Statistical properties of the delay time matrix

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The statistical properties of the delay time matrix for a simple, tractable model describing multichannel resonance scattering are studied. The cases of one to many open channels are examined first for a single resonance, and then for multiple resonances, including the limit of overlapping resonances. Numerical calculations for the delay time and its statistical properties in more complex yet realistic cases involving energy averaging are reported and analyzed.

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I. INTRODUCTION

There is a growing interest, ranging from nuclear to molecular physics, in understanding the manifestations of the chaotic behavior of quantum scattering systems. When the scattering is irregular in some sense, stochastic descriptions are appropriate. It is therefore worthwhile to investigate statistical properties of models where analytical results are available, as a reference and guide for more realistic cases and with the expectation that features of these properties are universal or generic [1]. Recently several such models whose dynamics are completely chaotic have been examined [2]. It is our aim here to study tractable multichannel scattering models supporting a mixture of regular and chaotic dynamics, as is typically the case in physical and chemical processes. Specifically, we shall focus on the properties of Smith's Hermitian "lifetime" or delay time matrix [3].

$$Q(E)=i\hbar S(E)\frac{dS(E)^{\dagger}}{dE} \tag{1}$$

in suitably chosen ensembles of scattering $S(E)$ matrices [4]. Only diagonal elements will be considered since nondiagonal elements have no direct physical meaning. $Q(E)_{aa}$ is interpreted as the average delay time experienced by a particle injected in channel a at energy E [5]. In the following the energy argument will be generally omitted.

Ours is not the first study of statistical properties of the delay time, but it differs from earlier work in significant ways: We focus on the delay time *distribution,* deriving and analyzing general expressions within the framework of a particular *unitary* S-matrix model; we evaluate our results analytically in standard limiting cases and numerically for other cases; we consider the implications of "double averaging" over ensembles of statistical model parameters and/or over a distribution of total energy. Delay time properties arising from statistical models for the scattering matrix have been studied previously and include the mean, the mean square deviation, and sum rules for overlapping and isolated resonance [6–9]. In the special case of decay of a "prepared" compound state arising from a sufficiently broad incoming wave packet, the distribution of delay times has been shown to be expressible as the Fourier transform of the energy autocorrelation function of the S matrix [8–10]. A systematic study of the relationship between our results and earlier work on delay time properties lies beyond the scope of the present paper and is left for future work. A limited number of obvious connections to other results is made at appropriate places throughout the paper.

An example of the type of behavior we wish to characterize is provided in Fig. 1, which depicts the energy

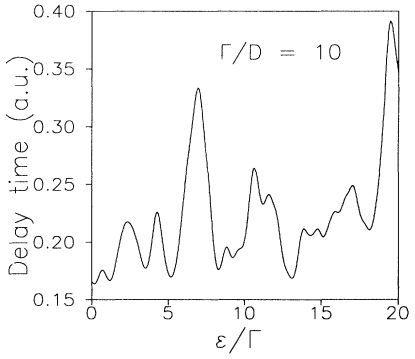

FIG. 1. Unaveraged Q_{aa} in atomic units (a.u.) versus ε/Γ for $\Gamma/D=10$, $N=30$, $D=1$ a.u., and a particular set of $n=20\,000$ resonances chosen at random according to the procedure described in Sec. V. The energy ε is $E-nD/2$, where $nD/2$ is approximately the center of the interval used in the numerical simulation; the same convention for ε applies to Figs. 2–4. There are approximately 200 resonances in the section of energy shown in the plot, but a much smaller number of peaks in Q_{aa}. The values of the amplitudes $\gamma_a^{(\lambda)}$ are taken randomly from a Gaussian distribution as described in Sec. III.

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dependence of a diagonal element of Q as obtained from a Breit-Wigner model for the S matrix. The model is described in Sec. II; specific details are provided in the figure caption. The essential feature of Fig. 1 is the irregular dependence of the delay time on energy: the energy range of the abscissa contains approximately 200 overlapping resonances of which the only remnants are the fluctuations associated with the 12 maxima and the 12 minima.

While it has been recognized that chaotic scattering is well characterized by the "circular orthogonal" ensemble of S matrices, these ensembles do not contain the energy-dependent information necessary for finding the statistical properties of Q [neither the derivative in (1) nor "two-point functions," i.e., moments depending on two energies, can be evaluated]. The same limitation applies to the maximum entropy ensembles [11]. The required ensembles must contain energy-dependent information. A possibility is to assume a "Gaussian orthogonal" ensemble of model Hamiltonians and, through their corresponding S matrices, obtain the proper statistics [12,13]. This program has been performed in part and has yielded average values for Q in the overlapping resonance regime [14]. But when higher moments are required, as in the autocorrelation function of Q, which involves four $S(E)$ matrix elements, the technical difficulties are formidable [13,15]. Another possible approach would use the semiclassical S matrix [16] to link classical irregular scattering to the corresponding quantum scattering [17]. However, the energy-dependent information required for the delay time autocorrelation function is lost when implementing this standard semiclassical treatment. Moreover, any assumption about properties of the classical delay time, such as the exponential and power-law distributions introduced in Refs. [17] and [18] to obtain the semiclassical S matrix autocorrelation function for hyperbolic and nonhyperbolic classical scattering, respectively, must certainly be avoided in an *a priori* semiclassical derivation of the analog of the corresponding quantum distribution of delay times.

This has motivated our search for simple, tractable models that retain the basic features of the more rigorous random matrix theory. Our model relies on a resonance description of an intermediate collision complex. The occurrence of resonances in nuclear reactions has been recognized for some time and has been exploited to develop compound nucleus models [19]. In chemical reactions there is growing computational evidence that resonances are ubiquitous and influence the reaction dynamics [20]. Asserting the intimate association between resonances and chaotic behavior, our approach rests on the basic model describing quantum resonances, namely, the Breit-Wigner model [21]. The path followed in the present paper is to specify our model for the scattering matrix and discuss the implications in Sec. II. We then characterize in full detail the statistical properties of the delay time associated with a single Briet-Wigner resonances for an arbitrary number of open channels, with fixed resonance parameters (Sec. III A), and subsequently extend the results in various directions, such as varying widths (Sec. III B) or multiple resonances (Sec. IV), by imposing the necessary simplifying assumptions to keep the model solvable, even though some of them may be unrealistic for a generic case: Most prominently, an independence condition between the various resonances is enforced. Finally, we consider more complex cases, involving averages over sets of resonance energies (henceforth denoted configurations) or over total energy, and report the results of numerical calculations thereof, in Sec. V. The paper concludes with a discussion in Sec. VI.

The use of an ensemble in the single resonance case rests on the possibility of examining either many different, isolated, resonances in the same system or individual resonances from a family of systems differing by the value of some perturbative parameter. Averaging over a large number of channels (for a selected system and energy) also justifies the statistical treatment. In general, when many resonances are present one may consider different averages, over energy or configurations, and it is important to know the relationship and/or possible equivalence of these averages. This aspect is addressed in Sec. V.

II. MODEL FOR SCATTERING MATRIX

Assume the following structure for the scattering operator S with multiple (n) resonances (see, e.g., [22]):

$$S=S(E)=1-\sum_{\lambda=1}^{n}\frac{iA^{(\lambda)}}{E-E_\lambda+i\Gamma_\lambda/2}\ . \tag{2}$$

This corresponds to a set of Breit-Wigner resonances with a collective background of unity. The case of nonunit background is discussed in Appendix A, where it is shown to have no effect on our final results. Threshold effects are neglected so that the number of channels N is constant for the range of energies of interest. It is assumed that all parameters are independent of energy and that the resonances are independent, i.e., for $\lambda\neq\lambda'$,

$$A^{(\lambda)}A^{(\lambda')}=0\ . \tag{3}$$

By imposing unitarity on S, one finds that the operator $A^{(\lambda)}$ is Hermitian,

$$A^{(\lambda)}=A^{(\lambda)\dagger} \tag{4}$$

and that it is related to Γ_λ,

$$A^{(\lambda)}A^{(\lambda)}=A^{(\lambda)}\Gamma_\lambda\ . \tag{5}$$

Furthermore, if the system is invariant under time reversal, S is symmetrical for a suitable choice of asymptotic states. This implies that A is symmetrical and, for a nondegenerate metastable state, it therefore factorizes

$$A^{(\lambda)}_{ab}=\gamma^{(\lambda)}_a\gamma^{(\lambda)}_b\ . \tag{6}$$

Because A is Hermitian and symmetrical the $\gamma^{(\lambda)}_a$'s can be chosen to be real. They are the amplitudes that connect the channel a with the resonant state. A discussion on the origin of this condition and the extension to complex values is provided in Appendix A. Combining conditions (5) and (6) one finds

$$\Gamma_\lambda=\sum_u [\gamma_u^{(\lambda)}]^2 . \tag{7}$$

The condition of "independent resonances" (3) is of course an unrealistic assumption in general. Indeed, for the case of a single open channel, the joint distribution of the resonance energies and widths has been obtained analytically for a unitary S-matrix model arising from statistical treatment of an effective non-Hermitian Hamiltonian [23]. The treatment of Ref. [23] reveals that resonance widths and energies are, in general, both intracorrelated and intercorrelated. There are, however, physically interesting cases where an independent resonance assumption does lead to reasonable results. An obvious case is the regime of well resolved resonances in which the poles do not mutually affect each other. Also, in the many overlapping resonances limit, the correlation effects between different resonances ("level-level" correlations) are frequently neglected on statistical grounds [24–26], although this assumption is not necessarily consistent with unitarity of the S matrix [27,28]. The absence of correlation between levels is here enforced at the "microscopic level," i.e., in the form of the S matrix, rather than statistically, so that all members of the S matrix ensemble are unitary.

From a more technical point of view, our model corresponds to the simplest possible case in the general parametrization proposed by Simonious for a unitary S matrix with an arbitrary number of poles [29]. (In his notation, defining the projector P_λ by $A^{(\lambda)}=\Gamma_\lambda P_\lambda$, the proposed structure corresponds to $P_\lambda P'_\lambda=\delta_{\lambda\lambda'}$.) We shall return to this point later.

Completion of the model (2) for S requires specification of E_λ, $\lambda=1,\ldots,n$, and $\gamma_a^{(\lambda)}$, $\lambda=1,\ldots,n$; $a=1,\ldots,N$. In all cases considered here, the γ's are treated as stochastic variables in a manner described in Sec. III. In Secs. III (single resonance) and IV (multiple resonances) results for the delay time are obtained and analyzed with the resonance positions E_λ fixed. In Sec. V the latter constraint is dropped and the E_λ's are selected from the Gaussian orthogonal ensemble. The set of parameters Γ_λ, $\lambda=1,\ldots,N$ is determined by selecting the γ's without constraint and then applying (7), or the Γ's can be fixed and (7) used as a constraint in the selection of the γ's both cases are considered in Sec. III.

