Generally covariant relativistic anisotropic magnetohydrodynamics

M. Gedalin and I. Oiberman
Ben-Gurion University, Beer-Sheva, Israel
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We derive generally covariant hydrodynamical equations for a plasma with an anisotropic pressure in the external electromagnetic field. The equations are formulated in terms of the variables defined in the local plasma rest frame, in which the electric field vanishes. Generally covariant generalization for the equation of state is derived, which reduces to the Chew-Goldberger-Low [Proc. R. Soc. London, Ser. A **236**, 112 (1956)] form when the plasma temperature is nonrelativistic in the plasma rest frame. Various ultrarelativistic limits are analyzed. The obtained equations are applied to the simplest monopole geometry of the relativistic stellar wind and to the analysis of the linear waves in the limit of geometrical optics.

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I. INTRODUCTION

It is widely accepted that the study of collisionless relativistic plasmas in strong magnetic fields is essential for the understanding of the physics of various astrophysical objects, such as pulsar winds, relativistic jets, active galactic nuclei, etc. Relativistic hydrodynamics is probably the most convenient way to describe the slow and large scale motion of such a plasma. In most models of plasma hydrodynamics the simplifying assumption of pressure tensor isotropy is used (see e.g., [1]). Although this approximation is quite reasonable in *collision-dominated* situations, it may be violated in a *collisionless* plasma with an imposed strong magnetic field, where there is no mechanism that could effectively ensure the energy redistribution among the parallel (with respect to the magnetic field) and perpendicular degrees of freedom. One can expect that in such systems the plasma pressure will not be isotropic.

The nonrelativistic limit of the anisotropic plasma theory is well known [2]. This Chew-Goldberger-Low (CGL) theory predicts different equations of state for parallel and perpendicular pressure. Earlier generalizations of CGL theory onto the case of relativistic velocities and temperatures were obtained by the direct application of the covariance principle [3,4] with some guess of the state equation, or by consideration of the *warm* plasma [5,6], which in fact corresponds to the nonrelativistic limit in the plasma rest frame.

Recently, relativistic generalization of CGL has been developed for Minkowski space [7] by taking the momenta of the Vlasov equation and applying the gyrokinetic expansion in the lowest order. The state equations were obtained, which can be reduced to the ordinary CGL state equations in the nonrelativistic limit.

Relativistic collisionless plasma is expected to be present in the vicinity of compact massive objects, such as black holes, and also at early stages of the universe evolution. Although there is no direct evidence that these plasmas are embedded in a strong external magnetic field, it is reasonable to think that such magnetic fields play a significant role in the plasma confinement in the equilibrium and also determine many of the wave properties and stability features. It is, therefore, necessary to generalize the anisotropic plasma theory onto generally relativistic cases also.

In [8] the results of [7] were reproduced for a plasma in a Kerr metric. That was achieved by an explicit 3+1 splitting of the space time and a definite choice of orthonormal tetrads for the so-called general comoving frames (GCMF). The derivation is based on the assumption that the three-electric-field vanishes in the chosen GCMF. On the other hand, the anisotropic plasma hydrodynamical theories, both nonrelativistic [2] and relativistic [7], are essentially local in the sense that the state equations are written in variables defined in the local plasma rest frame, so that the state of plasma "here and now" should not depend on what happens at global scales. Therefore, the very possibility of the splitting or global definition of GCMF should not be essential for the formulation of the generally relativistic (anisotropic) magnetohydrodynamics, which can be cast in a completely covariant form. This underlying idea that the local magnetic field determines the plasma anisotropy requires development of a local generally covariant theory of hydrodynamics of plasma with anisotropic pressure.

In the present paper we develop a generally relativistic generalization of the anisotropic hydrodynamics for a plasma in a strong magnetic field, based on the "locality principle." We show that the generally covariant equations can be formulated in a frame-independent form, without any assumptions about the metric. We also *derive* the distribution function in the axially symmetric form instead of postulating it (cf. [8]).

