eqs. (1) and (5), by

$$G_{fg}(t) = \int_{-\infty}^{\infty} dp \int_0^L dx f\left(x - \frac{pt}{m}, p\right) g(x,p). \tag{6}$$

We extend both f and g to the entire (x,p) plane, using Eqs. (2) and (3). The symmetry (2) implies that Eq. (6) can be rewritten as

$$G_{fg}(t) = \frac{1}{2}\int_{-\infty}^{\infty} dp \int_{-L}^{L} dx f\left(x - \frac{pt}{m}, p\right) g(x,p). \tag{7}$$

We now take advantage of the periodicity (3) by writing f and g as Fourier series

$$f(x,p) = \sum_{k=-\infty}^{\infty} f_k(p) e^{ik\pi x/L}, \tag{8}$$

where

$$f_k(p) = \frac{1}{2L}\int_{-L}^{L} dx f(x,p) e^{-ik\pi x/L} \tag{9}$$

and $g_k(p)$ is defined similarly. Replacing both f and g by Fourier series in Eq. (7) and evaluating a sum and an integral yields

$$G_{fg}(t) = L \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} dp f_k(p) g_{-k}(p) e^{-ik\pi pt/mL}. \tag{10}$$

Using the reverse transform (9) then gives

$$G_{fg}(t) = \frac{1}{4L}\sum_{k=-\infty}^{\infty} \int_{-L}^{L} dx_1 \int_{-L}^{L} dx_2 I_k(x_1,x_2,t) \times \exp\left[-\frac{ik\pi}{L}(x_1 - x_2)\right], \tag{11}$$

where

$$I_k(x_1,x_2,t) = \int_{-\infty}^{\infty} dp\ f(x_1,p) g(x_2,p) e^{-ik\pi pt/mL}. \tag{12}$$

These equations are more complicated than the original definition of the correlation function (6), but they have the advantage of incorporating the periodicity condition (3) and hence the walls of the container. They will therefore be used in the next two sections as the starting point of an analysis of the asymptotic behavior of $G_{fg}(t)$ for large t.

IV. ASYMPTOTIC BEHAVIOR OF FOURIER TRANSFORMS

To find an expression for the long-time behavior of the correlation function $G_{fg}(t)$, we must first find the limiting form of the integral $I_k(x_1,x_2,t)$ in Eq. (12). Since the integral is a Fourier transform in the variable p, the following well-known theorem will be useful

Theorem 1. Suppose that a function $h(p)$ is n times differentiable and that the nth derivative $h^{(n)}(p)$ is continuous, except at a finite number of points. Suppose also that for every integer k such that $0 \le k \le n$, the function satisfies

$$\lim_{p\to\pm\infty} h^{(k)}(p) = 0 \tag{13}$$

and

$$\int_{-\infty}^{\infty} dp \; \left| h^{(k)}(p) \right| < \infty, \tag{14}$$

where $h^{(k)}$ denotes the kth derivative.

Then for $s \neq 0$,

$$\left| \tilde{h}(s) \right| \leq \frac{M}{s^n}, \tag{15}$$

where $\tilde{h}$ is the Fourier transform

$$\tilde{h}(s) = \int_{-\infty}^{\infty} dp \; h(p) e^{-isp} \tag{16}$$

and M is a constant defined by

$$M = \int_{-\infty}^{\infty} dp \; \left| h^{(n)}(p) \right|. \tag{17}$$

This theorem can be proved by repeated integration by parts. See, for example, Ref. [15].

With the help of this theorem, we now describe a simple case where an asymptotic form for a Fourier integral can be found. Suppose that $h(p)$ is twice differentiable except at a single point $p = q$, where it is discontinuous. Then

$$h(p) = \sigma(p) + \gamma(p)\Theta(p - q), \tag{18}$$

where $\sigma(p)$ and $\gamma(p)$ are twice differentiable and $\Theta(p)$ is the Heaviside step function

$$\Theta(p) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases} \tag{19}$$

Using theorem 1, the Fourier transform of h can be written as

$$\tilde{h}(t) = \int_{q}^{\infty} dp \; \gamma(p) e^{-ipt} + O\left(\frac{1}{t^2}\right). \tag{20}$$

Integrating by parts twice then yields

$$\begin{aligned} \tilde{h}(t) = {} & \frac{1}{it}\gamma(q) e^{-itq} \\ & + \left(\frac{1}{it}\right)^2 \left[\gamma'(q) e^{-itq} + \int_{q}^{\infty} dp \; \gamma''(p) e^{-itp} \right] \\ & + O\left(\frac{1}{t^2}\right). \end{aligned} \tag{21}$$

Thus we have

$$\tilde{h}(t) = \frac{1}{it}\gamma(q) e^{-itq} + O\left(\frac{1}{t^2}\right). \tag{22}$$

This is an asymptotic expression for the Fourier integral (16), valid in the limit $t \to \infty$.