III. SINGLE BREIT-WIGNER RESONANCE

Throughout this section the resonance position, denoted E_0, is fixed; the fixed and the variable Γ cases are considered separately. For a single resonance ($n=1$), the S-matrix expression (2) reduces to

$$S=S(E)=1-\frac{iA}{E-E_0+i\Gamma/2} \tag{8}$$

with

$$\Gamma=\sum_a \gamma_a^2 . \tag{9}$$

A. Fixed width

Here we fix the value of Γ and treat the γ's as stochastic variables. What is the proper ensemble of S matrices compatible with unitarity and given values of Γ and E_0? Since the only variables are the γ's, the least biased choice is a distribution as homogeneous as possible, subject to the constraint (9). Although the general N-channel case could be directly described by making use of N-dimensional spherical geometry, it is a rewarding exercise to proceed by induction from one to many open channels. The distributions found for small N ($N<5$) are peculiar and differ qualitatively from the ones for higher values of N. Also, the reader will have a better ground for understanding the general coordinate system and results. The one-open-channel case is trivial in that there is only one member in the ensemble and $\Gamma=\gamma^2$.

1. Two channels

For two channels, labeled a and b, (9) is the equation of a circle of radius $\sqrt{\Gamma}$. The ensemble is chosen to be a homogeneous distribution of points on the circle. Writing

$$\gamma_a=\Gamma^{1/2}\sin\phi ,$$
$$\gamma_b=\Gamma^{1/2}\cos\phi , \tag{10}$$

we have

$$P_2(\phi)=\frac{1}{2\pi} \tag{11}$$

as the probability density for the angle ϕ. Equations (10) and (11) enable us to find arbitrary moments $\langle\gamma_a^m\gamma_b^n\rangle$; $m,n=0,1,2,\ldots$ ($\langle\ \rangle$ denotes an ensemble average) by integrating over the angle between 0 and and 2π. In particular

$$\langle\gamma_a^m\gamma_b^n\rangle=0,\quad m \text{ and/or } n=\text{odd} , \tag{12}$$

$$\langle\gamma^2\rangle=\frac{\Gamma}{2},\quad \langle\gamma^4\rangle=\frac{3\Gamma^2}{8} , \tag{13}$$

$$\langle\gamma_a^2\gamma_b^2\rangle=\frac{\Gamma^2}{8} , \tag{14}$$

where the absence of subscripts in the second- and the fourth-order moments denotes that the expressions are valid for arbitrary channels. Clearly, the joint distribution for γ_a and γ_b is not Gaussian, as there is a correlation between γ_a and γ_b, $\langle\gamma_a^2\gamma_b^2\rangle\neq\langle\gamma_a^2\rangle\langle\gamma_b^2\rangle$. This distribution can be explicitly obtained, e.g., by multiplying the marginal and the conditional distributions, but is not required for present purposes. For finding the distributions of the delay times, it is useful to write Q_{aa} in a compact form

$$\begin{aligned} Q_{aa}&=i\hbar\sum_u S_{au}\frac{dS_{au}^*}{dE} \\ &=\frac{-i\hbar[\gamma_a^4+i\gamma_a^2(E-E_0+i\Gamma/2)+\gamma_a^2\gamma_b^2]}{[(E-E_0)^2+\Gamma^2/4](E-E_0-i\Gamma/2)} \\ &=\frac{\hbar\gamma_a^2}{[(E-E_0)^2+\Gamma^2/4]} , \end{aligned} \tag{15}$$

where $u=a,b$ and in general represents an arbitrary

channel. The first two terms in the second line correspond to the diagonal contribution $u=a$ and the third one to the nondiagonal one $u=b$. The constraint (9) is used to obtain the last equality. Note that this result is independent of statistical averaging and is a direct consequence of the proposed structure of the unitary S matrix. Since only diagonal elements of the lifetime matrix are treated here [30] and all have the same functional expressions (15) and statistical properties, we shall use the notation q for the *values* allowed for diagonal elements Q_{uu}. The distribution of q for two channels $P_2(q)$ is then found to be

$$P_2(q)=\frac{1}{2\pi}\int_0^{2\pi}d\phi\,\delta(q-q_m\cos^2\phi)=\frac{1}{\pi(qq_m-q^2)^{1/2}}\ , \tag{16}$$

where q_m is the maximum value that q may take; this occurs when all γ's are equal to zero, except the one for the channel of interest. According to (15),

$$q_m=(\hbar\Gamma)/[(E-E_0)^2+\Gamma^2/4]\ . \tag{17}$$

2. Three channels

For three channels the constraint (9) is the equation of a sphere of radius $\Gamma^{1/2}$. In spherical coordinates the amplitudes are given by

$$\begin{aligned}\gamma_a&=\Gamma^{1/2}\sin\phi\sin\theta\ ,\\ \gamma_b&=\Gamma^{1/2}\cos\phi\sin\theta\ ,\\ \gamma_c&=\Gamma^{1/2}\cos\theta\ .\end{aligned} \tag{18}$$

Assuming a uniform distribution over the sphere, the probability distribution of angles is

$$P_3(\phi,\theta)=\frac{\sin\theta}{4\pi}\ . \tag{19}$$

The expression for q now is identical to the one for two channels, Eq. (15), as a result of partial cancellation between diagonal and nondiagonal contributions making use of the constraint (9). Indeed Eq. (15) is a general result independent of the number of channels; see the derivation of (50) below.

The distribution of q is

$$\begin{aligned}P_3(q)&=\frac{1}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\,\delta(q-q_m\cos^2\theta)\\ &=\frac{1}{2(qq_m)^{1/2}}\ ,\end{aligned} \tag{20}$$

where channel c [see (18)] has been used in the integral for convenience. The result does not depend on this particular choice because of the spherical symmetry.

3. Four channels

Examination of the four-channel case will allow us to write the general N-channel equations by induction. According to (9), the ensemble points are now constrained to a hypersphere of constant "radius" $\Gamma^{1/2}$. The appropriate coordinates are a generalization of standard spherical polar coordinates [31].

$$\begin{aligned}\gamma_a&=\Gamma^{1/2}\sin\phi\sin\theta\sin\chi\ ,\\ \gamma_b&=\Gamma^{1/2}\cos\phi\sin\theta\sin\chi\ ,\\ \gamma_c&=\Gamma^{1/2}\cos\theta\sin\chi\ ,\\ \gamma_d&=\Gamma^{1/2}\cos\chi\ ,\end{aligned} \tag{21}$$

where $0\le\phi\le2\pi$, $0\le\theta\le\pi$, and $0\le\chi=\pi$. The orthogonality of this system can be explicitly checked by computing the direction cosines in analogy to the three-dimensional case [32]. The scale factors $h_i=\sum_u(d\gamma_u/d\xi_i)^2$ (where $u=a,b,c,d$ and $\xi_i=\Gamma^{1/2},\phi,\theta,\chi$ for $i=1,2,3,4$, respectively) are

$$\begin{aligned}h_1&=1\ ,\\ h_2&=\Gamma^{1/2}\sin\theta\sin\chi\ ,\\ h_3&=\Gamma^{1/2}\sin\chi\ ,\\ h_4&=\Gamma^{1/2}\ .\end{aligned} \tag{22}$$

The differential element of hypersurface is then given by

$$dS_4=h_\phi h_\theta h_\chi\,d\phi\,d\theta\,d\chi=\Gamma^{3/2}\sin\theta\sin^2\chi\,d\phi\,d\theta\,d\chi\ , \tag{23}$$

and by integrating over the angles we obtain the total hypersurface "area" $S_4=2\pi^2\Gamma^{3/2}$. The distribution of angles is

$$P_4(\phi,\theta,\chi)=\frac{\sin\theta\sin^2\chi}{2\pi^2}\ . \tag{24}$$

The probability density of q is computed, as in the $N=3$ case, using the channel whose corresponding amplitude has only a cos function in polar coordinates, channel d in the present case:

$$\begin{aligned}P_4(q)&=\frac{1}{2\pi^2}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\int_0^{\pi}d\chi\sin\theta\sin^2\chi\\ &\qquad\times\delta(q-q_m\cos^2\chi)\\ &=\frac{2}{\pi q_m}(q_m/q-1)^{1/2}\ .\end{aligned} \tag{25}$$

4. N channels

In view of the above results we are ready to write general expressions for N open channels. When an extra channel is added, the appropriate N-dimensional spherical coordinate system is constructed by multiplying the $N-1$ coordinates of the previous set by the sin of a new angle ξ_N, varying between 0 and π. The remaining Nth coordinate is given by $\Gamma^{1/2}\cos\xi_N$. Using the notation $\{\xi_i\}$, $i=1,\ldots,N$, for the set of N spherical coordinates, with $\xi_1=\Gamma^{1/2}$, $\xi_2=\phi$, $\xi_3=\theta$, $\xi_4=\chi$, and so on, the scale factors take the form

$$\begin{aligned}h_1&=1\ ,\\ h_J&=\Gamma^{1/2}\sin\xi_{J+1}\cdots\sin\xi_N\quad(J=2,3,\ldots,N-1)\ ,\\ h_N&=\Gamma^{1/2}\ .\end{aligned} \tag{26}$$

The Jacobian $\mathcal{J}_N$ of the transformation from Cartesian to polar coordinates $\{\gamma\}\rightarrow\{\xi\}$ is the product of the N scale factors

$$\mathcal{J}_N=\Gamma^{(N-1)/2}\prod_{J=3}^{N}\sin^{J-2}\xi_J \tag{27}$$

and the differential of $(N-1)$-dimensional space available for the ensemble points ($dS_N=\prod_{i=2}^{N}h_i$) will be proportional to

$$F_N=\prod_{J=3}^{N}\sin^{J-2}\xi_J\ . \tag{28}$$

The normalization constant for the homogeneous distribution $P_N(\{\xi_2,\ldots,\xi_N\})=C_NF_N$ is therefore

$$C_N=\left[\int_0^{2\pi}d\phi\int_0^{\pi}d\xi_3\cdots\int_0^{\pi}d\xi_NF_N\right]^{-1}\ . \tag{29}$$