The paper is organized as follows. In Sec. II we make an invariant frame-independent splitting of the electromagnetic field tensor. The energy-momentum tensor of the electromagnetic field is further expressed in terms of the invariant magnetic field. In Sec. III we derive the distribution function from the Vlasov equation for the plasma in a strong magnetic field in the curved geometry, and obtain the basic hydrodynamical equations. In

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Sec. IV the state equations for anisotropic plasma are obtained in a most general form. Various limits are discussed. In Sec. V we apply the obtained results to the simplest case of the monopole relativistic stellar wind and consider ultrarelativistic limits. In Sec. VI we present the dispersion relations for the magnetohydrodynamic (MHD) waves obtained in the short wavelength limit of the geometrical optics.

II. INVARIANT ELECTROMAGNETIC FIELD SPLITTING

We assume that the description of the plasma can be reduced to one-fluid magnetohydrodynamics (problems that arise in these reduction procedures from the multifluid description to the one-fluid one are described in [9]). In this case the plasma motion is described by the set of MHD and Maxwell equations,

$$J^\mu;_\mu = 0, \tag{1}$$
$$T^{\mu\nu}_{;\nu} = j_\nu F^{\mu\nu}, \tag{2}$$
$$F^{\mu\nu}_{;\nu} = 4\pi j^\mu, \tag{3}$$
$$F_{\mu\nu;\sigma} + F_{\nu\sigma;\mu} + F_{\sigma\mu;\nu} = 0, \tag{4}$$

where J^μ is the mass flow density, j^μ is the current density, $F_{\mu\nu}$ is the electromagnetic tensor, $T^{\mu\nu}$ is the plasma energy-momentum tensor, and a semicolon denotes the usual covariant derivative. In what follows we adopt a positive metric signature, i.e., $(+,-,-,-)$. We also use the natural units in which $c = G = 1$.

The electromagnetic field tensor $F^{\mu\nu}$ can be decomposed in the usual way [10]. Namely, let $U^\mu(x^\mu)$ be a four-velocity field, $U^\mu U_\mu = 1$. Then $E_\mu = F_{\mu\nu}U^\nu$ and $B_\mu = (1/2)\varepsilon_{\mu\nu\alpha\beta}F^{\nu\alpha}U^\beta$ are the electric and magnetic fields, respectively, in the frame moving with the four-velocity U^μ. Here $\varepsilon_{\mu\nu\alpha\beta}$ is the completely antisymmetric four-tensor, $\varepsilon_{0123} = \sqrt{|g|}$, where $|g| = \det||g||$.

Both E_μ and B_μ are spacelike: $E^\mu E_\mu = -E^2 < 0$, $B^\mu B_\mu = -B^2 < 0$, and orthogonal to U^μ: $U^\mu E_\mu = U^\mu B_\mu = 0$.

The electromagnetic tensor can be expressed in terms of E_μ and B_μ as follows:

$$F_{\mu\nu} = (E^\mu U^\nu - E^\nu U^\mu) + \tfrac{1}{2}\varepsilon^{\mu\nu\beta\gamma}(U_\beta B_\gamma - U_\gamma B_\beta). \tag{5}$$

The electromagnetic invariants take the following form:

$$\tfrac{1}{2}F^{\mu\nu}F_{\mu\nu} = B_\mu B^\mu - E_\mu E^\mu = E^2 - B^2, \tag{6}$$

$$\tfrac{1}{4}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} = E_\mu B^\mu. \tag{7}$$

Bearing in mind magnetohydrodynamical applications we shall consider the magnetically dominated case $B^2 > E^2$ when there is no electric field along the magnetic field $E_\mu B^\mu = 0$. The last condition means $\mathbf{EB} = 0$ in the three-dimensional form.

In this case one can find a frame where $E_\mu \equiv 0$. Without loss of generality we may assume that the chosen U^μ corresponds to the frame, where electric field is absent. Introducing the unity vector in the direction of the magnetic field $n_\mu = B_\mu/B$, $n_\mu n^\mu = -1$, $n_\mu U^\mu = 0$, one can write the electromagnetic tensor in the following form:

$$F^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\beta\gamma}(U_\beta B_\gamma - U_\gamma B_\beta) = \tfrac{1}{2}\varepsilon^{\mu\nu\beta\gamma}B(U_\beta n_\gamma - U_\gamma n_\beta). \tag{8}$$

In any other frame that is determined by the four-velocity $\tilde{U}^\mu$ the corresponding electric and magnetic fields are

$$\tilde{E}_\mu = F_{\mu\nu}\tilde{U}^\nu = (\varepsilon_{\mu\nu\beta\gamma}U^\gamma\tilde{U}^\nu)B^\beta, \tag{9}$$
$$\tilde{B}_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\nu\alpha}\tilde{U}^\beta = (\tilde{U}^\beta U_\beta)B_\mu - (\tilde{U}^\beta B_\beta)U_\mu. \tag{10}$$

