Universal correlations for deterministic plus random Hamiltonians

E. Brézin,[1] S. Hikami,[2] and A. Zee[3]

[1]*Laboratoire de Physique Théorique, Unité propre du Centre National de la Recherche Scientifique, Associée à l'Ecole Normale Supérieure et à l'Université de Paris–Sud, Ecole Normale Supérieure, 24 rue Lhomond 75231, Paris Cedex 05, France*

[2]*Department of Pure and Applied Sciences, University of Tokyo, Meguro-ku, Komaba, 3-8-1, Tokyo 153, Japan*

[3]*Institute for Theoretical Physics, University of California, Santa Barbara, California 93106*

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We consider the (smoothed) average correlation between the density of energy levels of a disordered system, in which the Hamiltonian is equal to the sum of a deterministic H_0, and of a random potential φ. Remarkably, this correlation function may be explicitly determined in the limit of large matrices, for any unperturbed H_0 and for a class of probability distribution $P(\varphi)$ of the random potential. We find a compact representation of the correlation function. From this representation one readily obtains the short distance behavior, which has been conjectured in various contexts to be universal. Indeed we find that it is totally independent of both H_0 and $P(\varphi)$.

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I. INTRODUCTION

We consider a Hamiltonian $H = H_0 + \varphi$ with a deterministic H_0 and a random φ, in a finite-dimensional space in which H_0 and φ are Hermitian $N \times N$ matrices; we are interested in the large-N limit. The average number of eigenvalues in an interval is known to depend sensitively upon the spectrum of H_0 and upon the probability distribution $P(\varphi)$ of the disordered potential [1]. The one-point Green's function and the two-point connected Green's function are given by

$$G(z) = \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{z-H} \right\rangle, \tag{1.1}$$

$$G_{2c}(z,w) = \left\langle \left(\frac{1}{N} \mathrm{Tr} \frac{1}{z-H} \right) \left(\frac{1}{N} \mathrm{Tr} \frac{1}{w-H} \right) \right\rangle - \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{z-H} \right\rangle \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{w-H} \right\rangle . \tag{1.2}$$

We obtain the correlation function between two eigenvalues from $G_{2c}(z,w)$

$$\begin{aligned}\rho_{2c}(\lambda,\mu) = -\frac{1}{4\pi^2}[&G_{2c}(\lambda+i\epsilon,\mu+i\epsilon) \\ &+G_{2c}(\lambda-i\epsilon,\mu-i\epsilon) \\ &-G_{2c}(\lambda+i\epsilon,\mu-i\epsilon) - G_{2c}(\lambda-i\epsilon,\mu+i\epsilon)].\end{aligned} \tag{1.3}$$

In the simpler case in which H_0 is zero, it was proved in earlier papers [2, 3] that the resulting expression for the connected two-point Green's function is independent, up to a scale, of the probability distribution of the random potential φ for arbitrary values of λ and μ. One word of clarification is needed at this stage. While in Ref. [2] we computed $\rho_{2c}(\lambda,\mu)$ directly, in Ref. [3] we computed $G_{2c}(z_1,z_2)$ by letting N go to infinity first, with complex z_1 and z_2. To obtain the correlation function we then let the imaginary parts of z_1 and z_2 go to zero. By this procedure we obtain a correlation function $\rho_{2c}(\lambda,\mu)$ in which a smoothening of the eigenvalues in a range much larger than $1/N$, but small compared to the average point spacing, has been performed. This smoothening eliminates all the nonuniversal oscillations that are present in $\rho_{2c}(\lambda,\mu)$ if we had let the imaginary parts go to zero first [2] before letting N go to infinity.

For a given nonzero H_0, we could first question whether or not ρ_{2c} depends upon the probability distribution of φ and next how the result depends upon the spectrum of H_0. We shall answer these questions in several steps: first, we return for definiteness to the pure random case in which $H_0 = 0$, which was studied earlier [2]. Then we consider H_0 nonzero and a Gaussian $P(\varphi)$, on the basis of earlier work on the subject [4]. We consider next the non-Gaussian case with arbitrary H_0 and derive first a closed expression for the one-point Green's function. We then apply this closed expression for two specific examples in which the eigenvalues of H_0 are restricted to ± 1 and in which H_0 has uniformly spaced levels between -1 and $+1$. Further we derive the two-point connected Green's function $G_{2c}(z,w)$. We find that the short distance behavior is universal.

The universality of the short distance correlation is perhaps expected from the following heuristic argument. Imagine switching on adiabatically the potential φ. Starting from some initial eigenstate of H_0, the potential will induce a succession of transitions between those eigenstates. For finite time it is expected that this process depends sensitively on the nature of the spectrum of H_0. However, for long times the system will explore all the eigenstates of H_0 and one might wonder, or may even be suspect, whether the limit is insensitive to H_0. Therefore it is natural to consider the limit in which the eigenvalues λ and μ are near each other and we prove that in this limit ρ_{2c} is universal.

II. THE SIMPLE WIGNER CASE

We return to the simplest case in which H_0 is zero and $P(\varphi)$ is a Gaussian distribution characterized by

$$\langle \varphi_{ij} \rangle = 0,$$

$$\langle \varphi_{ij}\varphi_{kl} \rangle = \frac{v^2}{N}\delta_{jk}\delta_{il}. \tag{2.1}$$

Much is known about this Gaussian case of course [5, 6]. We have found that the correlation function in the large-N limit [2] may be written in the compact form

$$N^2 G_{2c}(z_1, z_2) = -\frac{\partial^2}{\partial z_1 \partial z_2}\ln[1 - G(z_1)G(z_2)]. \tag{2.2}$$

It is convenient for later generalization to define

$$u(z) = z - G(z), \tag{2.3}$$

given explicitly in this simple case by

$$u(z) = \frac{1}{2}[z + \sqrt{z^2 - 4}]. \tag{2.4}$$

Since $G(z) = 1/[z - G(z)]$, (2.2) is expressed also by

$$N^2 G_{2c}(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2}\ln\left[\frac{u(z_1) - u(z_2)}{z_1 - z_2}\right]. \tag{2.5}$$

It is useful to introduce also the polarization function

$$\Pi_0(z_1, z_2) = G(z_1)G(z_2). \tag{2.6}$$

We are working here in units in which v, the width of the Gaussian distribution, is one; it is easy by dimensional analysis to set it back to any desired value.

There are several ways of deriving (2.2). In Ref. [2] it was obtained through a study of the asymptotic form of the relevant orthogonal polynomials of high order. In Ref. [4] it was shown that it can be recovered simply by summing the diagrams obtained by expanding in powers of v: they consist of planar connected diagrams made of ladders and crossed ladders, rainbowlike vertex corrections, and full planar propagators.