V. ASYMPTOTIC FORM OF CORRELATION FUNCTIONS

Using the result of the preceding section, we will now derive a limiting form for the correlation function $G_{fg}(t)$ that is valid for large t. We assume that in the range $-\infty < p < \infty$, $-L \leq x < L$, f and g are $(n+1)$ times differentiable with respect to p, except for a discontinuity in the $(n-1)$th derivative of f, which is along the curve $p = q(x)$. We may then write

$$f^{(n-1)}(x,p) = S(x,p) + B(x,p)\Theta(p - q(x)), \tag{23}$$

where $S(x,p)$ and $B(x,p)$ are $(n+1)$ times differentiable with respect to p and partial derivatives with respect to p are written

$$f^{(n)}(x,p) := \left(\frac{\partial}{\partial p}\right)^n f(x,p). \tag{24}$$

We also assume that for $0 \leq j \leq n+1$, the functions f and g satisfy

$$\lim_{p \to \pm\infty} f^{(j)}(x,p) = \lim_{p \to \pm\infty} g^{(j)}(x,p) = 0. \tag{25}$$

These assumptions are not as restrictive as they may seem. Since the correlation function $G_{fg}(t)$ is a bilinear functional of the functions f and g, once we have dealt with functions of the class described above, we can just as easily work with linear combinations of such functions. This allows us to handle any function that is smooth apart from discontinuities that lie along smooth curves in the (x,y) plane. We can also switch the roles of f and g by using the symmetry

$$G_{fg}(t) = G_{gf}(-t). \tag{26}$$

Once the functions f and g are defined in the range $-\infty < p < \infty$, $-L \leq x < L$, the periodicity requirement (3) extends the definition to the whole (x,p) plane.

When f and g satisfy the above conditions and $n = 1$, $f(x,p)$ is discontinuous along the line $p = q(x)$. For any choice of x_1 and x_2 the product $f(x_1,p)g(x_2,p)$ can be written in a form similar to Eq. (18):

$$\begin{aligned} f(x_1,p)g(x_2,p) = {} & S(x_1,p)g(x_2,p) \\ & + B(x_1,p)g(x_2,p)\Theta(p - q(x_1)), \end{aligned} \tag{27}$$

where both the first term on the right-hand side and the prefactor of Θ are twice differentiable with respect to p. We can therefore use the result of Sec. IV [Eq. (22)] to find an asymptotic form of $I_k(x_1,x_2,t)$, valid for large t:

$$\begin{aligned} I_k(x_1,x_2,t) = {} & \frac{mL}{ik\pi t} B(x_1,q(x_1))g(x_2,q(x_1)) \\ & \times e^{-ik\pi q(x_1)t/mL} + O\left(\frac{1}{t^2}\right). \end{aligned} \tag{28}$$

Using this result and Eq. (11) and treating the $k = 0$ term separately, we find the asymptotic form for the correlation function

$$G_{fg}(t) \sim L\int_{-\infty}^{\infty} dp\ f_0(p)g_0(p) + \sum_{k\neq 0}\frac{m}{4\pi kit}\int_{-L}^{L} dx_1\int_{-L}^{L} dx_2\ \beta(x_1)g(x_2,q(x_1)) \times \exp\left[\frac{-ik\pi}{L}[x_1 - x_2 + q(x_1)t/m]\right], \tag{29}$$

where $\beta(x) = B(x,q(x))$ is a function that gives the size of the discontinuity in $f(x,p)$ at the point $(x,p) = (x,q(x))$.

In the Appendix we derive an equation that is a more general form of (29), valid for any integer $n \geq 0$. This equation is

$$G_{fg}(t) \sim L\int_{-\infty}^{\infty} dp\ f_0(p)g_0(p) + \frac{1}{4L}\sum_{k\neq 0}\left(\frac{mL}{\pi kit}\right)^n\int_{-L}^{L} dx_1 \times\int_{-L}^{L} dx_2\beta(x_1)g(x_2,q(x_1)) \times \exp\left[\frac{-ik\pi}{L}[x_1 - x_2 + q(x_1)t/m]\right], \tag{30}$$

where now $\beta(x)$ gives the discontinuity in $f^{(n-1)}(x,p)$ at $(x,q(x))$.