Equations (9) and (10) are the field transformation rules represented in the invariant form for the special case of the electromagnetic tensor in the form (8). Substituting the electromagnetic tensor in the form (8) into (2) we obtain the equation of motion in the form (see also [10,7])

$$T^{\mu\nu}_{;\nu} = (T^{\mu\nu}_f + T^{\mu\nu}_{el})_{;\nu} = 0, \tag{11}$$

where

$$T^{\mu\nu}_{el} = \frac{B^2}{8\pi}(2U^\mu U^\nu - g^{\mu\nu} - 2n^\mu n^\nu), \tag{12}$$

is the electromagnetic energy-momentum tensor, $T^{\mu\nu}_f$ is the fluid energy-momentum tensor, and $T^{\mu\nu}$ is now the *total* energy-momentum tensor. The spatial part of the electromagnetic energy-momentum tensor reduces to the usual MHD magnetic stress tensor in the nonrelativistic limit $U^\mu \to (1,0,0,0)$, $g_{\mu\nu} \to (1,-1,-1,-1)$.

The Maxwell equation (4) takes the following form:

$$(U^\mu B^\nu - U^\nu B^\mu)_{;\nu} = B(U^\mu n^\nu - U^\nu n^\mu)_{;\nu} + (U^\mu n^\nu - U^\nu n^\mu)B_{;\nu} = 0. \tag{13}$$

We shall use also the relations

$$n_\mu n^\mu_{;\nu} = U_\mu U^\mu_{;\nu} = 0, \quad U_\mu n^\mu_{;\nu} = -n_\mu U^\mu_{;\nu}, \tag{14}$$

which follow from the conditions $U_\mu U^\mu = 1$, $n_\mu n^\mu = -1$, and $U_\mu n^\mu = 0$.

III. GENERALLY COVARIANT HYDRODYNAMICS

We assume that the plasma is collisionless. In this case the distribution function f (for each plasma species) satisfies the relativistic Vlasov equation, which we write in the following form:

$$u^\mu\frac{\partial f}{\partial x^\mu} + \left(\frac{e}{m}F^{\mu\nu}u_\nu - \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda\right)\frac{\partial f}{\partial u^\mu} = 0, \tag{15}$$

where u_μ is the particle four-velocity, $u_\mu u^\mu = 1$, $\Gamma^\mu_{\nu\lambda}$ is the Christoffel symbol and

$$f = 2f_0\delta(u^\mu u_\mu - 1)\theta(u^0). \tag{16}$$

Here the Dirac δ function makes the distribution function f nonzero only on the mass shell $u^\mu u_\mu = 1$, while the θ function picks up future directed velocity.

The corresponding hydrodynamical series is obtained by multiplying (15) by u^μ, $u^\mu u^\nu$, $u^\mu u^\nu u^\alpha$, ..., with subsequent integration over the invariant volume $\sqrt{|g|}d^4u$ in the four-velocity space [11]. The equations for the first three momenta read (cf. [7])

$$J^\mu_{;\mu} = 0, \tag{17}$$
$$T^{\mu\nu}_{;\nu} = \tfrac{e}{m} J_\nu F^{\mu\nu}, \tag{18}$$
$$S^{\alpha\beta\mu}_{;\mu} = \tfrac{e}{m} g_{\mu\nu}(F^{\alpha\nu}T^{\beta\mu} + F^{\beta\nu}T^{\alpha\mu}). \tag{19}$$

The procedure of reduction of these multifluid equations to the one-fluid set and the conditions when it is possible are described in [9]. Here we simply assume that this reduction is already done. In this case, Eq. (18) transforms into Eq. (2) while two others remain unchanged. The only difference is that now J^μ, $T^{\mu\nu}$, $S^{\alpha\beta\mu}$ are the *total fluid* mass flow density, energy-momentum tensor, and heat flux four-tensor, respectively.