Thus each new added channel (for $N>2$) contributes to the normalization constant a factor

$$c_N=\frac{1}{\int_0^{\pi}\sin^{N-2}\xi_Nd\xi_N}=\frac{G(N/2)}{\pi^{1/2}G[(N-1)/2]}\ , \tag{30}$$

where G is the Gamma function. This gives for the constant C_N

$$C_N=\frac{G(N/2)}{2\pi^{N/2}}\ . \tag{31}$$

Combining the above results, the general expression for the probability distribution of the angles is

$$P_N(\xi_3,\ldots,\xi_N)=\frac{G(N/2)}{2\pi^{N/2}}\prod_{J=3}^{N}\sin^{J-2}\xi_J\ . \tag{32}$$

Note the independence with respect to $\xi_2=\phi$. From this expression a general formula for arbitrary even moments $\langle\gamma^{2\nu}\rangle$ can be obtained (odd moments vanish). The first two of these moments are required for evaluating the average of q and its autocorrelation function. It is convenient to use the amplitude corresponding to the Nth channel, the result being, however, independent of the particular channel

$$\begin{aligned}\langle\gamma^{2\nu}\rangle&=\int_0^{2\pi}d\phi\int_0^{\pi}d\xi_3\cdots\int_0^{\pi}d\xi_NP_N(\xi_3,\ldots,\xi_N)\\&\qquad\times\Gamma^{\nu}\cos^{2\nu}\xi_N\\&=c_N\Gamma^{\nu}\int_0^{\pi}d\xi_N\sin^{N-2}\xi_N\cos^{2\nu}\xi_N\\&=\Gamma^{\nu}\frac{G(N/2)G(\nu+\frac{1}{2})}{\pi^{1/2}G[(N/2)+\nu]}\ .\end{aligned} \tag{33}$$

The first partial integral is 2π and the rest, except for the last one, can be done with (30). These $(N-2)$ partial integrals cancel the corresponding normalization factors. The final expression in (33) is also valid for $N=2$. The result for the second moment

$$\langle\gamma^2\rangle=\Gamma/N \tag{34}$$

can also be obtained independently from the constraint (9) and the statistical equivalence of the channels. This leads to the average value

$$\langle q\rangle=\frac{\hbar\Gamma/N}{[(E-E_0)^2+\Gamma^2/4]}\ . \tag{35}$$

We are now ready to compute the autocorrelation function of q

$$\begin{aligned}F(\varepsilon)&=\langle q(E)q(E+\varepsilon)\rangle-\langle q(E)\rangle\langle q(E+\varepsilon)\rangle\\&=\left[\frac{2(N-1)}{N^2(N+2)}\right]\\&\quad\times\frac{\hbar^2\Gamma^2}{[(E-E_0)^2+\Gamma^2/4][(E+\varepsilon-E_0)^2+\Gamma^2/4]}\end{aligned} \tag{36}$$

or its normalized version

$$f(\varepsilon)=\frac{F(\varepsilon)}{F(0)}=\frac{(E-E_0)^2+\Gamma^2/4}{(E+\varepsilon-E_0)^2+\Gamma^2/4}\ . \tag{37}$$

To obtain the general distribution of delay times $P_N(q)$, the matrix element of Q for the Nth channel is again used, as was done for $N=2,3,4$,

$$\begin{aligned}P_N(q)=\int_0^{2\pi}d\phi\int_0^{\pi}d\xi_3\cdots\int_0^{\pi}d\xi_N\delta(q-q_m\cos^2\xi_N)\\\times P_N(\xi_3,\ldots,\xi_N)\ .\end{aligned} \tag{38}$$

Here the same cancellation described below Eq. (33) occurs and all that remains in the last integration is the factor c_N, the δ function, and $\sin^{N-2}\xi_N$. The result is $(N\geq3)$

$$P_N(q)=\frac{G(N/2)}{\pi^{1/2}G[(N-1)/2]}\frac{(1-q/q_m)^{(N-3)/2}}{(qq_m)^{1/2}}\ . \tag{39}$$

Some of the properties of this function are described below.

(i) Derivative $(N\geq3)+$ $dP_N(q)/dq\leq0$. Thus the distribution is a decreasing function of q from its maximum at $q=0$ to its minimum value at $q=q_m$. Moreover, for $N\geq5$, $dP_N(q)/dq|_{q_m}=0$. Lower N values are exceptional. For $N=2$ there is a minimum at $q=q_m/2$; see (16). Also, $dP_3(q)/dq|_{q_m}=-(2q_m)^{-2}$ and $dP_4(q)/dq|_{q_m}\rightarrow-\infty$.

(ii) Limits $P_N\rightarrow\infty$ when $q\rightarrow0$ $(N\geq2)$ and $P_N\rightarrow0$ when $q\rightarrow q_m$ $(N\geq4)$.

(iii) Second derivative: The condition to make the second derivative zero $(N\geq4)$ is given by the quadratic equation

$$\frac{(N-4)(N-\frac{9}{2})}{2q_m}q^2+(N-6)q+\tfrac{3}{2}q_m=0\ . \tag{40}$$

For $N=4$ this equation becomes linear and the solution $q=3q_m/4$ corresponds to an inflection point where the second derivative changes sign. For $N>4$ there are no real solutions of (40) so there are no inflection points and $d^2P_N(q)/dq^2\geq0$.

Incidentally, the same procedure used for the distribution of delay times $P_N(q)$ can be applied for obtaining the distribution of amplitudes $P_N(\gamma)$,

$$P_N(\gamma)=\int_0^{2\pi}d\phi\int_0^{\pi}d\xi_3\ldots\int_0^{\pi}d\xi_N\delta(\gamma-\Gamma^{1/2}\cos\xi_N)\times P_N(\xi_3,\ldots,\xi_N)$$
$$=\frac{G(N/2)}{(\pi\Gamma)^{1/2}G[(N-1)/2]}\times(1-\gamma^2/\Gamma)^{(N-3)/2}\ . \tag{41}$$

The corresponding distributions for the reduced variables $\gamma/\Gamma^{1/2}$ and q/q_m are well known in random matrix theory since they describe the distributions for the coefficients of the eigenstates and the squares of these coefficients in a GOE ensemble of Hamiltonians. (These eigenvectors uniformly cover the N-dimensional unit hypersphere [33]. Asymptotically, for large N, $P(\gamma/\Gamma^{1/2})$ tends to a Gaussian with zero mean and variance $1/N$ and $P(q/q_m)$ to a χ^2 distribution of *one* degree of freedom with mean $1/N$ [34]. Moreover, in the same limit, the correlations between the different channels tend to vanish.

The above single resonance treatment will play a basic role in the study of more complex situations, such as the overlapping resonance case to be dealt with later. This alone makes the effort worthwhile. In addition, however, it is possible to actually implement the ensembles considered in the present section, or justify their use, either by using different resonances in the same or different systems (in all cases shifting the energy scale so that the resonances are centered at E_0) or by comparing the outcomes for different channels in the same system for a given resonance and energy. The second option will be valid when a large number of channels is open, so that the probability $P(q)=P(Q_{uu})$, referred to the ensemble, is well approximated by the probability $P_{\text{single}}(q)$ of finding the value q, irrespective of the channel, in a single system. A statistical description of the outgoing channels is indeed a possible definition of chaos in scattering systems [35].

The restriction to a fixed value for Γ is clearly appropriate for a given multichannel system at a fixed energy, but will generally be too severe in other cases. This motivates the next subsection. We may say in advance, however, that in the many-channel limit, the variable Γ ensembles and the fixed Γ ensembles will give the same results.

B. Varying width

We now consider a different type of ensemble in which each γ is chosen independently from a Gaussian distribution with zero mean, the distribution for each channel being the same since the channels are assumed to be statistically equivalent (for a recent discussion on this assumption and its validity see [36]). The joint distribution for all γ is thus a product of Gaussians with equal width. Each set of N numbers $\gamma_a,\gamma_b,\ldots$ determined in this way will have, in general, a different value for the sum of the squares, i.e., for the total width Γ associated with that particular member of the ensemble. Averaging the expression (9), which now is not a constraint, over the ensemble one finds, for the second moment

$$\langle\gamma^2\rangle=\langle\Gamma\rangle/N\ . \tag{42}$$

Noticing that the total width Γ, here a random variable, appears in the denominator of the S and Q matrices and that the statistical properties of this quantity are also of interest, it is convenient to use the set of spherical coordinates $\{\xi\}$ described in Sec. III A 4. Since the reader is already familiar with the N-channel results and coordinates, we shall directly describe the N-channel case without recourse to induction.

Using (9) and (42), the joint probability density $P(\{\gamma\})$ is given by

$$P(\gamma_a,\gamma_b,\ldots)=\frac{e^{-N\Gamma/\langle 2\Gamma\rangle}}{(2\pi\langle\Gamma\rangle/N)^{N/2}}\ . \tag{43}$$

It is independent of the angles. Since there is a one to one relation between the γ's and the spherical coordinates $\{\xi\}$, the joint distribution in the new variables is, using the appropriate Jacobian, $P(\{\xi\})=P(\{\gamma\})\mathcal{T}_N$ [see (27)],

$$P(\Gamma^{1/2},\xi_2,\ldots,\xi_N)=\frac{e^{-\Gamma N/(2\langle\Gamma\rangle)}\Gamma^{(N-1)/2}}{(2\pi\langle\Gamma\rangle/N)^{N/2}}\times\prod_{J=3}^{N}\sin^{J-2}\xi_J\ . \tag{44}$$

Note that the width and the angles are independent variables. Integrating over $\Gamma^{1/2}$, one obtains for the angles the same distributions considered in the fixed-Γ case. Thus, in general, any average will be computed first for Γ fixed and then the integral over $\Gamma^{1/2}$ is carried out. The procedure will be denoted by double angular brackets $\langle\langle\ \rangle\rangle$.

By integrating over the angles and multiplying by the Jacobian $1/(2\Gamma^{1/2})$, the marginal distribution for Γ is found to be

$$P(\Gamma)=\frac{e^{-\Gamma N/(2\langle\Gamma\rangle)}\Gamma^{(N-2)/2}}{(2\langle\Gamma\rangle/N)^{N/2}G(N/2)}\ , \tag{45}$$

which is a Gamma or χ^2 distribution for N degrees of freedom, as was to be expected from the start, since Γ is a sum of N^2 Gaussian variables with equal width. The route we have followed, however, has the advantage of showing explicitly the connection with the fixed-Γ ensemble.