The form that we give here is not the one that appears in Ref. [2], but it is easy to carry out the differentiations and verify that it is same. By differentiating (2.2), we have

$$N^2 G_{2c}(z_1, z_2)$$
$$= \frac{1}{2(z_1 - z_2)^2}\left(\frac{z_1 z_2 - 4}{[(z_1^2 - 4)(z_2^2 - 4)]^{1/2}} - 1\right), \tag{2.7}$$

from which, using (1.3), we obtain [2, 7]

$$\rho_{2c}(\lambda, \mu) = -\frac{1}{2N^2\pi^2}\frac{1}{(\lambda - \mu)^2}\frac{4 - \lambda\mu}{[(4 - \lambda^2)(4 - \mu^2)]^{1/2}}. \tag{2.8}$$

For λ close to μ it has a singular behavior

$$\rho_{2c}(\lambda, \mu) \simeq -\frac{1}{2\pi^2 N^2(\lambda - \mu)^2}. \tag{2.9}$$

It is instructive to compare it with the exact expression [2], i.e., without smoothening. In the limit $\lambda - \mu$ goes to zero with

$$x = 2\pi N(\lambda - \mu)\rho(\tfrac{1}{2}(\lambda + \mu)) \tag{2.10}$$

fixed. In this limit the exact correlation function becomes [6]

$$\rho_{2c}(\lambda, \mu) \simeq -\frac{1}{\pi^2 N^2}\frac{\sin^2 x}{(\lambda - \mu)^2}. \tag{2.11}$$

The smoothening has thus replaced $\sin^2 x$ by $1/2$, thereby producing a double pole.

III. GENERAL PROBABILITY DISTRIBUTION WITHOUT H_0

We now go to an arbitrary probability distribution for the disorder. The density of eigenvalues is no longer semicircular; we limit ourselves here to spectra that extend over a single segment of length $4a$ on the real axis. We choose the origin at the center of this segment. This is automatic when the probability distribution $P(\varphi)$ is even; otherwise one would have to shift the eigenvalues. Then we know from earlier work [2, 7, 8] that, rather remarkably, (2.5) is still valid with

$$u(z) = \frac{1}{2}[z + \sqrt{z^2 - 4a^2}] \tag{3.1}$$

and hence

$$\rho_{2c}(\lambda, \mu) = -\frac{1}{2N^2\pi^2}\frac{1}{(\lambda - \mu)^2} \times \frac{4a^2 - \lambda\mu}{[(4a^2 - \lambda^2)(4a^2 - \mu^2)]^{1/2}}. \tag{3.2}$$

The non-Gaussian terms of $P(\varphi)$ have stretched the support of the spectral density from $[-2, 2]$ to $[-2a, 2a]$ and also deformed the density of eigenvalues from Wigner's semicircle to a polynomial multiplying a squareroot [1]. However, the only effect of these non-Gaussian terms on $G_{2c}(z_1, z_2)$ and $\rho_{2c}(\lambda, \mu)$ is a simple rescaling $z_{1,2} \to \frac{z_{1,2}}{a}$, $\lambda \to \frac{\lambda}{a}$, $\mu \to \frac{\mu}{a}$, followed by multiplying G_{2c} and ρ_{2c} by $(1/a)^2$.

To derive these results we had to proceed in two steps. First [2], we considered a probability distribution

$$P(\varphi) = \frac{1}{Z}\exp[-N\mathrm{Tr}V(\varphi)] \tag{3.3}$$

in which $V(\varphi)$ was an arbitrary polynomial of φ. In these cases we could diagonalize φ, integrate out the unitary group, and handle the eigenvalues of φ by orthogonal polynomial techniques [6, 10] (or Dyson's two-dimensional electrostatic approach [9, 11]). It was then extended to nonunitary invariant measures or equivalently to a measure in which $\mathrm{Tr}V(\varphi)$ was replaced by a sum of arbitrary products of traces of powers of φ [3].

If we were to try again to calculate G_{2c} by expanding the non-Gaussian terms of the measure and summing the planar diagrams, we would have to deal with expectation values of the type $\langle \mathrm{Tr}\varphi^n \mathrm{Tr}\varphi^m \rangle$ expanded in powers of the non-Gaussian terms. An equivalent way of stating the results above is the following. Call $\alpha_{n,m}$ the Gaussian connected expectation value

$$\alpha_{n,m} = \langle \mathrm{Tr}\varphi^n \mathrm{Tr}\varphi^m \rangle_{c,0}. \tag{3.4}$$

For arbitrary n and m it is rather cumbersome to determine these numbers. They are of order one in the large-N limit, whereas the disconnected part is of order N^2. If needed explicitly, we could obtain them by expanding their generating function

$$N^2 G_{2c}(z_1, z_2) = \sum_{n,m} \alpha_{n,m} z_1^{-n-1} z_2^{-m-1} \tag{3.5}$$

given by (2.5), in inverse powers of the z's.

We now consider the same expectation value $\langle \mathrm{Tr}\varphi^n \mathrm{Tr}\varphi^m \rangle_c$ for the non-Gaussian case. The result of (3.2) is equivalent to the statement

$$\langle \mathrm{Tr}\varphi^n \mathrm{Tr}\varphi^m \rangle_c = a^{n+m} \alpha_{n,m} \tag{3.6}$$

in which $2a$ is again the end point of the support of the spectral density.

A direct derivation of these identities is nontrivial. For specific values of m and n they may be extracted from what is known about $G(z)$. See Appendix A.

IV. ONE-POINT GREEN'S FUNCTION WITH NONZERO H_0

We now have a given unperturbed Hamiltonian H_0, with eigenvalues ϵ_i, $i = 1, ..., N$, with N large as always. The one-point Green's function, or the average resolvent, was first found by Pastur [12] for the Gaussian distribution $P(\varphi)$. Define the deterministic unperturbed one-point Green's function

$$G_0(z) = \frac{1}{N} \mathrm{Tr} \frac{1}{z - H_0} = \frac{1}{N} \sum_i \frac{1}{z - \epsilon_i}. \tag{4.1}$$

Then the full one-point Green's function

$$G(z) = \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{z - H} \right\rangle \tag{4.2}$$

is obtained by solving the implicit equation

$$G(z) = G_0(z - G(z)). \tag{4.3}$$

If one insists on writing an explicit formula for $G(z)$ in terms of $G_0(z)$, one can write down the integral representation

$$G(z) = \oint \frac{du}{2\pi i} \left(u \frac{d}{du} \ln[u - G_0(z - u)] \right). \tag{4.4}$$

The contour circles around the solution of the equation, which behaves as $1/z$ at infinity. In other words, the one-point Green's function is determined by

$$G(z) = \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{z - H_0 - \varphi} \right\rangle = \frac{1}{N} \sum_i \frac{1}{z - \epsilon_i - G(z)}. \tag{4.5}$$

In the simplest case, in which $H_0 = 0$, Eq. (4.5) becomes a simple quadratic equation for G, from which one recovers immediately Wigner's semicircle law for the imaginary part of G, which is proportional to the density of eigenvalues. It is easy to give a diagrammatic proof [4] of (4.5). In the large-N limit, the planar diagrams are the rainbow diagrams whose sum leads directly to (4.5).