These momenta themselves are defined as distribution averages

$$J^\mu = m\langle u^\mu\rangle, \quad T^{\mu\nu} = m\langle u^\mu u^\nu\rangle, \quad S^{\alpha\beta\mu} = m\langle u^\alpha u^\beta u^\mu\rangle, \tag{20}$$

with the following averaging prescription

$$\langle X(u^\mu)\rangle = \int \sqrt{|g|}\, d^4u f X. \tag{21}$$

This averaging requires knowledge of the distribution function f. We shall specify the distribution function in the MHD approximation, in which the inhomogeneity scale is assumed to be much larger than the particle gyroradius, while the time variation scale is assumed to be much larger than the gyroperiod. For the relativistic particles this approximation can be written qualitatively as follows:

$$\Omega L \gg u, \qquad \Omega\tau \gg 1 \qquad \Gamma/\Omega \ll 1, \tag{22}$$

where $\Omega = qB/m$ (we work in natural units $c \equiv 1$) is the charged particle gyrofrequency, L and τ are typical spatial and temporal scales of the plasma motion, $u \approx 1$ is the typical particle velocity, and Γ is the typical value of the Christoffel symbol.

The last condition imposes constraints on the gravitational field strength. It can be violated in strong gravitational fields, e.g., near the horizon $r \sim r_g$ in Schwarzschild metrics, where $\Gamma \propto r_g/r(r-r_g)$. A more exact general condition cannot be obtained and in each specific case the validity of the approximation should be verified separately.

In the MHD approximation the Vlasov equation (15) takes the following form:

$$\varepsilon^{\mu\nu\alpha\beta}(n_\alpha U_\beta - U_\alpha n_\beta)u_\nu \frac{\partial f}{\partial u^\mu} = 0, \tag{23}$$

and the solution is $f_0 = f_0(u_t, u_\parallel, u_\perp)$, where the following *invariant* decomposition is used:

$$u^\mu = u_t U^\mu + u_\parallel n^\mu + u^\mu_\perp, \quad u^\mu_\perp U_\mu = u^\mu_\perp n_\mu = 0,$$
$$u^\mu_\perp u_{\perp\mu} = -u^2_\perp, \tag{24}$$
$$u^\mu_\perp = u_\perp(e^\mu_1 \cos\phi + e^\mu_2 \sin\phi), \quad e^\mu_1 e_{1\mu} = e^\mu_2 e_{2\mu} = -1,$$
$$e^\mu_1 e_{2\mu} = 0. \tag{25}$$

Of course, since $u^\mu u_\mu = u_t^2 - u_\parallel^2 - u_\perp^2 = 1$, u_t is not an independent variable.

The invariant volume in the velocity space now can be written as

$$\varepsilon_{\alpha\beta\gamma\delta}du_0^\alpha du_1^\beta du_2^\gamma du_3^\delta = \varepsilon_{\alpha\beta\gamma\delta}U^\alpha n^\beta e^\gamma_1 e^\delta_2 du_t du_\parallel u_\perp du_\perp d\phi$$
$$= du_t du_\parallel u_\perp du_\perp d\phi, \tag{26}$$

and

$$2\sqrt{|g|}d^4u\delta(u^\mu u_\mu - 1)\theta(u^0) \to u_t^{-1}u_\perp du_\parallel du_\perp d\phi\theta(u_t). \tag{27}$$

Now the mass flow density can be written as

$$J^\mu = m(U^\mu\langle u_t\rangle + n^\mu\langle u_\parallel\rangle), \tag{28}$$

and is not parallel to the four-velocity U^μ unless $\langle u_\parallel\rangle = 0$.

The fluid velocity $\bar{U}^\mu = J^\mu/\sqrt{J^\mu J_\mu}$ now is

$$\bar{U}^\mu = (U^\mu\langle u_t\rangle + n^\mu\langle u_\parallel\rangle)/\sqrt{\langle u_t\rangle^2 - \langle u_\parallel\rangle^2}. \tag{29}$$

The transformation rules (9), (10) immediately give the fields in the fluid rest frame $\bar{E}_\mu = 0$ and

$$\bar{B}_\mu = B(\langle u_t\rangle n_\mu + \langle u_\parallel\rangle U_\mu)/\sqrt{\langle u_t\rangle^2 - \langle u_\parallel\rangle^2}. \tag{30}$$

Since $\bar{E}_\mu = 0$ we may assume (without loss of generality) that $\langle u_\parallel\rangle = 0$ and $\bar{U}^\mu = U^\mu$. More specifically, we assume that $f = (1/2\pi)f_0(u_\parallel^2, u_\perp^2)$. Then all the cross correlated terms $\propto \langle u_\parallel u_t\rangle$, $\propto \langle u_\parallel u_\perp\rangle$, $\propto \langle u^\mu u_t\rangle$, and $\propto \langle u^\mu u_\parallel\rangle$ vanish.