By taking into account the symmetry of f and g, we can modify Eq. (30) so that the integrals over x_1 and x_2 do not include values outside the range $0 \leq x \leq L$, which defines the box. The symmetry requirement of Eq. (2) implies that

$$f^{(n-1)}(x,p) = (-1)^{n-1}f^{(n-1)}(-x,-p) \tag{31}$$

and hence that

$$\beta(x) = (-1)^n\beta(-x). \tag{32}$$

Similarly, $g(x,p) = g(-x,-p)$ and $q(x) = -q(-x)$. Using these symmetries, we can rewrite Eq. (30) as

$$G_{fg}(t) \sim L\int_{-\infty}^{\infty} dp\ f_0(p)g_0(p) + \frac{1}{2L}\sum_{k\neq 0}\left(\frac{mL}{\pi kit}\right)^n \times\int_{0}^{L} dx_1\ \beta(x_1)\exp\left[\frac{-ik\pi[mx_1 + q(x_1)t]}{mL}\right] \times\bar{g}_k(q(x_1)), \tag{33}$$

where

$$\bar{g}_k(p) = \int_0^L dx_2\ (g(x_2,p) + g(x_2,-p))\cos\left(\frac{k\pi x_2}{L}\right). \tag{34}$$

Equation (33) is the central result of this paper. From it we can see that the asymptotic rate of decay of the correlation function $G_{fg}(t)$ depends on the parameter n, which gives the degree of differentiability of the phase space function f. This result has been derived under the assumption that g is differentiable to a higher order than f. However, by using Eq. (26), we can interchange f and g. This implies that discontinuities in g and its derivatives have a similar effect on the long-time behavior of $G_{fg}(t)$ to those of f.

VI. SPECIAL CASES OF CORRELATION FUNCTIONS

In this section we will look at two special cases where the expression (33) for the long-time limit of the correlation function simplifies. The first case is where $q(x)$, the locus of the discontinuity, is independent of x. We then obtain

$$G_{fg}(t) \sim L\int_{-\infty}^{\infty} dp\ f_0(p)g_0(p) + \frac{1}{2L}\sum_{k\neq 0}\left(\frac{mL}{ik\pi t}\right)^n\bar{g}_k(q)\beta_k e^{-ik\pi qt/mL}, \tag{35}$$

where

$$\beta_k = \int_0^L dx\ \beta(x)e^{-ik\pi x/L} \tag{36}$$

and $\bar{g}_k(p)$ is given by Eq. (34).

The second case where the asymptotic form of the correlation function can be simplified is when the function $q(x)$ has a stationary point $x_1 = X$ such that $q'(X) = 0$. In this case, we can find a limiting form for the integral over x_1 in Eq. (33) using a stationary-phase approximation. The integral is

$$J_k(x_2,t) = \int_0^L dx_1\ \beta(x_1)\bar{g}_k(q(x_1)) \times\exp\left[\frac{-ik\pi[mx_1 + q(x_1)t]}{mL}\right]. \tag{37}$$

As t becomes large, the phase of the integrand varies rapidly with x_1, except where $q(x_1)$ has a stationary point. Contributions to the integral from points not near the stationary point $x_1 = X$ will therefore be small in the limit $t \to \infty$.

The stationary-phase approximation consists of replacing the integrand in Eq. (37) by an approximate form that is valid close to the stationary point. The form is

$$\beta(X)\bar{g}_k(q(X)) \times\exp\frac{-ik\pi}{mL}\left(mX + q(X)t + \frac{1}{2}t(x_1 - X)^2q''(X)\right). \tag{38}$$

We also ignore the effect of the end points of the integral by extending the range of integration to $\pm\infty$. The integral can then be evaluated, yielding

$$J_k(x_2,t) \sim \left(\frac{2mL}{iktq''(X)}\right)^{1/2}\beta(X)\bar{g}_k(q(X)) \times\exp\left[\frac{-ik\pi[mX + q(X)t]}{mL}\right]. \tag{39}$$