The self-energy part $\Sigma(z)$ of the one-point Green's function $G(z)$ is simply given for a Gaussian distribution $P(\varphi)$ by

$$\Sigma(z) = G(z). \tag{4.6}$$

For the non-Gaussian case, the self-energy part $\Sigma(z)$, defined by

$$G(z) = \frac{1}{N} \sum_i \frac{1}{z - \epsilon_i - \Sigma(z)}, \tag{4.7}$$

is of course no longer simply equal to $G(z)$.

Here we will restrict ourselves to the non-Gaussian probability distribution for

$$P(\varphi) = \frac{1}{Z} \exp[-N(\tfrac{1}{2}\mathrm{Tr}\varphi^2 + g\mathrm{Tr}\varphi^4)]. \tag{4.8}$$

To treat the non-Gaussian term $g\mathrm{Tr}\varphi^4$, it is convenient to consider an "equation of motion" obtained by shifting the random matrix by an arbitrary matrix φ to $\varphi + \epsilon$ in the integral, which gives $\langle [1/(z - H_0 - \varphi)]_{ij} \rangle$. The invariance of this integral under this shift tells us that to first order in ϵ, the coefficient of ϵ_{nm} must vanish:

$$\left\langle \left(\frac{1}{z - H_0 - \varphi} \right)_{in} \left(\frac{1}{z - H_0 - \varphi} \right)_{mj} \right\rangle - N \left\langle \left(\frac{1}{z - H_0 - \varphi} \right)_{ij} [\varphi_{mn} + 4g(\varphi^3)_{mn}] \right\rangle = 0. \tag{4.9}$$

Setting $n = i$ and $m = j$ and summing over these indices, we obtain the following equation with $H = H_0 + \varphi$:

$$\left\langle \left(\frac{1}{N} \mathrm{Tr} \frac{1}{z - H} \right)^2 \right\rangle = \left\langle \frac{1}{N} \mathrm{Tr} \left[\frac{1}{z - H} (\varphi + 4g\varphi^3) \right] \right\rangle. \tag{4.10}$$

By the definition of two-point connected Green's function, we write the above equation as

$$NG^2(z) + NG_{2c}(z, z) = \left\langle \mathrm{Tr} \left(\frac{1}{z - H} \varphi \right) \right\rangle + 4g \left\langle \mathrm{Tr} \left(\frac{1}{z - H} \varphi^3 \right) \right\rangle, \tag{4.11}$$

where $G_{2c}(z,z)$ is order of $1/N^2$. The first term on the right-hand side of this equation may be rearranged by writing $\varphi = (\varphi + H_0 - z) + (z - H_0)$:

$$\frac{1}{N}\left\langle \mathrm{Tr}\left(\frac{1}{z-H_0-\varphi}\varphi\right)\right\rangle$$

$$= -1 + \frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{z-H_0-\varphi}(z-H_0)\right\rangle$$

$$= -1 + \frac{1}{N}\sum_i \frac{z-\epsilon_i}{z-\epsilon_i-\Sigma(z)}$$

$$= \Sigma(z)G(z), \tag{4.12}$$

with the self-energy $\Sigma(z)$ defined above. Repeating the same procedure, we write the second term on the right-hand side of (4.6)

$$\left\langle \mathrm{Tr}\left(\frac{1}{z-H_0-\varphi}\varphi^3\right)\right\rangle$$

$$= -\left\langle \mathrm{Tr}\varphi^2\right\rangle + \left\langle \mathrm{Tr}\left[\frac{1}{z-H_0-\varphi}(z-H_0)\,\varphi(z-H_0)\right]\right\rangle . \tag{4.13}$$

We have omitted the term $\langle \mathrm{Tr}\varphi(z-H_0)\rangle$ since it is zero by being odd in φ.

We will now consider the relevant Feynman diagrams. As in Ref. [4], we find it convenient to use the language of large-N QCD to describe the diagrams. Let us introduce the gluon-quark-quark vertex function $\Gamma(z)$ defined to include gluon self-energy corrections. In other hands, it is one-particle irreducible in the quark lines, but not in the gluon line. Then we can write the second term in Eq. (4.13) as

$$\sum_i\sum_j(z-\epsilon_i)(z-\epsilon_j)\left\langle\left(\frac{1}{z-H_0-\varphi}\right)_{ij}\varphi_{ji}\right\rangle$$

$$= \frac{1}{N}\sum_i\sum_j(z-\epsilon_i)(z-\epsilon_j) \times\frac{1}{[z-\epsilon_i-\Sigma(z)][z-\epsilon_j-\Sigma(z)]}\Gamma(z)$$

$$= N\Gamma(z)[1+\Sigma(z)G(z)]^2. \tag{4.14}$$

As shown in Fig. 1, in the planar limit the vertex function $\Gamma(z)$, the (quark) self-energy $\Sigma(z)$ and the Green's function or quark propagator $G(z)$ are related by

$$\Sigma(z) = \Gamma(z)G(z). \tag{4.15}$$

Using this fact and putting everything together, we find that $G(z)$, for arbitrary nonzero H_0 and for a non-Gaussian probability distribution for φ, is determined by

$$G^2(z) = \Sigma(z)G(z) - 4gx + 4g\frac{\Sigma(z)}{G(z)}[1+\Sigma(z)G(z)]^2, \tag{4.16}$$

where we denote $\langle \mathrm{Tr}\varphi^2\rangle$ by x. This quantity x does not depend on H_0 and is simply related to the end point $a(g)$ of the spectrum for the $H_0 = 0$ case by

$$x = \frac{a^2}{3}(4-a^2), \tag{4.17}$$

where $a(g)$ satisfies the equation [1]

$$12ga^4 + a^2 = 1. \tag{4.18}$$

Thus we have accomplished our goal of finding Eq. (4.16), which determines the one-point Green's function $G(z)$ in general. Note that the self-energy Σ that appears in (4.16) is itself related to $G(z)$ as described in (4.7). We could replace $G(z)$ in (4.16) by $G_0(z-\Sigma)$ and solve explicitly (4.16) for Σ by a contour integral, but it is not clear that this would be the easiest way to numerically handle these equations. In any case, one does not expect any interesting behavior of the density of state for generic probability distributions. It depends very sensitively upon H_0 and $P(\varphi)$, except near the edge, at which one obtains in general a square root behavior as for Wigner's semicircle law. This will be illustrated numerically in two examples.