Because of the additional integral over Γ, not all results of Sec. III A can be generalized in the form of simple expressions. The average value $\langle\langle q\rangle\rangle$, for instance, can be written in terms of cosine and sine integrals and trigonometric functions. Since the general form is not very illuminating, we shall report the simpler case where $E=E_0$ (denoted by the subindex 0 in the average) that also allows analytical treatment of arbitrary moments. At resonance, the angle average with Γ fixed takes the form

$$\langle q^\nu\rangle_0=(4\hbar/\Gamma)^\nu\frac{G(N/2)G(\nu+\frac{1}{2})}{\pi^{1/2}G[(N/2)+\nu]}\ . \tag{46}$$

By performing the Γ average with (45) one obtains

$$\langle\langle q^{\nu}\rangle_0\rangle=\left[\frac{2\hbar N}{\langle\Gamma\rangle}\right]^{\nu}\frac{G(\nu+\frac{1}{2}G(N/2-\nu)}{\pi^{1/2}G(N/2+\nu)}\ . \tag{47}$$

In the limit $N\to\infty$, (47) is simply related to (46) by replacing in (46) Γ by $\langle\Gamma\rangle$. This is because the marginal Γ distribution (45) tends to a delta function centered at $\langle\Gamma\rangle$ as $N\to\infty$, so the average of any property in the Γ-fixed ensemble calculated at $\Gamma=\langle\Gamma\rangle$ is the same as that of the varying-Γ ensemble in the limit of many channels.

It should be recognized, however, that the statistical assumptions involved (the transition amplitudes are treated as equivalent Gaussian independent random variables) are only valid as a limiting case and that a realistic distribution of Γ will be characterized by a number of "effective degrees of freedom" smaller than N [33]. However, since little is known about the general properties of the delay time even under the present and somewhat extreme conditions and since the results will be in any case a reference for more realistic treatments, we shall in the following assume the validity of the above strong statistical assumptions and concentrate for simplicity on fixed Γ ensembles.

IV. MULTIPLE RESONANCES

A. General properties of the delay time

To compute Q we use (2) and (4) to write the derivative of $S^{\dagger}$ with respect to energy as

$$\frac{dS^{\dagger}}{dE}=\sum_{\lambda}\frac{-iA^{(\lambda)}}{(E-E_{\lambda}-i\Gamma_{\lambda}/2)^2}\ . \tag{48}$$

Then conditions (3) and (5) lead to

$$Q=\hbar\sum_{\lambda}A^{(\lambda)}/[(E-E_{\lambda})^2+\Gamma_{\lambda}^2/4] \tag{49}$$

for the lifetime operator, with the diagonal element

$$Q_{aa}=\hbar\sum_{\lambda}\frac{[\gamma_a^{(\lambda)}]^2}{[(E-E_{\lambda})^2+\Gamma_{\lambda}^2/4]}=\sum_{\lambda}Q_{aa}^{(\lambda)}\ . \tag{50}$$

As shown by Lyuboshitz [7], this is precisely the structure obtained for Q from the general unitary S-matrix model of Simonius [29], considering that the factors in the numerator for the general Simonious model are not the γ's but different functions, say α's, that can be related to the original γ's in a definite although somewhat intricate way (see Eq. (10) in [7]). Since the new functions α satisfy the same constraint as the original γ [Eq. (7)], one may also regard (50), with α instead of γ, as the starting point for a statistical treatment.

If we further assume that all resonances are statistically equivalent and that the widths Γ_{λ} are fixed numbers, the probability distribution in spherical coordinates (one set of coordinates $\{\xi^{(\lambda)}\}$ for each resonance) reads

$$P(\{^{(1)}\},\ldots,\{\xi^{(n)}\})=P_N(\{\xi^{(1)}\})\cdots P_N(\{\xi^{(n)}\}) \tag{51}$$

and the distribution of q is now

$$P_{N,n}(q)=\int\cdots\int d\{\xi^{(1)}\}\cdots d\{\xi^{(n)}\}\times\delta\left[q-\sum_{\lambda}q_m^{(\lambda)}\cos^2\xi_N^{(\lambda)}\right]\times P(\{\xi^{(1)}\},\ldots,\{\xi^{(n)}\})\ , \tag{52}$$

where $q_m^{(\lambda)}=(\hbar\Gamma_{\lambda})/[(E-E_{\lambda})^2+\Gamma_{\lambda}^2/4]$ is the maximum value of q allowed for the resonance λ. This distribution has characteristic function $\Phi_{N,n}(k)=\int\exp(ikq)P_{N,n}(q)dq$. Because of the assumed independence of the resonances, the total delay time is a sum of the independent stochastic variables $Q_{aa}^{(\lambda)}$ as expressed by (50). (The central limit theorem cannot be applied because neither the average nor the variance of these variables agrees.) Therefore $\Phi_{N,n}(k)$ is the product of the characteristic functions $\Phi_N^{(\lambda)}(k)=\int\exp(ikq)P_N^{(\lambda)}(q)$ of each resonance's distribution $P_N^{(\lambda)}(q)$. [The distributions for the individual resonances $P_N^{(\lambda)}(q)$ are given by (39) with q_m replaced by the corresponding λ-dependent quantity $q_m^{(\lambda)}$.] Accordingly,

$$\ln\Phi_{N,n}=\sum_{\lambda=1}^{n}\ln\Phi_N^{(\lambda)}\ . \tag{53}$$

By expanding both sides in a Taylor series around $ik=0$ [i.e., $\ln\Phi(k)=\sum_{\nu=0}^{\infty}\kappa_{\nu}(ik)^{\nu}/\nu!$], it is found that *the cumulants κ_{ν} of the total delay time distribution are the sum of the cumulants $\kappa_{\nu}^{(\lambda)}$ of the individual resonance distributions.* In particular, the first three cumulants are equal to the mean and to the second and the third *central* moments. They are simply the sum over the resonances λ of the results given in Sec. III for the isolated resonance case. The autocorrelation function also takes the form of a sum over resonances.

B. Many overlapping resonances

We now assume two further simplifications, which are aimed at describing the regime of overlapping resonances:

$$\Gamma_{\lambda}=\Gamma,\quad \lambda=1,2,\ldots,n\ , \tag{54}$$

$$\Gamma\gg D\ . \tag{55}$$

Here D is the mean spacing between the real part of the resonance energies E_{λ}. This provides a model in the spirit of Ericson's theory of cross section fluctuations [24,37]. That this is so can be checked by computing the "two-point" moment ratio $\langle S_{ab}(E)S_{ab}^{*}(E+\varepsilon)\rangle/\langle|S_{ab}(E)|^2\rangle$ $(a\neq b)$ for the S matrix (2). The numerator is given by

$$\langle S_{ab}(E)S_{ab}^{*}(E+\varepsilon)\rangle=\sum_{\lambda}\frac{\langle[\gamma_a^{(\lambda)}]^2[\gamma_b^{(\lambda)}]^2\rangle}{(E_{\lambda}-E-i\Gamma/2)(E_{\lambda}-E-\varepsilon+i\Gamma/2)}=\frac{2-i\pi\Gamma^2}{N(N+2)(D(\varepsilon-i\Gamma)}\ . \tag{56}$$

In the last equality the angle integration has been performed using (32) and the sum over discrete resonances has been replaced by an integral over a continuous vari-

able E_λ,

$$\sum_\lambda \rightarrow \frac{1}{D}\int dE_\lambda \ , \tag{57}$$

which is a good approximation because of (55). This integral can then be performed by contour integration in the complex plane. For the ratio one obtains

$$\frac{\langle S_{ab}(E)S_{ab}^*(E+\varepsilon)\rangle}{\langle |S_{ab}(E)|^2\rangle} = \frac{-i\Gamma}{\varepsilon - i\Gamma} \ , \tag{58}$$

which has the dependence found in Ericson's theory.

For the average of the delay time we find, replacing sums by integrals in (50) and using (34),

$$\langle q\rangle = \frac{h}{ND} \ , \tag{59}$$

a well known result, which is actually independent of the peculiarities of the model [38]. Notably, $\langle q\rangle$ is independent of the resonance width Γ and depends only on the mean resonance spacing D. This is essentially the result obtained from the classical spectral property for bimolecular collisions, which relates the delay time to the excess state density associated with the interaction region of the potential energy function [39]. In the limit of long delay times, which one anticipates to be associated with resonance scattering, $\langle q\rangle$ is approximately the mean classical transit time of the interaction region for a set of trajectories. It is known that this mean has precisely the form of Eq. (59) if N is interpreted as the number of states below energy E on dividing surfaces that partition the interaction region from the remainder of phase space and D^{-1} as the state density in the interaction region [40].

An arbitrary cumulant of the delay time distribution can be readily obtained using the results of IV A and noting, by examining tables of cumulants in terms of moments, that the νth cumulant for the λ resonance can be written as $\kappa_\nu^{(\lambda)} = f_\nu (q_m^{(\lambda)})^\nu$, where the proportionality factor f_ν depends on the order of the cumulant ν, but is independent of the particular resonance λ. The f_ν consist of combinations of moments $\langle \gamma^{2j}\rangle$, $j \le \nu$, which are known [see (33)]. Here is a table of the first three factors:

$$\begin{aligned} f_1 &= \frac{\langle\gamma^2\rangle}{\Gamma} = \frac{1}{N} \ , \\ f_2 &= \Gamma^{-2}[\langle\gamma^4\rangle - \langle\gamma^2\rangle^2] = \frac{2(N-1)}{N^2(N+2)} \ , \\ f_3 &= \Gamma^{-3}[\langle\gamma^6\rangle - 3\langle\gamma^4\rangle\langle\gamma^2\rangle + 2\langle\gamma^2\rangle^3] \\ &= \frac{8(N-2)(N-1)}{(N+4)(N+2)N^3} \ . \end{aligned} \tag{60}$$

For large N the factors are simply proportional to $N^{-\nu}$. (Although no *simple* formula exists for the cumulants in terms of moments and therefore for the factors f_ν, there are "recipes" that can be used to generate the expressions for an arbitrary order; see, e.g., [41].) Thus the sum $\kappa_\nu = \sum_\lambda \kappa_\nu^{(\lambda)}$ can be replaced by an integral over $q_m^{(\lambda)}$ that involves νth-order poles. The general result is obtained by induction

$$\begin{aligned} \kappa_\nu &= f_\nu \sum_\lambda (q_m^{(\lambda)})^\nu \\ &= \frac{f_\nu(\hbar\Gamma)^\nu}{D}\int \frac{dE_\lambda}{(E_\lambda - E + i\Gamma/2)^\nu (E_\lambda - E - i\Gamma/2)^\nu} \\ &= \frac{2\pi(2\nu-2)!\hbar^\nu f_\nu}{[(\nu-1)!]^2 D\Gamma^{\nu-1}} \ . \end{aligned} \tag{61}$$