The resulting form for the correlation function is

$$G_{fg}(t) \sim L\int_{-\infty}^{\infty} dp\ f_0(p)g_0(p) + \frac{1}{2L}\left(\frac{2\pi}{q''(X)}\right)^{1/2}\sum_{k\neq 0}\left(\frac{mL}{ik\pi t}\right)^{n+1/2} \times \bar{g}_k(q(X))\beta(X)\exp\left[\frac{-ik\pi[mX+q(X)t]}{mL}\right]. \tag{40}$$

VII. AN EXAMPLE OF A CORRELATION FUNCTION

We now give a simple example of a correlation function whose long-time limit can be calculated using the results we have derived. Let $g(x,p)$ be defined in the range $0 < x < L$ by

$$g(x,p) = \left(\frac{2}{mL^2\pi k_BT}\right)^{1/2}\exp\left(\frac{-p^2}{2mk_BT}\right)\chi(x), \tag{41}$$

where

$$\chi(x) = \begin{cases} 1, & x \leq L/2 \\ 0, & x > L/2. \end{cases} \tag{42}$$

This is the equilibrium ensemble for a gas at temperature T, confined in the left half of the box by a partition at $x = L/2$. If at time $t = 0$ we remove the partition, then the ensemble will evolve according to the equations that we have given. We wish to estimate the fraction of the gas molecules that will be moving to the right in the box at time t. We therefore use the function

$$f(x,p) = \Theta(p). \tag{43}$$

Since $f(x,p)$ has a discontinuity along the line $p = 0$, we can use Eq. (35) to find the long-time limit of the correlation function $G_{fg}(t)$. We need the coefficients $\bar{g}_k(0)$, which for $k \neq 0$ are

$$\bar{g}_k(0) = \begin{cases} \frac{1}{k\pi}\left(\frac{2}{m\pi k_BT}\right)^{1/2}(-1)^{(k-1)/2}, & \text{odd } k \\ 0, & \text{even } k, \end{cases} \tag{44}$$

and the coefficients defined by equation (36),

$$\beta_k = \begin{cases} \frac{2L}{ik\pi}, & \text{odd } k \\ 0, & \text{even } k. \end{cases} \tag{45}$$

The sum in Eq. (35) is therefore

$$\frac{L}{t}\left(\frac{m}{k_BT}\right)^{1/2}C, \tag{46}$$

where C is a numerical constant, which can be evaluated with the help of tables [16]

$$C = -\left(\frac{2}{\pi^7}\right)^{1/2}\sum_{n=-\infty}^{\infty}(2n+1)^{-3}(-1)^n = 0.050. \tag{47}$$

To evaluate the time-independent term in Eq. (35), we need $f_0(p)$ and $g_0(p)$. The integrals are easy to evaluate, and remembering the symmetry (2) that applies to both f and g, we find

$$L\int_{-\infty}^{\infty} dp\ f_0(p)g_0(p) = \frac{1}{2}. \tag{48}$$

We can now write down the limiting form of the correlation function $G_{fg}(t)$ for large t:

$$G_{fg}(t) \sim \frac{1}{2} + \frac{L}{t}\left(\frac{m}{k_BT}\right)^{1/2}C. \tag{49}$$

The first term tells us that at large times, approximately half the molecules in the ensemble are traveling to the left. The second term shows that the relaxation to equilibrium of this variable is nonexponential, going as t^{-1}.

VIII. CONCLUSION

We have seen that even in this simple model, the long-time behavior of correlation functions is not trivial and has some of the features that we expect to find in more complex, chaotic systems. The decay rate at long times depends on the smoothness of the phase-space functions f and g involved in the correlation function.

An important feature that the one-dimensional ideal gas shares with chaotic dynamical systems is the trajectory instability described in Sec. II. The ideal gas is not chaotic because the separation between two adjacent trajectories increases linearly with time, while in truly chaotic systems, the separation increases exponentially with time. However, in both cases, the trajectory instability causes a dynamical system to act on ensembles as a microscope, taking features of an ensemble that are on a very small scale and "magnifying" them with the passage of time. This is the reason why "microscopic" properties of an ensemble function, such as its smoothness, have an influence on the long-time behavior.

This reasoning and the examples provided by the current paper and by results for chaotic dynamical systems [5,8] lead us to a conjecture. In general, we expect that a dynamical system that shows trajectory instability, whether or not it is truly chaotic, should also have the property that the long-time behavior of its correlation functions $G_{fg}(t)$ depends upon the smoothness of the phase-space functions f and g.