We now consider two simple cases as examples, specified by the eigenvalues ϵ_i of H_0: (i) $\epsilon_i = \epsilon$ for $i = 1,...,N/2$ and $\epsilon_i = -\epsilon$ for $i = (N/2)+1,...,N$, with N even, and (ii) the ϵ_i are uniformly spaced over an interval from $-\alpha$ to α.

For the first case (i), we have, from (4.7),

$$G = \frac{1}{2}\left(\frac{1}{z-\epsilon-\Sigma}+\frac{1}{z+\epsilon-\Sigma}\right) . \tag{4.19}$$

In the case $g = 0$, (4.16) tells us simply $\Sigma = G$ and thus we obtain the cubic equation

$$(z-G)^2G + (1-\epsilon^2)G - z = 0. \tag{4.20}$$

The imaginary part of $G(z)$ is represented in Fig. 2 by a solid line for the particular case $\epsilon = 1$. The end points of the spectrum occur at $z = \pm\sqrt{27}/2$. The point $z = 0$ becomes the end point also for $g = 0$. For $g \neq 0$, we consider the following representative values for $\epsilon = 1$: (a) $g = 1/2$, $x = 11/27$ and (b) $g = -1/48$, $x = 4/3$, and solve numerically. We represent the imaginary parts of $G(z)$ in Fig. 2 by a dotted line and by a dashed line, respectively. We have a fifth-order polynomial equation for Σ by substituting (4.19) into (4.16). At the critical value $g = -1/48$, we find the imaginary parts of Σ and G, which vanish at $z_c \cong 3.225$ with the exponent 3/2,

$$\mathrm{Im}G(z) \simeq (z_c - z)^{3/2}, \tag{4.21}$$

which shows the same singularity as the $H_0 = 0$ case at the edge [1, 14].

In the second example (ii), we have, from (4.7),

$$z = \Sigma(z) + \alpha\coth[\alpha G(z)]. \tag{4.22}$$

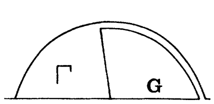

FIG. 1. Self-energy $\Sigma(z)$ expressed by the product of the vertex parts $\Gamma(z)$ and $G(z)$.

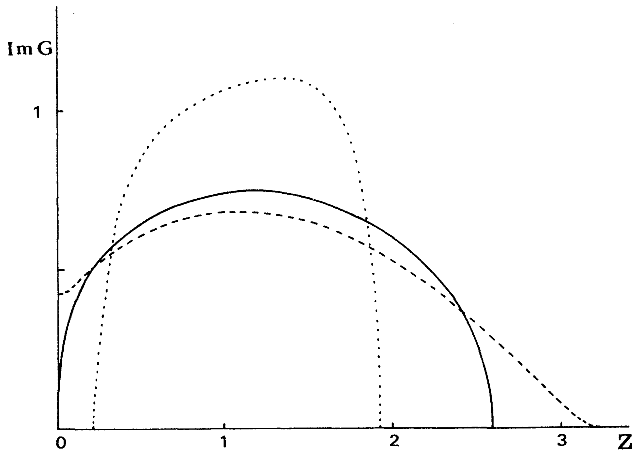

FIG. 2. Solid line, imaginary part of the one-point Green's function $G(z)$, for $g = 0$; dotted line, imaginary $G(z)$ for $g = 1/2$; dashed line, imaginary $G(z)$ for $g = -1/48$. In all three cases, the eigenvalues of H_0 are ± 1. There is a symmetric counterpart for $z < 0$.

We consider further the case $\alpha = 1$.The imaginary part of $G(z)$ is evaluated numerically and represented in Fig. 3. At the critical value $g = -1/48$, the imaginary part of $G(z)$ shows the same singularity at the edge as before. Note that the value $g = -1/48$ always remains critical, independent of H_0. This is due to the singular behavior of x given by (4.16) near $g_c = -1/48$, which is independent of H_0. We have considered the non-Gaussian probability distribution given by (4.8). It is easy to apply the present method to the other non-Gaussian $P(\varphi)$ case.

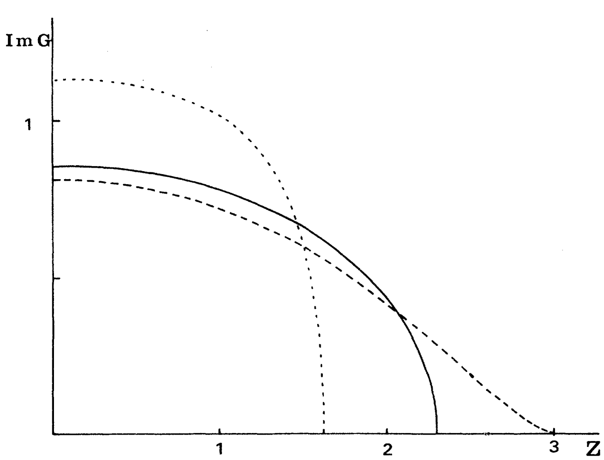

FIG. 3. Imaginary part of the one-point Green's function $G(z)$, for the case in which the eigenvalues of H_0 are given uniformly between -1 and 1. The solid line shows the case of $g = 0$, the dotted line corresponds to $g = 1/2$ and $x = 11/27$, and the dashed line shows the case $g = -1/48$ and $x = 4/3$.

V. TWO-POINT CORRELATION FUNCTION WITH NONZERO H_0 AND φ GAUSSIAN

We now finally come to the two-point connected Green's function, or correlation function, $G_{2c}(z,w)$. In an earlier work [4] we have shown that for a Gaussian distribution $P(\varphi)$ we can determine $G_{2c}(z,w)$ as follows. First, we define

$$g_i(z) = \frac{1}{z - \epsilon_i - \Sigma(z)}. \tag{5.1}$$

For the Gaussian case, the self-energy $\Sigma = G(z)$. For notational convenience we also define the "circle product"

$$G^n(z) \circ G^m(w) = \frac{1}{N} \sum_{i=1}^{N} g_i^n(z) g_i^m(w). \tag{5.2}$$

Then in the large-N limit, we found

$$N^2 G_{2c}(z,w) = \left(\frac{G^2(z) \circ G^2(w)}{1 - G(z) \circ G(w)} + \frac{[G^2(z) \circ G(w)][G(z) \circ G^2(w)]}{[1 - G(z) \circ G(w)]^2} \right) \times \frac{1}{1 - G(z) \circ G(z)} \frac{1}{1 - G(w) \circ G(w)}. \tag{5.3}$$

This result was derived by summing all the relevant planar diagrams, consisting of generalized ladders, with arbitrary cyclic permutations of the rungs, with fully dressed planar propagators and planar, i.e., rainbowlike, vertex corrections. (We have again set v, the width of the random distribution, equal to one.)