A comparison is made next between the first three cumulants [42] of the exponential, χ^2 with $2p$ degrees of freedom, Gaussian, and present distributions (denoted, respectively, as $\kappa^{\exp}$, κ^{χ^2}, κ^{Gau}, and κ). The first cumulant, the average, is set equal in all the distributions

$$\kappa_1 = \kappa_1^{\exp} = \kappa_1^{\chi^2} = \kappa_1^{\mathrm{Gau}} = \frac{h}{ND} \ . \tag{62}$$

The second one reads

$$\begin{aligned} &\kappa_2 = \frac{2(N-1)h^2}{\pi D\Gamma N^2(N+2)}, \quad \kappa_2^{\exp} = \left[\frac{h}{ND}\right]^2 , \\ &\kappa_2^{\chi^2} = \left[\frac{h}{ND}\right]^2 \Big/ p, \quad \kappa_2^{\mathrm{Gau}} = \sigma^2 \ . \end{aligned} \tag{63}$$

The variances of the Gaussian and the χ^2 distributions may be set equal to κ_2. This fixes the value of σ^2 for the Gaussian and the parameter p for the χ^2 distribution, $p = (N+2)\pi\Gamma/[(N-1)2D]$. The third-order cumulants are then

$$\begin{aligned} &\kappa_3 = \frac{(N-1)(N-2)}{(N+2)(N+4)}\frac{12h^3}{\pi^2 D\Gamma^2 N^3}, \quad \kappa_3^{\exp} = 2\left[\frac{h}{ND}\right]^3 , \\ &\kappa_3^{\chi^2} = \frac{2(N-1)(N+4)}{3(N-2)(N+2)}\kappa_3, \quad \kappa_3^{\mathrm{Gau}} = 0 \ . \end{aligned} \tag{64}$$

Thus, even though the χ^2 distribution is the closest one, the dependence on N^{-3} of κ_3 makes the Gaussian (with a vanishing third cumulant) a good approximation to $P(q)$ in the limit of large N and Γ/D. This will not be the case in general. Note among other differences that the range for the Gaussian is the full real axis, all other distributions, including the one derived here, being restricted to the positive real axis. All except the Gaussian show some asymmetry, as reflected by a nonvanishing third cumulant. The exponential distribution, in fact, a χ^2 distribution with $p = 1$, reproduces the dependence of $P(q)$ with respect to (large) N but fails otherwise. For the χ^2 distribution, the number of degrees of freedom for large N is $2p \rightarrow \pi\Gamma/D \sim \Gamma/D$.

The energy autocorrelation function takes in the overlapping limit the form

$$\begin{aligned} F(\varepsilon) &= \langle q(E)q(E+\varepsilon)\rangle - \langle q(E)\rangle\langle q(E+\varepsilon)\rangle \\ &= \frac{\hbar^2\Gamma^2}{D}\left[\frac{2(N-1)}{N^2(N+2)}\right] \\ &\quad \times \int \frac{dE_\lambda}{[(E-E_\lambda)^2 + \Gamma^2/4][(E+\varepsilon-E_\lambda)^2 + \Gamma^2/4]} \\ &= \left[\frac{2(N-1)}{N^2(N+2)}\right]\frac{4\pi\hbar^2\Gamma}{D(\varepsilon^2+\Gamma^2)} \ , \end{aligned} \tag{65}$$

a Lorentzian function that scales as N^{-2} and whose half-width Γ corresponds to the width of the resonances. Note that this expression does not depend on the energy E and tends to zero for large ε. We conclude that Q_{uu} is ergodic in that energy averages of q are equivalent to angle averages of q (see Ref. [43] for a similar analysis of the cross section). Dividing by $F(0)$, we obtain the same functional dependence found by Ericson for the cross section,

$$\frac{F(\varepsilon)}{F(0)}=\frac{1}{1+(\varepsilon/\Gamma)^2} . \tag{66}$$

Any conclusions about the ergodicity of other functions of q, e.g., the energy autocorrelation function, require an additional analysis, which has not been undertaken.

V. FLUCTUATIONS AND DOUBLE AVERAGING

Up to this point the positions of the resonance energies, i.e., the *configuration* or set of numbers $\{E_\lambda\}$, have been assumed to be fixed for a given ensemble. In fact, in the most general case one would consider S-matrix ensembles where these positions differ from one member of the ensemble to the other. In the spirit of previous approximations, we shall restrict ourselves to ensembles where the configurations (resonance energies) and the angles (partial width amplitudes) are independent stochastic variables. (See [44,36] and references therein for a justification and applications of this approach. The difference with these earlier works is that they make use of a K-matrix representation of the scattering matrix, while we deal directly with a properly parametrized S matrix. However, in both ways unitarity of S is ensured for all members of the ensemble.) In complex scattering systems the distribution of resonance energy spacings is usually assumed to agree with that for the spacings between the eigenstates of random real Gaussian orthogonal matrices. The distribution of the ratio x of spacings between adjacent levels to the mean distance D can be given for most practical purposes by the Wigner distribution,

$$P(x)=(\pi/2)x\exp(-\pi x^2/4) . \tag{67}$$

There are also correlations between nearest-neighbor spacings expressed in terms of certain correlation coefficients [44]. In our computations the Wigner distribution is imposed and the first four of these coefficients are approximately accounted for following the prescription given by Hofmann, Richert, and Tepel [44]. This method allows random sets $\{E_\lambda\}$ to be obtained without explicitly diagonalizing matrices, a forbidding task when a large number of resonances is required.

The configurational average is thus to be combined with the angle average described in the previous sections, the latter being operationally associated with a channel average. We shall assign the symbols $\mathcal{C}$ and $\mathcal{A}$, respectively, to these averages. As discussed by Muga and Levine [45], when two different averages are involved, there are different types of fluctuations. For the autocorrelation function Δ, essentially a fluctuation at two different energies, we find, following the notation used in [45] and with $q=q(E)$ and $q'=q(E+\varepsilon)$,

$$\Delta_{\mathcal{CA}}\equiv\mathcal{CA}(qq')-\mathcal{CA}q\mathcal{CA}q' , \tag{68}$$

$$\Delta_{\mathcal{C}}\mathcal{A}\equiv\mathcal{C}(\mathcal{A}q\mathcal{A}q')-\mathcal{CA}q\mathcal{CA}q' , \tag{69}$$

$$\mathcal{C}\Delta_{\mathcal{A}}\equiv\mathcal{C}[\mathcal{A}(qq')-\mathcal{A}q\mathcal{A}q'] , \tag{70}$$

$$\Delta_{\mathcal{A}}\mathcal{C}\equiv\mathcal{A}(\mathcal{C}q\mathcal{C}q')-\mathcal{A}\mathcal{C}q\mathcal{A}\mathcal{C}q' , \tag{71}$$

$$\mathcal{A}\Delta_{\mathcal{C}}\equiv\mathcal{A}[\mathcal{C}(qq')-\mathcal{C}q\mathcal{C}q'] , \tag{72}$$

which are related by

$$\Delta_{\mathcal{CA}}=\Delta_{\mathcal{C}}\mathcal{A}+\mathcal{C}\Delta_{\mathcal{A}} \tag{73}$$

$$=\Delta_{\mathcal{A}}\mathcal{C}+\mathcal{A}\Delta_{\mathcal{C}} \tag{74}$$

and where $\mathcal{C}$ denotes a configurational average and $\mathcal{A}$ denotes an angle average. Equation (68) is the fluctuation in a single *dual* $\mathcal{CA}$ ensemble where both angles and configurations vary independently. Equation (69) describes the configurational fluctuation of angle averaged values and Eq. (70) is a configurational average of the fluctuations evaluated at a given configuration due to the distribution of angles. Equations (71) and (72) admit interpretations similar to Eqs. (69) and (70) by reversing the order of the averages. Note that $\mathcal{CA}=\mathcal{AC}$.

Even though configurational averages have theoretical significance and may even be implemented by examination of different systems [1], experiments are more likely to involve energy averages for a given system. These can, however, have different origins: One possibility is that the collisional wave packets are broad in energy so that the experiment directly provides broad energy averages (coherent mixture) or the wave packets are sharply defined in energy and have a distribution of centroid energies (monochromatic incoherent mixture). These two scenarios require different treatments [6] and involve different fluctuations if a further average, for example, over amplitudes, is performed. In the following we do not explicitly restrict our analysis to one scenario or the other, but do point out, in due course, how the order of the angle and energy averaging determines whether the coherent or incoherent case is being modeled.

The energy average $\mathcal{E}$, in combination with $\mathcal{A}$, leads to the family of quantities

$$\Delta_{\mathcal{EA}}\equiv\mathcal{EA}(qq')-\mathcal{EA}q\mathcal{EA}q' , \tag{75}$$

$$\Delta_{\mathcal{E}}\mathcal{A}\equiv\mathcal{E}(\mathcal{A}q\mathcal{A}q')-\mathcal{EA}q\mathcal{EA}q' , \tag{76}$$

$$\mathcal{E}\Delta_{\mathcal{E}}\equiv\mathcal{E}[\mathcal{A}(qq')-\mathcal{A}q\mathcal{A}q'] , \tag{77}$$

$$\Delta_{\mathcal{A}}\mathcal{E}\equiv\mathcal{A}(\mathcal{E}q\mathcal{E}q')-\mathcal{A}\mathcal{E}q\mathcal{A}\mathcal{E}q' , \tag{78}$$

$$\mathcal{A}\Delta_{\varepsilon}\equiv A[\mathcal{E}(qq')-\mathcal{E}q\mathcal{E}q'] , \tag{79}$$

which obey the relations

$$\Delta_{\mathcal{EA}}=\Delta_{\mathcal{E}}\mathcal{A}+\mathcal{E}\Delta_{\mathcal{A}} \tag{80}$$

$$=\Delta_{\mathcal{A}}\mathcal{E}+\mathcal{A}\Delta_{\mathcal{E}} . \tag{81}$$

Different mathematical implementations of the energy average are possible. We have used a Lorentzian of width β at half-height centered at E_L,

$$\mathcal{L}(E';E_L,\beta)=\frac{1}{2\pi}\frac{\beta}{(E'-E_L)^2+\beta^2/4} . \tag{82}$$

This particular choice allows some of the integrals to be done analytically and has been used elsewhere [6] for this reason.