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APPENDIX: THE CASE WHERE $f(x,p)$ HAS A DISCONTINUITY IN THE $(n-1)$th DERIVATIVE

In this appendix we generalize the treatment of the long-time behavior of $G_{fg}(t)$ given in Sec. V to include the case where $f(x,p)$ has a discontinuity in the $(n-1)$th derivative. Suppose that f and g satisfy the conditions given in Sec. (V). Integrating Eq. (12) by parts $(n-1)$ times, we find

$$I_k(x_1,x_2,t) = \left(\frac{mL}{ik\pi t}\right)^{(n-1)} \int_{-\infty}^{\infty} dp\; e^{-ik\pi pt/mL} \times \left(\frac{\partial}{\partial p}\right)^{(n-1)} [f(x_1,p)g(x_2,p)]. \quad \text{(A1)}$$

By Leibnitz's theorem,

$$\left(\frac{\partial}{\partial p}\right)^{(n-1)} [f(x_1,p)g(x_2,p)] = \sum_{r=0}^{n-1} {}^{(n-1)}C_r f^{(r)}(x_1,p)g^{(n-1-r)}(x_2,p), \quad \text{(A2)}$$

where

$$f^{(r)}(x_1,p) := \left(\frac{\partial}{\partial p}\right)^r f(x_1,p) \quad \text{(A3)}$$

and ${}^nC_r$ is the binomial coefficient

$${}^nC_r = \frac{n!}{r!(n-r)!}. \quad \text{(A4)}$$

Since f is $(n-1)$ times differentiable with respect to p and g is $(n+1)$ times differentiable, the right-hand side of Eq. (A2) can be rewritten as

$$S(x_1,x_2,p) + (n-1)f^{(n-2)}(x_1,p)g'(x_2,p) + f^{(n-1)}(x_1,p)g(x_2,p), \quad \text{(A5)}$$

where $S(x_1,x_2,p)$ is twice differentiable with respect to p, so that its Fourier transform $\tilde{S}(x_1,x_2,\tau)$ [see Eq. (16)] is bounded by τ^{-2}. To find a bound on the Fourier transform of the second term in (A5), we integrate by parts twice:

$$\int_{-\infty}^{\infty} dp\; f^{(n-2)}(x_1,p)g'(x_2,p)e^{-i\tau p} = \frac{1}{i\tau}\int_{-\infty}^{\infty} dp\; f^{(n-1)}(x_1,p)g'(x_2,p)e^{-i\tau p} + \left(\frac{1}{i\tau}\right)^2 \int_{-\infty}^{\infty} dp\; \frac{d}{dp}\left[f^{(n-2)}(x_1,p)g''(x_2,p)\right] e^{-i\tau p}. \quad \text{(A6)}$$

Since $f^{(n-1)}$ satisfies Eq. (23), the long-time limit of the first integral on the right-hand side of Eq. (A6) can be found in the same way that the integral $I_k(x_1,x_2,t)$ was evaluated in Sec. V, giving a result bounded by t^{-1}. This means the Fourier transforms of all terms but the last in (A5) are bounded by τ^{-2}.

In the same way, we can find a limiting expression for the Fourier transform of the last term in (A5):

$$\int_{-\infty}^{\infty} dp\; f^{(n-1)}(x_1,p)g(x_2,p)e^{-i\tau p} = \frac{1}{i\tau}B(x_1,q(x_1))g(x_2,q(x_1))e^{-i\tau q(x_1)} + O\left(\frac{1}{\tau^2}\right). \quad \text{(A7)}$$

We therefore have

$$I_k(x_1,x_2,t) = \left(\frac{mL}{ik\pi t}\right)^n B(x_1,q(x_1))g(x_2,q(x_1)) \times e^{-ik\pi q(x_1)t/mL} + O\left(\frac{1}{t^{n+1}}\right). \quad \text{(A8)}$$

Placing this result in Eq. (11) gives the result that we quoted in Sec. V, as Eq. (30):

$$G_{fg}(t) \sim L\int_{-\infty}^{\infty} dp\; f_0(p)g_0(p) + \frac{1}{4L}\sum_{k\neq 0}\left(\frac{mL}{\pi kit}\right)^n \int_{-L}^{L} dx_1 \times \int_{-L}^{L} dx_2\; B(x_1,q(x_1))g(x_2,q(x_1)) \times \exp\left[\frac{-ik\pi}{L}\left(x_1 - x_2 + q(x_1)t/m\right)\right]. \quad \text{(A9)}$$

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