Let us define a projection operator

$$P_{\mu\nu} = g_{\mu\nu} - U_\mu U_\nu + n_\mu n_\nu, \tag{31}$$

with the features

$$P_{\mu\nu}P^\nu_\sigma = P_{\mu\sigma}, \quad P_{\mu\nu}U^\nu = P_{\mu\nu}n^\nu. \tag{32}$$

Since $u_\mu u_\nu U^\nu = u_\mu u_\nu n^\nu = 0$, the symmetry properties require that $\langle u_\mu u_\nu\rangle \propto P_{\mu\nu}$, and the fluid energy-momentum tensor takes the following form:

$$T^{\mu\nu} = m\langle u^\mu u^\nu\rangle = (\epsilon + p_\perp)U^\mu U^\nu - p_\perp g^{\mu\nu}$$
$$+(p_\parallel - p_\perp)n^\mu n^\nu, \tag{33}$$

where

$$p_\perp = m\tfrac{1}{2}\langle u_\perp^2\rangle, \quad p_\parallel = m\langle u_\parallel^2\rangle, \quad \epsilon = m\langle u_t^2\rangle, \quad \rho = m\langle u_t\rangle, \tag{34}$$

and the averaging takes the form

$$\langle X\rangle = \int du_{\parallel}u_{\perp}du_{\perp}u_t^{-1}Xf_0(u_{\parallel}^2,u_{\perp}^2). \quad (35)$$

Similarly, for the heat flux tensor one has

$$S^{\mu\nu\alpha} = (\rho + q_{\parallel} + q_{\perp})U^{\mu}U^{\nu}U^{\alpha} + q_{\parallel}\mathrm{Sym}(U^{\mu}n^{\nu}n^{\alpha}) -q_{\perp}\mathrm{Sym}(U^{\mu}P^{\nu\alpha}), \quad (36)$$

where

$$q_{\perp} = \tfrac{m}{2}\langle u_t u_{\perp}^2\rangle, \quad q_{\parallel} = m\langle u_t u_{\parallel}^2\rangle, \quad (37)$$

and Sym denotes symmetrization over indices as follows:

$$\mathrm{Sym}(A^{\mu}B^{\nu}C^{\beta}) = A^{\mu}B^{\nu}C^{\beta} + B^{\mu}C^{\nu}A^{\beta} + C^{\mu}A^{\nu}B^{\beta}. \quad (38)$$

Substituting the obtained expression for the energy-momentum tensor into (11) one has eventually

$$T^{\mu\nu}_{;\nu} = \left[\left(\epsilon + p_{\perp} + \frac{B^2}{4\pi}\right)U^{\mu}U^{\nu} - \left(p_{\perp} + \frac{B^2}{8\pi}\right)g^{\mu\nu} + \left(p_{\parallel} - p_{\perp} - \frac{B^2}{4\pi}\right)n^{\mu}n^{\nu}\right]_{;\nu} = 0, \quad (39)$$

where $T^{\mu\nu}$ is the *total* energy-momentum tensor for plasma and magnetic fields.

One can see that the derived equation can be obtained from its analog in special relativity [7] by direct substitution of the ordinary derivative to covariant derivative, when the proper ensemble averaging procedure (21), (35) is applied.

IV. STATE EQUATION

The obtained set of Eqs. (1), (13), and (39) requires a closure in the form of a state equation. In the simplest isotropic nonrelativistic case this state equation is found in the form $p = p(\rho)$ from the energy conservation equation. In our case the corresponding equation is obtained by taking a projection of the energy-momentum balance equation (39) onto the flow velocity direction U^{μ}. Let $D = U^{\mu}\partial_{\mu}$ be the usual convective derivative. Then the continuity equation (1) gives

$$U^{\mu}_{;\mu} = -D\ln\rho. \quad (40)$$

Another useful relation is obtained by projecting (13) onto n_{μ} direction and using (40),