We have performed numerical computations of the quantities (68)–(70) and (75)–(79) [the computation of (71) and (72) is time consuming and has not been performed] for fixed Γ ensembles with 30 channels $N=30$ in the limits of (a) overlapping resonances $\Gamma/D=10$ and (b) quasi-isolated resonances $\Gamma/D=0.1$. The intermediate case (c) $\Gamma/D=1$ has also been examined. The first energy of the configuration is always taken as $E_1=0$ and q is evaluated at $E=nD/2$, the approximate midpoint of the energy range spanned by n resonances. In all cases $D=1$ (atomic units are used throughout), $E_L=E$, and $\beta=10$, except where indicated. The number of configurations n_c and the number of resonances n (see Table I) are chosen so that a further increase does not significantly change the results. The main findings are as follows.

(i) Energy and configurational averages give essentially the same results for the autocorrelation functions when combined with the angle average in all cases; see Figs. 2(a)–2(d). When normalizing each component, however, the detailed structures of $\Delta_{\mathcal{E}\mathcal{A}}(\varepsilon)/\Delta_{\mathcal{E}\mathcal{A}}(0)$ and $\Delta_{\mathcal{C}\mathcal{A}}(\varepsilon)/\Delta_{\mathcal{C}\mathcal{A}}(0)$ may be different [see Figs. 3(a)–3(d)], even though in the total fluctuation depicted in Figs. 2(a)–2(d) the contribution of this difference is minimal. The range of the energy average β is important in this respect. A large β tends to wash out the configuration-dependent features, while a small β will yield configuration-dependent structure. For example, Fig. 4 shows the average $\mathcal{E}\mathcal{A}q$ for a particular configuration at different values of β in the case $\Gamma/D=0.1$. The peaks at $\beta=0.01$ clearly indicate the position of the resonances, but, at $\beta=10$, $\mathcal{E}\mathcal{A}q$ is completely flat. Of course, while the present model is built so that β can be arbitrarily large, in real systems the basic hypothesis of independence of resonance parameters with respect to energy will be only approximately true in a limited energy range.

(ii) The normalized quantities $\mathcal{C}\Delta_{\mathcal{A}}(\varepsilon)/\mathcal{C}\Delta_{\mathcal{A}}(0)$ and $\mathcal{E}\Delta_{\mathcal{A}}(\varepsilon)/\mathcal{E}\Delta_{\mathcal{A}}(0)$ (dashed curves in Fig. 3), accurately agree in all cases (even for $\Gamma/D=0.1$) with the normalized autocorrelation function obtained analytically for the overlapping limit Eq. (66). This means that the decompositions in (73) and (80) are particularly suitable to analyze the total fluctuations $\Delta_{\mathcal{C}\mathcal{A}}$ or $\Delta_{\mathcal{E}\mathcal{A}}$. In general, these will consist of a universal Lorentzian component plus a possible oscillating contribution from $\Delta_{\mathcal{E}\mathcal{A}}$ or $\Delta_{\mathcal{C}\mathcal{A}}$ that will become increasingly important as the ratio Γ/D decreases. This is in practice useful to avoid the numerical computation of the universal component.

(iii) $\mathcal{A}\Delta_{\mathcal{E}}$ is of importance for an experiment where the delay time is obtained for wave packets of width β. This implies an energy average *first*. The "global," nonparametrized, Q operator is energy conserving [46] and thus the expectation values for given wave packets are averages of the energy-dependent results weighted with the corresponding probabilities of the different energies in the packet of interest. Estimates can be easily obtained by assuming the Lorentzian energy distribution (82). The delay time for a packet with this energy distribution, in an arbitrary entrance channel u, is

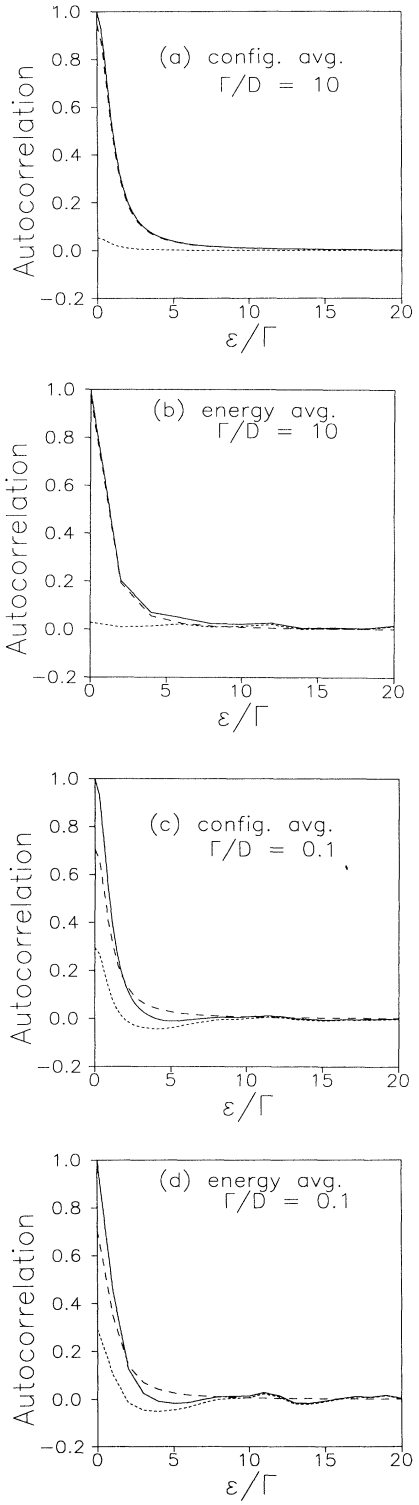

FIG. 2. (a) $\Delta_{\mathcal{C}\mathcal{A}}/\Delta_{\mathcal{C}\mathcal{A}}(\varepsilon=0)=[\Delta_{\mathcal{C}\mathcal{A}}+\mathcal{C}\Delta_{\mathcal{A}}]/\Delta_{\mathcal{C}\mathcal{A}}(\varepsilon=0)$ (solid), $\Delta_{\mathcal{C}\mathcal{A}}/\Delta_{\mathcal{C}\mathcal{A}}(\varepsilon=0)$ (dotted), and $\mathcal{C}\Delta_{\mathcal{A}}/\Delta_{\mathcal{C}\mathcal{A}}(\varepsilon=0)$ (dashed) versus ε/Γ for $\Gamma/D=10$. (b) $\Delta_{\mathcal{E}\mathcal{A}}/\Delta_{\mathcal{E}\mathcal{A}}(\varepsilon=0)$ $=[\Delta_{\mathcal{E}\mathcal{A}}+\mathcal{E}\Delta_{\mathcal{A}}]/\Delta_{\mathcal{E}\mathcal{A}}(\varepsilon=0)$ (solid), $\Delta_{\mathcal{E}\mathcal{A}}/\Delta_{\mathcal{E}\mathcal{A}}(\varepsilon=0)$ (dotted), and $\mathcal{E}\Delta_{\mathcal{A}}/\Delta_{\mathcal{E}\mathcal{A}}(\varepsilon=0)$ (dashed) versus ε/Γ for $\Gamma/D=10$. (c) Same as (a), but for $\Gamma/D=0.1$. (d) Same as (b), but for $\Gamma/D=0.1$.

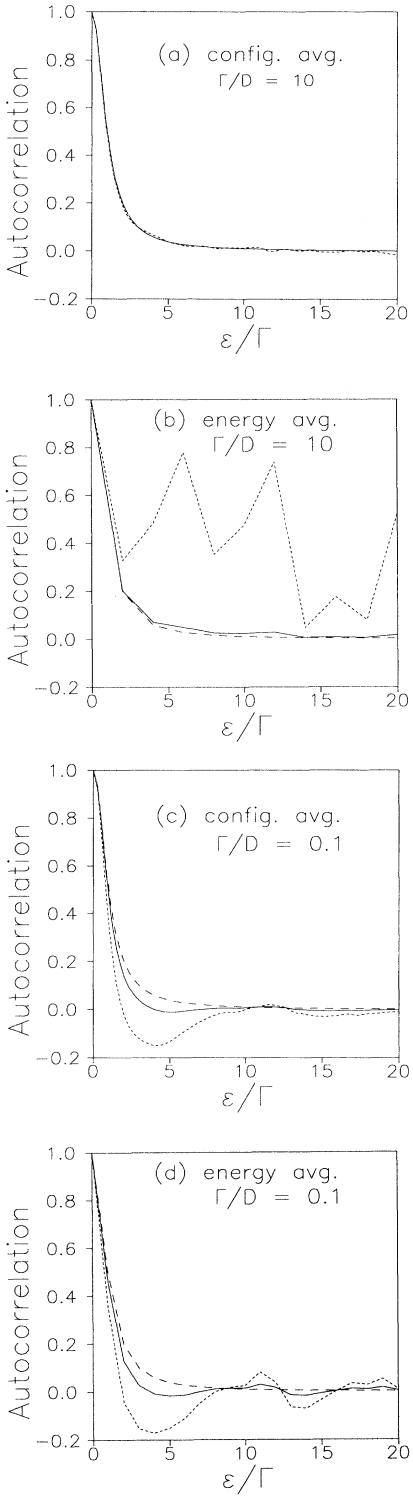

FIG. 3. (a) $\Delta_{\mathcal{C}\mathcal{A}}/\Delta_{\mathcal{C}\mathcal{A}}(\varepsilon=0)$ (solid), $\Delta_{\mathcal{C}\mathcal{A}}/\Delta_{\mathcal{C}\mathcal{A}}(\varepsilon=0)$ (dotted), and $\mathcal{C}\Delta_{\mathcal{A}}/\mathcal{C}\Delta_{\mathcal{A}}(\varepsilon=0)$ (dashed) versus ε/Γ for $\Gamma/D=10$. The dashed curve is not distinguishable from the solid curve on the scale of the plot. (b) $\Delta_{\mathcal{E}\mathcal{A}}/\Delta_{\mathcal{E}\mathcal{A}}(\varepsilon=0)$ (solid), $\Delta_{\mathcal{E}\mathcal{A}}/\Delta_{\mathcal{E}\mathcal{A}}$ $(\varepsilon=0)$ (dotted), and $\mathcal{E}\Delta_{\mathcal{A}}/\mathcal{E}\Delta_{\mathcal{A}}(\varepsilon=0)$ (dashed) versus ε/Γ for $\Gamma/D=10$. (c) Same as (a), but for $\Gamma/D=0.1$ and that the solid and dashed curves are now distinguishable. (d) Same as (b), but for $\Gamma/D=0.1$.