Surprisingly enough, we can show that this rather complicated expression can again be written simply in the form

$$N^2 G_{2c}(z_1, z_2) = -\frac{\partial^2}{\partial z_1 \partial z_2} \ln[1 - G(z_1) \circ G(z_2)]. \tag{5.4}$$

Note that this expression has the same form as (2.2), but with the ordinary product replaced by the circle product. Defining $u(z) = z - G(z)$ as before and using the identity

$$1 - G(z) \circ G(w) = \frac{z - w}{u(z) - u(w)}, \tag{5.5}$$

we can also write

$$N^2 G_{2c}(z_1, z_2) = \frac{\partial^2}{\partial z_1 \partial z_2} \ln \left(\frac{u(z_1) - u(z_2)}{z_1 - z_2} \right), \tag{5.6}$$

just as in (2.5). To derive these compact representations, we simply differentiate (5.4) to obtain the previous result (5.3), using the identity

$$\frac{dG(z)}{dz} = -\frac{G(z) \circ G(z)}{1 - G(z) \circ G(z)}. \tag{5.7}$$

VI. TWO-POINT CORRELATION FUNCTION WITH NONZERO H_0 AND φ NON-GAUSSIAN

The remarkable existence of such a compact representation as (5.6) naturally prompts us to ask whether this representation also holds for a general non-Gaussian distribution $P(\varphi)$. In this section, we answer this question partially by studying $G_{2c}(z,w)$ for the distribution

$$P(\varphi) = \frac{1}{Z}\exp[-N(\tfrac{1}{2}\mathrm{Tr}\varphi^2 + g\mathrm{Tr}\varphi^4)] \tag{6.1}$$

to first order in g.

As contributions of order g, there are six different classes of diagrams, with their typical representatives shown in Fig. 4. The contributions of these six classes of diagrams to $G_{2c}(z,w)$ are

$$\begin{aligned}
D_a &= -8g\frac{\partial^2}{\partial w\partial z}\left[\frac{G(z)\circ G(w)}{1-G(z)\circ G(w)}\right],\\
D_b &= -4g\frac{\partial^2}{\partial w\partial z}[G(z)G(w)],\\
D_c &= -4g\frac{\partial^2}{\partial w\partial z}\left[\frac{G(z)\circ G(w)}{1-G(z)\circ G(w)}G(z)G(w)\right],\\
D_d &= -4g\frac{\partial^2}{\partial w\partial z}\left[\frac{G(z)\circ G(w)}{1-G(z)\circ G(w)}[G^2(z)+G^2(w)]\right],\\
D_e &= -8g\frac{\partial^2}{\partial w\partial z}\left[\frac{G(z)}{1-G(z)\circ G(z)}\frac{G^2(z)\circ G(w)}{1-G(z)\circ G(w)}\right.\\
&\quad\left.+\frac{G(w)}{1-G(w)\circ G(w)}\frac{G^2(w)\circ G(z)}{1-G(z)\circ G(w)}\right],\\
D_f &= -4g\frac{\partial^2}{\partial w\partial z}\left[\frac{G^3(z)}{1-G(z)\circ G(z)}\frac{G^2(z)\circ G(w)}{1-G(z)\circ G(w)}\right.\\
&\quad\left.+\frac{G^3(w)}{1-G(w)\circ G(w)}\frac{G^2(w)\circ G(z)}{1-G(z)\circ G(w)}\right].
\end{aligned}\tag{6.2}$$

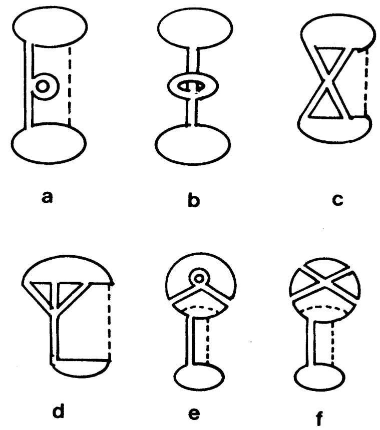

FIG. 4. Diagrams of order g for $G_{2c}(z,w)$. The dashed line represents the ladder of φ propagators.

To order g, the one-point Green's function $G(z)$ that appears here can of course be taken to be the lowest-order Green's function, which we denote by $G_0(z)$ [note that this $G_0(z)$ is different from the one defined in (4.1), where we used it for the deterministic unperturbed one-point Green function]. For the sake of notational simplicity, however, we will omit the subscript 0 here. In other words, $G(z)$ is determined by

$$G(z) = \frac{1}{N}\sum_i \frac{1}{z-\epsilon_i - G(z)}. \tag{6.3}$$

Summing these terms $D_a, ..., D_f$ and adding this to the unperturbed term $N^2G_{2c}(z,w) = -\frac{\partial^2}{\partial w\partial z}\ln[1-G(z)\circ G(w)]$, we obtain

$$\begin{aligned}
N^2G_{2c}(z,w) = -\frac{\partial^2}{\partial w\partial z}\ln\Bigg\{1 - G(z)\circ G(w) + 4g\Bigg[&G(z)\circ G(w)[2+G^2(z)+G^2(w)]\\
&+G(z)G(w) + G^2(z)\circ G(w)\left(\frac{2G(z)+G^3(z)}{1-G(z)\circ G(z)}\right) + G^2(w)\circ G(z)\left(\frac{2G(w)+G^3(w)}{1-G(w)\circ G(w)}\right)\Bigg]\Bigg\} + O(g^2).
\end{aligned}\tag{6.4}$$

We note, that to lowest order in g,

$$G^2(z)\circ G(w) = \frac{G(z)\circ G(z) - G(z)\circ G(w)}{w-z-G(w)+G(z)} = \frac{G(z)\circ G(z) - 1}{w-z-G(w)+G(z)} + \frac{w-z}{[w-z-G(w)+G(z)]^2} \tag{6.5}$$

and

$$G(z)\circ G(w) = \frac{G(z)-G(w)}{w-z-G(w)+G(z)}. \tag{6.6}$$

Then (6.4) may be written as

$$\begin{aligned}
N^2G_{2c}(z,w) = -\frac{\partial^2}{\partial w\partial z}\ln\Bigg[\left(\frac{z-w}{z-w+G(w)-G(z)}\right)\Bigg(&1 + 4gG(z)G(w) - 4g\frac{G(z)[2+G^2(z)]}{[z-w-G(z)+G(w)][1-G(z)\circ G(z)]}\\
&+4g\frac{G(w)[2+G^2(w)]}{[z-w-G(z)+G(w)][1-G(w)\circ G(w)]}\Bigg)\Bigg].
\end{aligned}\tag{6.7}$$

The last two terms of Eq. (6.7) come from parts of diagrams of classes (e) and (f) in Fig. 4.