$$n^{\nu}n_{\mu}U^{\mu}_{;\nu} = D\ln(\rho/B). \quad (41)$$

Projecting (39) onto U_{μ} and using (14), (40), and (41) after very easy algebra one arrives at the following state equation:

$$D\left(\frac{\epsilon}{\rho}\right) = \frac{p_{\parallel}}{\rho^2}D\rho + \frac{p_{\perp} - p_{\parallel}}{\rho B}DB, \quad (42)$$

where internal energy density $\epsilon = \rho e(\rho, B)$, and pressure is related to the internal energy as follows:

$$p_{\parallel} = \rho^2\frac{\partial e}{\partial\rho}, \qquad p_{\perp} = p_{\parallel} + \rho B\frac{\partial e}{\partial B}. \quad (43)$$

The state equation can be written in an another useful form as follows:

$$\frac{p_{\parallel}}{\rho} = \frac{d(\epsilon/\rho)}{d\ln(\rho/B)}, \quad (44)$$

$$\frac{p_{\perp}}{\rho} = \frac{d(\epsilon/\rho)}{d\ln B}. \quad (45)$$

In agreement with the "locality" and covariance principles the generally covariant state equation has the same form as its analog in special relativity [7].

One can easily see that in the isotropic case $p_{\parallel} = p_{\perp} = p$ the internal energy becomes a function of ρ solely and the state equation $p = p(\rho)$ is recovered.

The state equation (42) determines actually a class of state equations that are compatible with the relativistic hydrodynamical equations for the anisotropic plasma. It is a generalization of the relativistic isotropic state equation onto the generally relativistic anisotropic case in the same sense as the *nonrelativistic* Chew-Goldberger-Low [2] state equations

$$p_{\perp}/\rho B = \mathrm{const}, \quad p_{\parallel}B^2/\rho^3 = \mathrm{const}, \quad (46)$$

generalize the isotropic state equation $p = p(n)$ onto the *nonrelativistic* anisotropic case. It is worthwhile to note that the CGL form of state equations (46) is recovered as a special case of (42) when

$$\epsilon = \rho\left(1 + k_1\frac{\rho^2}{B^2} + k_2 B\right), \quad (47)$$

where the coefficients k_1 and k_2 are constants.

Multiplying Eq. (19) by $n_{\alpha}n_{\beta}$, after some simple algebra one obtains (cf. [7]; in [8] this relation is lost)

$$D\frac{q_{\parallel}B^2}{\rho^3} = 0, \quad \Rightarrow \quad \frac{q_{\parallel}B^2}{\rho^3} = \mathrm{const}. \quad (48)$$

Multiplying Eq. (19) by $U_{\alpha}U_{\beta}$ and taking into account (48) one has

$$\frac{q_{\perp}^2 q_{\parallel}}{\rho^5} = \mathrm{const}, \quad \Rightarrow \quad \frac{q_{\perp}}{\rho B} = \mathrm{const}. \quad (49)$$

Equations (48) and (49) look exactly as the CGL state equations (46) with the only substitution $p \to q$. However, they cannot be considered as a generalization of the state equation because of the substantial difference between p and q. Indeed, let us consider an isotropic case $p_{\parallel} = p_{\perp} = p$. In this case also $q_{\parallel} = q_{\perp} = q$. Equation (48) implies $q \propto \rho^3/B^2$, while (43) shows that the pressure and internal energy do not depend on B at all.

On the other hand (49) gives $q \propto \rho^{5/3}$ in the isotropic case both in the nonrelativistic and ultrarelativistic limits. At the same time it is easy to see that $p \propto \rho^{5/3}$ in the nonrelativistic case, but $p \propto \rho^{4/3}$ in the ultrarelativistic case. We, therefore (in contrast with [8]), consider (42) and (43) as a most general form of the state equation for an anisotropic plasma. On the other hand (48) and (49) are useful in the determination of the state equations in various limits.