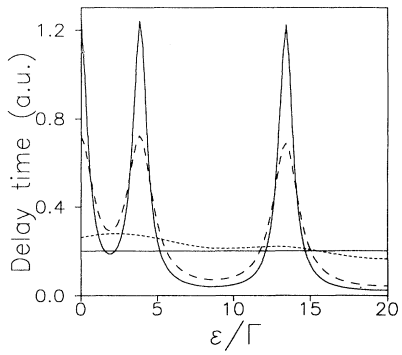

FIG. 4. $\mathcal{E}\mathcal{A}q$ (a.u.) versus ε/Γ for a particular configuration of resonances. $\Gamma=0.1$ a.u. and $D=1$ a.u.; $\beta=0.01$, 0.10, 1.00, and 10 a.u. (peaked solid, dashed, dotted, and straight solid lines, respectively).

$$\mathcal{E}q=\int dE\,\mathcal{L}(E;E_L,\beta)q(E)\ , \tag{83}$$

where $q(E)$ is the energy parametrized value in (50). By averaging over the angles,

$$\mathcal{A}\mathcal{E}q=\frac{\hbar}{N}\sum_{\lambda}\frac{\beta+\Gamma}{(E_L-E_\lambda)^2+(\beta+\Gamma)^2/4}\ . \tag{84}$$

For very broad packets $\beta\gg\Gamma$ and a single resonance (only one member in the sum), the packet moves freely without being affected by the resonance. In this case the delay tends to 0 as $\beta\to\infty$. However, when multiple resonances are present a large β on the scale of the average spacing D implies that sums can be substituted by integrals in (84), so that (59) is recovered, i.e., the delay will depend on the inverse of the product of the average spacing and number of channels. For overlapping resonances $\Gamma\gg D$, $\langle q\rangle$ is an energy independent quantity, Eq. (59), and $\mathcal{A}\mathcal{E}q=\langle q\rangle$.

Similarly, the autocorrelation function $\Delta_{\mathcal{A}\mathcal{E}}$ for wave packets is

$$\begin{aligned}\mathcal{A}(\mathcal{E}q\mathcal{E}q')-\mathcal{A}\mathcal{E}q\mathcal{A}\mathcal{E}q'&=\frac{2\hbar^2(\beta+\Gamma)^2(N-1)}{N^2(N+2)}\\&\times\sum_{\lambda}\frac{1}{(E_L-E_\lambda)^2+(\beta+\Gamma)^2/4}\\&\times\frac{1}{(E_L+\varepsilon-E_\lambda)^2+(\beta+\Gamma)^2/4}\ .\end{aligned} \tag{85}$$

Replacing sums by integrals in the overlapping limit the autocorrelation function in (85) takes the same form as for the fixed energy case Eq. (65) with Γ replaced by $\Gamma+\beta$. Indeed, if $\beta+\Gamma\gg D$, the substitution of integrals for sums can always be done in (85) whether or not the condition $\Gamma\gg D$ is satisfied, i.e., also far from the overlapping limit. The fixed energy results are recovered as $\beta\to 0$.

(iv) The averages $\mathcal{C}\mathcal{A}q$ or $\mathcal{E}\mathcal{A}q$ are in all cases equal to

TABLE I. First two cumulants (κ_1,κ_2) and skewness coefficient (s) calculated for different ratios Γ/D in the dual $\mathcal{CA}$ ensemble. For comparison, the values obtained analytically for the single $\mathcal{A}$ ensemble in the limit of overlapping resonances are also provided [quantities with superscript (0)].

Γ/D	κ_1, $\kappa_1^{(0)}$	κ_2, $\kappa_2^{(0)}$	$s=\kappa_3^2/\kappa_2^3$, $s^{(0)}$	n_c	n
10.0	0.21, 0.21	0.0027, 0.0025	0.66, 0.65	20 000	7500
1.0	0.21, 0.21	0.029, 0.025	2.16, 2.07	20 000	2500
0.1	0.21, 0.21	0.37, 0.25	7.04, 6.55	27 000	2000

the result predicted for the overlapping limit Eq. (59).

(v) The first three cumulants of the total lifetime distribution at fixed energy E have been evaluated for the $\mathcal{CA}$ ensemble. The results are summarized in Table I. Note the increase of the second cumulant and of the skewness of the distribution when Γ/D decreases, as well as the progressive deviation from the corresponding overlapping limit results. It is possible to visualize approximately this distributions as follows: The exact expression is

$$P(q)=\int d\{E_\lambda\}P(\{E_\lambda\})P(q|\{E_\lambda\}) , \tag{86}$$

where $P(q|\{E_\lambda\})$ is the conditional probability for having delay time q in the configuration $\{E_\lambda\}$. The exact expression for $P(q|\{E_\lambda\})$ is unknown, but we can calculate its cumulants by summing over the cumulants of the individual resonances as described in Sec. IV A. The conditional probability in terms of its cumultants takes the form

$$P(q|\{E_\lambda\})=\frac{1}{2\pi}\int dk\, e^{-ikq}\exp\left[\sum_{\nu=1}^{\infty}\kappa_\nu(ik)^\nu/(\nu!)\right] . \tag{87}$$

If, for the purpose of obtaining a graphic representation, third- and higher-order cumulants of $P(q|\{E_\lambda\})$ are neglected, the integral yields a Gaussian

$$P(q|\{E_\lambda\})\approx\frac{1}{(2\pi\kappa_2)^{1/2}}e^{-(q-\kappa_1)^2/(2\kappa_2)} . \tag{88}$$

This approximate result is then substituted in the integral (86), which is evaluated numerically. The approximation will be poor very close to $q=0$, where large values of k and therefore higher-order cumulants contribute in (87) and allows for negative values of q, but gives a reasonable pictorial view of the main features of the distribution and, in particular, the tail of the distribution at large q. The results are shown in Fig. 5 for the following three cases: (a) $\Gamma/D=10$ (overlapping resonances) (b) $\Gamma/D=0.1$ (quasi-isolated resonances), and (c) $\Gamma/D=1$. Note that even though the third cumulants of the components have been neglected, the approximate distribution may have nonvanishing third cumulants. Also note that this approximation has *not* been used for the computations in Table I.

For case (a) Fig. 5 confirms that the distribution of delay times is essentially a Gaussian distribution. For case (b) we note that $P(q)$ has a slowly decaying long-time tail whose functional dependence on q is of interest (dotted curve in Fig. 5). A double logarithmic plot (Fig. 6) reveals that the decay is algebraic $P(q)\sim q^{-z}$, with $z\sim -1.6$. The origins of this behavior and of this particular value of the decay exponent are not apparent to us. For a discussion of conditions leading to algebraic decay of the population of (quasi)bound states in chaotic quantum systems and a relation between such decay functions and Smith's delay time, the reader is referred to Ref. [10]. We emphasize that the decay functions considered in Ref. [10] are not the same as the delay time probability

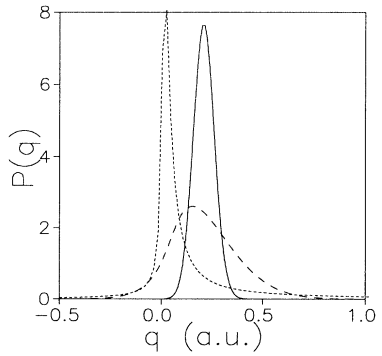

FIG. 5. Approximate $P(q)$ versus q (a.u.) for $\Gamma/D=10$ (solid), $\Gamma/D=1$ (dashed), and $\Gamma/D=0.1$ (dotted). The negative tails should be disregarded.

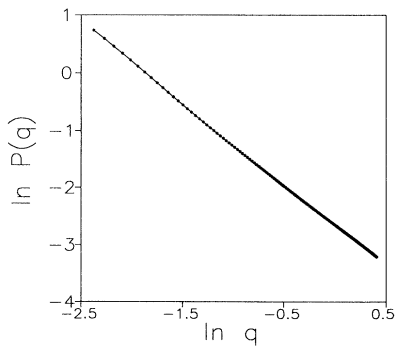

FIG. 6. Double logarithmic plot of the decaying part of $P(q)$ versus q (a.u.) for $\Gamma/D=0.1$.

distributions studied here. In classical chaotic scattering, algebraic decay is associated with nonhyperbolicity arising from the presence of stable or marginally stable periodic orbits in the interaction region.

VI. DISCUSSION

Some statistical properties of the delay time matrix for ensembles of unitary, multichannel, S-matrix models based on Breit-Wigner resonances have been examined from single resonances to the limit of many overlapping resonances, from few to many open channels, and from fixed to varying resonance widths. Explicit expressions have been obtained for autocorrelation functions, probability distributions, and/or cumulants that will serve as a simple reference for understanding more realistic cases. A key ingredient in preserving unitarity and allowing for analytical results is the formal treatment given to the independent-resonance assumption.

It is worth clarifying a common misapprehension: The quantities examined, Smith's delay times, are not decay times of prepared states, although they are related to them. The study of the decay of a prepared state involves a broad wave packet in energy whose survival probability (approximately an exponentially decaying function at intermediate times) is followed [47]. In general, the delay time of such a packet, as defined by the expectation value of Q, can be regarded as the difference between the *integral* over time of the survival probability with and without interaction [46,47,10]. While the statistical nature of the exponentially decaying survival probability is due to the possibility of preparing many systems under identical conditions, i.e., to ensembles of quantum mechanical systems described by the same *wave functions,* the dispersion of delay times in the present paper refers to the average values of Q calculated for members of ensembles of S *matrices* characterized by a variation in the transition amplitudes, resonance widths, or resonance positions.

Analytical expressions for the delay time probability distribution have been obtained for a single resonance. They tend, in the limit of large number of channels, to a χ^2 distribution with one degree of freedom. For multiple resonances the time delay distribution can be characterized by means of its cumulants, which are obtained as the sum of the individual resonance cumulants. Approximate analytical forms have been given in the overlapping limit. A particularly good fit in this case is obtained by means of a χ^2 distribution with a number of degrees of freedom proportional to Γ/D and essentially independent of the number of open channels N for large N values (a Gaussian is also a good approximation for large N). N is, however, the number of degrees of freedom assigned to the χ^2 distribution of the widths Γ when the transition amplitudes are treated as Gaussian-independent random variables (Sec. III B).