We would now like to reexpress this result using the exact one-point Green's function as far as possible. Using the notation $G_0(z)$ for the one-point Green function for nonzero H_0 and for the Gaussian distribution ($g = 0$), we find that the self-energy $\Sigma(z)$ of the non-Gaussian distribution $P(\varphi)$ is given by

$$\Sigma(z) = G_0(z) - \frac{8gG_0(z) + 4gG_0^3(z)}{1 - G_0(z) \circ G_0(z)} + O(g^2). \quad (6.8)$$

Then the last two terms of (6.7) are absorbed by the expression of the self-energy $\Sigma(w)$ and $\Sigma(z)$. Therefore, we finally obtain the following simple expression, up to order g,

$$N^2 G_{2c}(z,w) = -\frac{\partial^2}{\partial z \partial w} \ln\left[\left(\frac{z-w}{z-w-\Sigma(z)+\Sigma(w)}\right) \times [1 + 4gG(z)G(w)]\right]$$
$$= -\frac{\partial^2}{\partial z \partial w} \ln\left[\left(\frac{z-w}{u(z)-u(w)}\right) \times [1 + 4gG(z)G(w)]\right] + O(g^2), \quad (6.9)$$

where we now define

$$u(z) = z - \Sigma(z) \quad (6.10)$$

as the appropriate generalization of (2.3).

We have thus obtained the same form as in (5.6), except for an extra factor of $[1 + 4gG(z)G(w)]$. Although this form is quite involved, it makes it clear that the short distance singularity of the smoothened $\rho_{2c}(z,w)$ is not affected by the non-Gaussian character of the distribution. It is thus truly universal, in the sense that it does not depend upon the spectrum of H_0 or the distribution of φ. In Appendix B we show that this expression is consistent with the known result for the $H_0 = 0$ case. Since this factor appears inside ln, we separate this term as $-4g[\partial_z G(z)][\partial_w G(w)]$. This term seems to be the unconnected part for two-point Green's function. Thus if we neglect this factorized term, we have the same form as (2.5). The general expression for the connected two-point Green's function may be written as

$$G_{2c}(z,w) = -\frac{\partial}{\partial z}\frac{\partial}{\partial w} \ln[1 - G(z) \circ G(w)\Gamma^{(1)}(z,w)] + \frac{\partial}{\partial z}\frac{\partial}{\partial w} G(z)G(w)\Gamma^{(2)}(z,w). \quad (6.11)$$

The second term represents the diagram class (b) in Fig. 4. The fact that the terms in (6.2) all collect into the relatively simple form of (6.9) suggests to us that some variation of this form may in fact hold to all orders in g and possibly even to arbitrary $P(\varphi)$. We do not have a proof of all this tempting conjecture at this point.

VII. DISCUSSION

Thus far we have considered the correlation function ρ_{2c} for arbitrary values of λ and μ, in which case all four terms on the right-hand side of (1.3) contribute. In the limit in which $\lambda - \mu$ tends to zero, one gets a singularity that is entirely due to the Green's functions with opposite signs of their infinitesimal imaginary parts. Indeed $[u(z) - u(w)]/(z - w)$ is not singular when w approaches z, with z and w on the same side of the cut. However, if the imaginary parts of z and w are opposite, and when they approach the real axis on the support of the spectral density of the resolvent, the ratio drastically increases at a short distance. The residue of the singularity depends upon the imaginary part of u, but it disappears when we take the derivative of the logarithm. Therefore if λ and μ are such that $\rho(\lambda)$ and $\rho(\mu)$ are nonzero, when μ is close to λ we have a double pole with a universal residue minus one:

$$N^2 G_{2c}(\lambda \pm i\epsilon, \mu \mp i\epsilon) \cong \frac{-1}{(\lambda-\mu)^2}. \quad (7.1)$$

Consequently, for arbitrary H_0, we have, as in the simple Wigner ensemble, a behavior

$$\rho_{2c}(\lambda,\mu) \cong -\frac{1}{2\pi^2 N^2 (\lambda-\mu)^2}, \quad (7.2)$$

which, remarkably enough, is totally independent of any specific characteristic of the problem. We have proved this for a distribution $P(\varphi)$ defined with a quartic V to lowest order in the quartic coupling, but we are tempted to conjecture that this short distance universality in fact holds to all orders and perhaps even for arbitrary V. We close with a few concluding remarks.

We have checked that the same result (2.2) holds for a model that we have considered recently [15], in which random matrices are made of independent random blocks, as when they are attached to a lattice. In that case the result comes out immediately from the explicit expression given in that work. We obtain

$$(CN)^2 G_{2c}(z_1,z_2) = -\frac{\partial^2}{\partial z_1 \partial z_2} \sum_k \ln[1 - \epsilon_k G(z_1)G(z_2)], \quad (7.3)$$

with C the number of lattice sites and ϵ_k the single-particle Bloch energies as defined in Ref. [15]. Similarly, in Ref. [4] we considered a situation in which the random matrix φ depends on a parameter called time. Again, the explicit expression given in that work may be written in the form

$$N^2 G_{2c}(z_1,z_2) = -\frac{\partial^2}{\partial z_1 \partial z_2} \ln[1 - e^{-u(t)} G(z_1)G(z_2)], \quad (7.4)$$

where $u(t)$ is a function of time defined in Ref. [4]. In fact, the universal form discussed in this paper seems to hold in every problem with some random features.

As noted for the simple Wigner case the short distance singularity that one finds in (7.1) for the smoothened correlation function is spurious: the true $\rho_{2c}(\lambda,\mu)$ is finite at short distance, but the smoothening has replaced some vanishing numerator by a constant average. This justifies the procedure followed in the literature in computing the universal fluctuations in mesoscopic systems: the smoothened function correlation is used with some cutoff at short distance. An integration by parts [16], particularly easy since G_{2c} takes the form of a second derivative, allows one to return to the integrable logarithmic singularity. The short distance cutoff may then be removed.

Consequently, the interesting feature of these results is not so much the universal nature of the short distance spurious singularity, but the general representation (2.2) of the two-point Green's function.