As an example of the application of the obtained expressions we consider a two-parametric distribution function of the form

$$f(u_\parallel, u_\perp) = f\left(\frac{u_\parallel}{\lambda_\parallel}, \frac{u_\perp}{\lambda_\perp}\right). \tag{50}$$

The momenta take the following form:

$$\rho = \lambda_\parallel \lambda_\perp^2 \int \xi_2 d\xi_1 d\xi_2 \, f(\xi_1, \xi_2), \tag{51}$$

$$\epsilon = \lambda_\parallel \lambda_\perp^2 \int (1 + \lambda_\parallel^2 \xi_1^2 + \lambda_\perp^2 \xi_2^2)^{1/2} \xi_2 d\xi_1 d\xi_2 \, f(\xi_1, \xi_2), \tag{52}$$

$$p_\parallel = \lambda_\parallel^3 \lambda_\perp^2 \int (1 + \lambda_\parallel^2 \xi_1^2 + \lambda_\perp^2 \xi_2^2)^{-1/2} \xi_2 d\xi_1 d\xi_2 \, f(\xi_1, \xi_2), \tag{53}$$

$$p_\perp = \lambda_\parallel \lambda_\perp^4 \int (1 + \lambda_\parallel^2 \xi_1^2 + \lambda_\perp^2 \xi_2^2)^{-1/2} \xi_2 d\xi_1 d\xi_2 \, f(\xi_1, \xi_2), \tag{54}$$

$$q_\parallel = \lambda_\parallel^3 \lambda_\perp^2 \int \xi_1^2 \xi_2 d\xi_1 d\xi_2 \, f(\xi_1, \xi_2), \tag{55}$$

$$q_\perp = \lambda_\parallel \lambda_\perp^4 \int \xi_2^3 d\xi_1 d\xi_2 \, f(\xi_1, \xi_2), \tag{56}$$

where we switched to the integration over the dimensionless variables $\xi_1 = u_\parallel/\lambda_\parallel$, $\xi_2 = u_\perp/\lambda_\perp$. One can see that in this case $\rho \propto \lambda_\parallel \lambda_\perp^2$, $q_\parallel \propto \lambda_\parallel^3 \lambda_\perp^2$, and $q_\perp \propto \lambda_\parallel \lambda_\perp^4$. Substituting this into (48) and (49) one immediately has

$$\lambda_\perp \propto \sqrt{B}, \qquad \lambda_\parallel \propto \rho/B. \tag{57}$$

The energy and pressure do not have such a simple form in the general case. However, it is easy to verify that Eqs. (44) and (45) are satisfied automatically when substituting $d/d\ln(\rho/B) = d/d\ln\lambda_\parallel$, $d/d\ln B = 2d/d\ln\lambda_\perp$.

Great simplification can be made in several limits. In the nonrelativistic case (in the plasma rest frame) $\lambda_\parallel, \lambda_\perp \ll 1$ one has

$$p_\parallel \propto \lambda_\parallel^2 \lambda_\perp^2 \propto \rho^2/B, \quad p_\perp \propto \lambda_\parallel \lambda_\perp^4 \propto \rho B, \tag{58}$$

which are the ordinary CGL state equations in the form (46).

In the ultrarelativistic limit when $\lambda_\parallel \gg \lambda_\perp \gg 1$

$$p_\parallel \propto \lambda_\parallel^2 \lambda_\perp^2 \propto \frac{\rho^2}{B}, \tag{59}$$

$$p_\perp \propto \lambda_\perp^4 \propto B^2, \tag{60}$$

$$\epsilon \propto \lambda_\parallel^2 \lambda_\perp^2 \approx p_\parallel. \tag{61}$$

This case is typical for pulsar magnetospheres, where the transverse momenta are rapidly radiated out due to synchrotron radiation in a strong magnetic field (see e.g., [12]).

In the opposite case $\lambda_\perp \gg \lambda_\parallel \gg 1$ and one obtains

$$p_\parallel \propto \lambda_\parallel^3 \lambda_\perp \propto \frac{\rho^3}{B^{5/2}}, \tag{62}$$

$$p_\perp \propto \lambda_\parallel \lambda_\perp^3 \propto \rho B^{1/2}, \tag{63}$$

$$\epsilon \propto \lambda_\parallel \lambda_\perp^3 \approx p_\perp. \tag{64}$$

This case may be relevant if the plasma is strongly heated in the perpendicular direction due to a rapid magnetic compression.