In the overlapping resonances case, the Lorentzian obtained for the autocorrelation function of the diagonal elements of the delay time is a real function, as it should be, because of the Hermiticity of Q. This Hermitian property derives from the unitarity of S as shown by Smith [3]. In our derivation we have made use of assumptions present in earlier models to describe "Ericson fluctuations." It is worth emphasizing, however, that no version of the Ericson model for overlapping resonances would lead to a purely real autocorrelation function. Indeed, we have found that the autocorrelation function for another simplified model proposed by Bauer, Mello, and McVoy [6] is a complex quantity, because in that model the unitarity of S is not ensured for all members of the ensemble. The form found for the autocorrelation function implies ergodicity in the sense that angle (transition amplitude) and energy (resonance position) averages of the delay times are equivalent.

While most analytical results are obtained for an ensemble where the "configuration," or set of resonance energies, is fixed, the more general ensembles where the configuration varies have also been numerically studied, as well as the relations between configurational and energy averages when each is accompanied by an "angle" average over the amplitudes. The presence of the two averages (angle and configuration, or angle and energy) leads to different partitions of the total autocorrelation function. It has been found that one of the two components (when normalized) is universal, i.e., under all conditions examined (not only in the overlapping limit) it is the same Lorentzian function, precisely the one predicted by Ericson for the cross section. This universal component is the configurational (or energy) average of the fluctuation of delay times due to the amplitudes. The other component, a configurational (or energy) fluctuation of quantities averaged first over the amplitudes, is negligible in the overlapping limit, but can be of importance in other cases, yielding oscillations attributable to the resonance structure. The same can be said qualitatively about the cross section (see Appendix B), with the difference that the contribution of the oscillating component appears, on the basis of numerical investigation, to be much less important for the cross section. Thus the delay time appears to be more sensitive than the cross section to the underlying resonance structure. Energy averages and configurational averages behave similarly in most cases when combined with the average over amplitudes (angles), provided the energy average is performed over a large energy interval.

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APPENDIX A: EFFECT OF THE BACKGROUND

The most general S-matrix form compatible with a simple pole includes a unitary "background" matrix S^{bg} [22],

$$S = S(E) = S^{bg}\left[1 - \frac{iA}{E - E_0 + i\Gamma/2}\right], \tag{A1}$$

where the term in large parentheses is also unitarity. We shall assume that in the energy interval of interest, all parameters A and the background are energy independent. If S is to be symmetric, the rank-one matrix $(S^{bg}A)_{ac}=\gamma_a\delta_c^*$ is also symmetric. The phases can be chosen so that $\gamma_c=\delta_c^*$ and thus $(S^{bg}A)_{ac}=\gamma_a\gamma_c$. However, unless $S^{bg}=1$, the γ's will generally be complex. The constraint (9) takes therefore the form $\Gamma=\sum_u|\gamma_u|^2$ and the expression for the delay time becomes

$$Q_{aa}=\frac{\hbar|\gamma_a|^2}{[(E-E_0)^2+\Gamma/4]}\ . \tag{A2}$$

This does not essentially modify our reasoning and conclusions in the main text. The ensembles are constructed by a similar procedure: Now the moduli of the γ's are homogeneously distributed over the hypersphere, while the phases are randomly chosen between 0 and 2π. This involves an extra set of N independent random variables (the phases) and integrations, but the final results for the delay time distributions remain the same. The multiple resonance case is generalized similarly.

APPENDIX B: CROSS SECTION

The inelastic cross section is proportional to $\sigma=|S_{ab}|^2$ $(a\neq b)$. In this appendix, expressions for σ, its average, and its autocorrelation function are provided. n resonances with equal width Γ are assumed.

Using Eq. (2) one finds

$$\sigma=\sum_\lambda\sum_{\lambda'}\frac{\gamma_a^{(\lambda)}\gamma_b^{(\lambda)}\gamma_a^{(\lambda')}\gamma_b^{(\lambda')}}{(E-E_\lambda+i\Gamma/2)(E-E'_\lambda-i\Gamma/2)}\ . \tag{B1}$$

Note that $\mathcal{A}S_{ab}$ $(a\neq b)$ is zero, as well as $\mathcal{E}S_{ab}$ for large enough β, i.e., we are dealing with a purely statistical model, devoid of direct reactions. The averaged cross sections accordingly describe only "compound" processes, without a "direct" contribution from rapid processes.

The angle average gives zero unless $\lambda=\lambda'$. Using the distribution (32) we find

$$\mathcal{A}\sigma=k_1Y_1\ . \tag{B2}$$

See (B5) and (B7) below for the new notation. To evaluate autocorrelation functions the product

$$\sigma\sigma'\equiv\sigma(E)\sigma(E+\varepsilon)=\sum_\lambda\sum_{\lambda'}\sum_{\lambda''}\sum_{\lambda'''}\frac{\gamma_a^{(\lambda)}\gamma_b^{(\lambda)}\gamma_a^{(\lambda')}\gamma_b^{(\lambda')}\gamma_a^{(\lambda'')}\gamma_b^{(\lambda'')}\gamma_a^{(\lambda''')}\gamma_b^{(\lambda''')}}{(E-E_\lambda+i\Gamma/2)(E-E'_\lambda-i\Gamma/2)(E+\varepsilon-E''_\lambda+i\Gamma/2)(E+\varepsilon-E'''_\lambda-i\Gamma/2)} \tag{B3}$$

is required. The angle average will again give zero unless the resonance indices are at least pairwise equal: The possible terms are $(\lambda=\lambda',\ \lambda''=\lambda''',\ \lambda\neq\lambda'')$, $(\lambda=\lambda'',\ \lambda'=\lambda''',\ \lambda\neq\lambda')$, $(\lambda=\lambda''',\ \lambda'=\lambda'',\ \lambda\neq\lambda')$, and $(\lambda=\lambda'=\lambda''=\lambda''')$, each of which requires separate treatment.

Combining all terms, the autocorrelation function $\Delta_{\mathcal{A}}$ (this symbol and similar ones containing Δ in the present appendix refer to cross section fluctuations) can be written as

$$\begin{aligned}\Delta_{\mathcal{A}}&=\mathcal{A}[\sigma(E)\sigma(E+\varepsilon)]-\mathcal{A}\sigma(E)\mathcal{A}\sigma(E+\varepsilon)\\&=k_1^2[|Y_4(\varepsilon)|^2+|Y_3(\varepsilon)|^2]+Y_2(\varepsilon)(k_2-3k_1^2)\ ,\end{aligned} \tag{B4}$$

where

$$k_1=\mathcal{A}(\gamma_a^2\gamma_b^2)=\frac{\Gamma^2}{N(N+2)}\ , \tag{B5}$$

$$k_2=\mathcal{A}\gamma_a^4\gamma_b^4=\frac{9\Gamma^4}{N(N+2)(N+4)(N+6)}\ , \tag{B6}$$

$$Y_1=\sum_\lambda\frac{1}{(E-E_\lambda)^2+\Gamma^2/4}\ , \tag{B7}$$

$$Y_2=\sum_\lambda\frac{1}{[(E-E_\lambda)^2+\Gamma^2/4][(E+\varepsilon-E_\lambda)^2+\Gamma^2/4]}\ , \tag{B8}$$

$$Y_3=\sum_\lambda\frac{1}{(E-E_\lambda+i\Gamma/2)(E+\varepsilon-E_\lambda+i\Gamma/2)}\ , \tag{B9}$$

$$Y_4=\sum_\lambda\frac{1}{(E-E_\lambda-i\Gamma/2)(E+\varepsilon-E_\lambda+i\Gamma/2)}\ . \tag{B10}$$

The overlapping limit is obtained by replacing sums by integrals, as in (57). The normalized autocorrelation function for the cross section $\Delta_{\mathcal{A}}(\varepsilon)/\Delta_{\mathcal{A}}(0)$ is then found to be the same Lorentzian as for the delay time Eq. (66).

In general, it is not possible to replace sums by integrals in (B4) unless an additional average over configurations or energy is performed. (We shall discuss next the $\mathcal{C}$ average, but the same results are valid for $\mathcal{E}$.) The effect of the average $\mathcal{C}$ on $\Delta_{\mathcal{A}}$ is actually equivalent to the replacement of sums by integrals. This makes the term proportional to $|Y_3|^2$ vanish. Since the energy dependence of the remaining terms is Lorentzian, the cross section fulfills the same property found for the delay time, namely, that the normalized quantity $\mathcal{C}\Delta_{\mathcal{A}}(\varepsilon)/\mathcal{C}\Delta_{\mathcal{A}}(0)$ is a Lorentzian [Eq. (66)], for any value of the Γ/D ratio.

Indeed the results for the normalized components are completely parallel to the delay time since the other component of the dual ensemble fluctuation is given by

$$\Delta_{\mathcal{C}\mathcal{A}}=\mathcal{C}[\mathcal{A}\sigma(E)\mathcal{A}\sigma(E+\varepsilon)]-\mathcal{C}\mathcal{A}\sigma(E)\mathcal{C}\mathcal{A}\sigma(E+\varepsilon) \tag{B11}$$

$$=k_1^2\{\mathcal{C}[Y_1(0)Y_1(\varepsilon)]-\mathcal{C}Y_1(0)\mathcal{C}Y_1(\varepsilon)\}\ , \tag{B12}$$

which, up to the constant k_1, is the same as for the delay time result. Thus $\Delta_{\mathcal{C}\mathcal{A}}(\varepsilon)/\Delta_{\mathcal{C}\mathcal{A}}(0)$ is the same for the delay time and the cross section. Of course the agreement with the delay time functions need not be found for $\mathcal{C}\Delta_{\mathcal{A}}(\varepsilon)/\Delta_{\mathcal{C}\mathcal{A}}(0)$ or $\Delta_{\mathcal{C}\mathcal{A}}(\varepsilon)/\Delta_{\mathcal{C}\mathcal{A}}(0)$. The latter quantity is found (via numerical calculation) to be much smaller than the former for the cross section, so that the total normalized cross section fluctuation $\Delta_{\mathcal{C}\mathcal{A}}(\varepsilon)/\Delta_{\mathcal{C}\mathcal{A}}(0)$ $=\mathcal{C}\Delta_{\mathcal{A}}(\varepsilon)/\Delta_{\mathcal{C}\mathcal{A}}(0)+\Delta_{\mathcal{C}\mathcal{A}}(\varepsilon)/\Delta_{\mathcal{C}\mathcal{A}}(0)$ is actually coincident with the Lorentzian (66) in all cases examined.

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