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APPENDIX A: CALCULATION OF $\langle \mathrm{Tr}\varphi^{2n}\mathrm{Tr}\varphi^4\rangle_c$

Here we will calculate $\langle \mathrm{Tr}\varphi^{2n}\mathrm{Tr}\varphi^4\rangle_c$ for the probability distribution

$$P(\varphi)=\frac{1}{Z}\exp[-N\mathrm{Tr}(\tfrac{1}{2}\varphi^2+g\varphi^4)]. \tag{A1}$$

The one-point Green's function

$$G(z)=\left\langle \frac{1}{N}\mathrm{Tr}\left(\frac{1}{z-\varphi}\right)\right\rangle \tag{A2}$$

satisfies the equation [13]

$$G^2(z)-(z+4gz^2)G(z)+4gz^2+\frac{(a^2-2)^2}{9a^2}=0, \tag{A3}$$

with the end points $\pm 2a$ of the spectrum given by

$$12ga^4+a^2-1=0. \tag{A4}$$

From $G(z)$ we obtain, after a tedious but simple calculation,

$$\left\langle \frac{1}{N}\mathrm{Tr}\varphi^{2n}\right\rangle=\frac{(2n)!}{n!(n+2)!}a^{2n}[2n+2-na^2]. \tag{A5}$$

Then one has

$$\begin{aligned}-\frac{\partial}{\partial g}\left\langle \frac{1}{N}\mathrm{Tr}\varphi^{2n}\right\rangle&=\langle \mathrm{Tr}\varphi^{2n}\mathrm{Tr}\varphi^4\rangle_c\\&=\frac{(2n)!}{n!(n+2)!}n(n+1)(2-a^2)\\&\quad\times a^{2n-2}\frac{\partial a^2}{\partial g}.\end{aligned} \tag{A6}$$

Taking $\partial a^2/\partial g=-12a^6/(2-a^2)$ from (3.10), we obtain, as announced

$$\begin{aligned}\langle \mathrm{Tr}\varphi^{2n}\mathrm{Tr}\varphi^4\rangle_c&=a^{2n+4}\frac{12}{n+2}\frac{(2n)!}{(n!)^2}\\&=a^{2n+4}\langle \mathrm{Tr}\varphi^{2n}\mathrm{Tr}\varphi^4\rangle_{c0}.\end{aligned} \tag{A7}$$

The method extends to higher powers of m, but becomes very tedious. The indirect derivations of Refs. [7,10] are of course easier.

APPENDIX B: TWO-POINT GREEN'S FUNCTION

The result of two-point correlation function (5.4) with nonzero H_0 and with a Gaussian distribution $P(\varphi)$ is also derived by considering an equation of motion, which is obtained by shifting the arbitrary random matrix φ to $\varphi+\epsilon$. The propagator $[1/(z-H_0-\varphi)]_{ij}[1/(w-H_0-\varphi)]_{kl}$ has the corrections of order ϵ by this shift and they become

$$\begin{aligned}&\left(\frac{1}{z-H}\right)_{in}\epsilon_{nm}\left(\frac{1}{z-H}\right)_{mj}\left(\frac{1}{w-H}\right)_{kl}\\&\quad+\left(\frac{1}{z-H}\right)_{ij}\left(\frac{1}{w-H}\right)_{kn}\epsilon_{nm}\left(\frac{1}{w-H}\right)_{ml}-\left(\frac{1}{z-H}\right)_{ij}\left(\frac{1}{w-H}\right)_{kl}[\epsilon_{nm}\varphi_{mn}+4g\epsilon_{nm}(\varphi^3)_{mn}].\end{aligned} \tag{B1}$$

By setting $n=i$, $m=j$, and $k=l$ and summing over these indices, we obtain

$$\left\langle\left(\frac{1}{N}\mathrm{Tr}\frac{1}{z-H}\right)^2\left(\frac{1}{N}\mathrm{Tr}\frac{1}{w-H}\right)\right\rangle+\left\langle\frac{1}{N}\mathrm{Tr}\frac{1}{z-H}\left(\frac{1}{w-H}\right)^2\right\rangle-\left\langle\frac{1}{N}\mathrm{Tr}\left(\frac{1}{z-H}(\varphi+4g\varphi^3)\right)\frac{1}{N}\mathrm{Tr}\frac{1}{w-H}\right\rangle=0. \tag{B2}$$

The second term of Eq. (B2) is written as

$$\left\langle \mathrm{Tr}\left[\frac{1}{z-H}\left(\frac{1}{w-H}\right)^2\right]\right\rangle$$

$$= -\frac{\partial}{\partial w}\left\langle \mathrm{Tr}\left(\frac{1}{z-H}\frac{1}{w-H}\right)\right\rangle$$

$$= -\frac{\partial}{\partial w}\left(\frac{G(z)-G(w)}{w-z}\right). \qquad \text{(B3)}$$

By the factorization in the large-N limit, the first term of (B2) is given by

$$\left\langle \left(\frac{1}{N}\mathrm{Tr}\frac{1}{z-H}\right)^2\left(\frac{1}{N}\mathrm{Tr}\frac{1}{w-H}\right)\right\rangle$$

$$= G^2(z)G(w) + 2G(z)G_{2c}(z,w) + G_{2c}(z,z)G(w) + O\left(\frac{1}{N^2}\right). \qquad \text{(B4)}$$

From (4.11) and (B2), we have

$$2G(z)G_{2c}(z,w) + \frac{\partial}{\partial w}\left(\frac{G(z)-G(w)}{z-w}\right) - \left\langle \mathrm{Tr}\left(\frac{1}{z-H}(\varphi+4g\varphi^3)\right)\mathrm{Tr}\frac{1}{w-H}\right\rangle_c = 0. \qquad \text{(B5)}$$

We consider the two-point Green's function $G_{2c}(z,w)$ based on this equation. First, we consider the simplest case $H_0 = 0$ and $g = 0$. In this case, we write the equation for $G_{2c}(z,w)$,

$$[2G(z)-z]G_{2c}(z,w) = -\frac{\partial}{\partial w}\left(\frac{G(z)-G(w)}{z-w}\right). \qquad \text{(B6)}$$

Since $G(z) = 1/[z-G(z)]$ in the case $H_0 = 0$ and $g = 0$, we have

$$\frac{1}{2G(z)-z} = \frac{G(z)}{G^2(z)-1} = -\frac{1}{\sqrt{z^2-4}}. \qquad \text{(B7)}$$

Taking the derivative of w in (B6), we have

$$G_{2c}(z,w) = \frac{1}{\sqrt{z^2-4}}\frac{1}{(z-w)^2} \times \left(G(z) - G(w) + (z-w)\frac{G^2(w)}{1-G^2(w)}\right)$$

$$= \frac{1}{2(z-w)^2}\left(\frac{zw-4}{\sqrt{z^2-4}\sqrt{w^2-4}} - 1\right). \qquad \text{(B8)}$$

This coincides with the previous expression given by (2.7).

For the $H_0 \neq 0$ and $g = 0$ case, we have, from (B2),

$$[2G(z)-z]G_{2c}(z,w) + \frac{\partial}{\partial w}\left(\frac{G(z)-G(w)}{z-w}\right) + \left\langle \mathrm{Tr}\left(\frac{1}{z-H_0-\varphi}H_0\right)\mathrm{Tr}\frac{1}{w-H_0-\varphi}\right\rangle_c = 0. \qquad \text{(B9)}$$

We diagonalize the matrix H_0 with the eigenvalues ϵ_i in the last term. The diagrams of this term are classified into four different types (Fig. 5). We note that if we replace ϵ_i by $[\epsilon_i + \Sigma(z) - z] + [z - \Sigma(z)]$ in all four types of diagrams in Fig. 5, we have only an identity without any result for $G_{2c}(z,w)$. Namely, in this case we have, after calculations,

$$\left\langle \mathrm{Tr}\left(\frac{1}{z-H_0-\varphi}H_0\right)\mathrm{Tr}\frac{1}{w-H_0-\varphi}\right\rangle_c$$

$$= [z-2G(z)]G_{2c}(z,w) + \frac{1}{1-G(z)\circ G(w)}\frac{\partial}{\partial w}\ln[1-G(z)\circ G(w)]. \qquad \text{(B10)}$$

Inserting this into (B9), we find that $G_{2c}(z,w)$ cancels out and obtain the identity equation

$$\frac{\partial}{\partial w}\left(\frac{G(z)-G(w)}{z-w}\right) = \frac{1}{1-G(z)\circ G(w)}\frac{\partial}{\partial w}\ln[1-G(z)\circ G(w)]. \qquad \text{(B11)}$$

To get the expression for $G_{2c}(z,w)$, we write ϵ_i separately for the diagrams of Figs. 5(a) and 5(c) as $\epsilon_i = [\epsilon_i + \Sigma(w) - w] + [w - \Sigma(w)]$ and for Figs. 5(b) and 5(d) $\epsilon_i = [\epsilon_i + \Sigma(z) - z] + [z - \Sigma(z)]$. Then we have the following expression by summing the contributions of Figs. 5(a)–5(d):

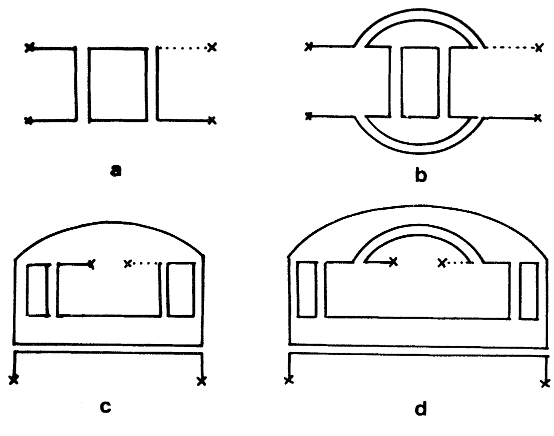

FIG. 5. Four different diagrams of $\langle \mathrm{Tr}(\frac{1}{z-H_0-\varphi}H_0)\mathrm{Tr}\frac{1}{w-H_0-\varphi}\rangle$. The dotted line denotes the eigenvalue ϵ_i of H_0.

$$\left\langle \mathrm{Tr}\left(\frac{1}{z-H_0-\varphi}H_0\right)\mathrm{Tr}\frac{1}{w-H_0-\varphi}\right\rangle_c$$

$$= [w-\Sigma(w)-G(z)]G_{2c}(z,w)+[z-w-\Sigma(z)+\Sigma(w)]G(z)\circ G(z)G_{2c}(z,w) - \frac{G^2(z)\circ G(w)}{[1-G(z)\circ G(w)][1-G(w)\circ G(w)]} - \frac{[G^2(w)\circ G(z)][G(z)\circ G(z)]}{[1-G(z)\circ G(w)]^2[1-G(w)\circ G(w)]}. \quad \text{(B12)}$$

Using the previous identity of (B11), we write the equation for $G_{2c}(z,w)$ using (B10) and (B12),

$$[G(z)-G(w)-z+w][1-G(z)\circ G(z)]G_{2c}(z,w)$$
$$= -\frac{1}{[1-G(z)\circ G(w)]^2[1-G(w)\circ G(w)]} \times\{G(z)\circ G^2(w)[1-G(z)\circ G(z)] - G^2(z)\circ G(w)[1-G(z)\circ G(w)]\}. \quad \text{(B13)}$$

It is easy to see that this equation leads to the previous result (5.3), which was derived by a diagrammatic analysis [4]. To see this equivalence, we write the following two quantities as

$$G^2(z)\circ G^2(w) = \frac{1}{N}\sum_i \frac{1}{[z-\epsilon_i-G(z)]^2}\frac{1}{[w-\epsilon_i-G(w)]^2} = \frac{G^2(z)\circ G(w)-G(z)\circ G^2(w)}{w-z-G(w)+G(z)}, \quad \text{(B14)}$$

$$G^2(z)\circ G(w) = \frac{G(z)\circ G(z)-G(z)\circ G(w)}{w-z-G(w)+G(z)} \quad \text{(B15)}$$

and replace these quantities in (B13); then we obtain the same result as (5.3).

For the non-Gaussian distribution $P(\varphi)$, we obtain the expression of $G_{2c}(z,w)$, up to order g, as (6.9). When we set $H_0 = 0$, this expression should be consistent with the known result, discussed in Sec. III. We will show in the following that indeed the expression (6.9) becomes consistent with the known result when $H_0 = 0$.

When $H_0 = 0$, we have by the scaling $z \to z/a$ and $w \to w/a$, from the expression of $G_{2c}(z,w)$ for the Gaussian distribution $P(\varphi)$,

$$G_{2c}(z,w) = -\frac{\partial^2}{\partial z\partial w}\ln\left[1-G_0\left(\frac{z}{a}\right)G_0\left(\frac{w}{a}\right)\right] = \frac{\partial^2}{\partial z\partial w}\left(\frac{z-w}{z-w-\frac{z-\sqrt{z^2-4a^2}}{2}+\frac{w-\sqrt{w^2-4a^2}}{2}}\right). \quad \text{(B16)}$$

Now we set $H_0 = 0$ in the expression (6.9). Then all circle products become usual products and we have also

$$G(z)G(w) = \frac{G(z)-G(w)}{w-z-\Sigma(w)+\Sigma(z)}. \quad \text{(B17)}$$

Then we write (6.9) in the form

$$G_{2c}(z,w) = -\frac{\partial^2}{\partial z\partial w}\ln\left(\frac{z-w}{z-w+\Sigma(w)-\Sigma(z)+4gG(z)-4gG(w)}\right), \quad \text{(B18)}$$

where the self-energy $\Sigma(z)$ is given as (6.8)

$$\Sigma(z) = G_0(z) - \frac{8gG_0(z)}{1-G_0^2(z)} - \frac{4gG_0^3(z)}{1-G_0^2(z)} + O(g^2), \quad \text{(B19)}$$

with

$$G_0(z) = \frac{z-\sqrt{z^2-4}}{2}. \quad \text{(B20)}$$

We expand the quantity a in (B16) as $a^2 = 1-12g+O(g^2)$ and compare (B16) and (B18). Then we find that (6.9) is consistent with (B16) when we set $H_0 = 0$.

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