V. GENERALLY RELATIVISTIC STELLAR WIND

As an example of an application of the obtained hydrodynamical equations we shall derive a set of flow constants (see e.g., [13,14]) for a generally relativistic stellar wind with anisotropic pressure. We assume a spherically symmetric metric of the form

$$ds^2 = g_{tt}(r)dt^2 - g_{rr}(r)dr^2 - g_{\theta\theta}(r)d\theta^2 - g_{\phi\phi}(r,\theta)d\phi^2$$

in the spherical coordinates (note the notation where the sign of $g_{\mu\nu}$ is now given explicitly). We also assume time stationarity. We shall analyze only the equatorial plane where a monopole magnetic field geometry can be assumed [13]

$$B^\mu = (B^t, B^r, 0, B^\phi), \qquad U^\mu = (U^t, U^r, 0, U^\phi), \tag{65}$$

and all variables do not depend on θ. The condition $B_\mu U^\mu = 0$ requires

$$g_{tt} B^t U^t - g_{rr} B^r U^r - g_{\phi\phi} B^\phi U^\phi = 0. \tag{66}$$

Let us consider the simplest case $B^\phi = U^\phi = 0$. In this case (66) immediately gives

$$B^t = B^r \frac{g_{rr} U^r}{g_{tt} U^t}, \tag{67}$$

and taking into account the definition $B^2 = -B^\mu B_\mu$, one obtains

$$B^r = B U^t \left(\frac{g_{tt}}{g_{rr}}\right)^{1/2}. \tag{68}$$

The continuity equation (1) reduces to

$$\sqrt{|g|}\rho U^r = \text{const}, \tag{69}$$

where $|g| = g_{tt} g_{rr} g_{\theta\theta} g_{\phi\phi}$. In a similar way (13) gives

$$\sqrt{|g|}(B^r U^t - B^t U^r) = \text{const} \quad \Rightarrow \quad B(g_{\theta\theta} g_{\phi\phi})^{1/2} = \text{const}, \tag{70}$$

where we have taken into account the relations (67) and (68). In the equatorial plane of the Schwarzschild metric $g_{\theta\theta} = g_{\phi\phi} = r^2$ and Eq. (70) takes a simple form

$$r^2 B = \text{const},$$

which looks like the ordinary r dependence of the radial magnetic field [13]. It should be noted, however, that our B is the magnetic field in the fluid frame. The distant observer's magnetic field is obtained with the help of the transformation rule (10), where $\tilde{U}^\mu = (g_{tt}^{-1/2}, 0, 0, 0)$. Taking into account the relations (67) and (68) one finds $\tilde{B}^t = 0$, $\tilde{B}^r = B/\sqrt{g_{tt}}$.

The t component of (2) gives after simple transformations

$$g_{tt}\sqrt{|g|}T^{tr} = g_{tt}\sqrt{|g|}U^t U^r(\epsilon + p_\|) = \text{const}. \quad (71)$$

Let us consider the parallel pressure dominated case, in which (see above) $\epsilon \approx p_\| \propto \rho^2/B$. Combination of the derived constants easily gives

$$U^t\sqrt{g_{tt}}/U^r\sqrt{g_{rr}} = \text{const}$$

and taking into account the normalization $g_{tt}U^tU^t - g_{rr}U^rU^r = 1$, one has

$$U^r \propto (g_{rr})^{-1/2}, \quad U^t \propto (g_{tt})^{-1/2}, \quad \rho \propto 1/\sqrt{g_{tt}g_{\theta\theta}g_{\phi\phi}}, \quad (72)$$

$$p_\| \propto 1/g_{tt}\sqrt{g_{\theta\theta}g_{\phi\phi}}, \quad p_\perp \propto B^2 \propto 1/g_{\theta\theta}g_{\phi\phi}. \quad (73)$$

In the far zone of the Schwarzschild geometry the main dependence is $g_{\theta\theta} \propto g_{\phi\phi} \propto r^2$, and one has

$$U^t \approx \text{const}, \quad \rho \propto p_\| \propto B \propto r^{-2}, \quad (74)$$

$$p_\perp \propto r^{-4}, \quad p_\perp/p_\| \propto r^{-2}, \quad p_\|/B^2 \propto r^2. \quad (75)$$

The last two relations show that the fluid remains "transversely cold" and that the ratio of the kinetic to magnetic pressure rapidly increases.

In the case of the perpendicularly dominated pressure $\epsilon \approx p_\perp \propto \rho B^{1/2}$ one obtains

$$U^t \propto (g_{\theta\theta}g_{\phi\phi})^{1/4}/g_{tt}. \quad (76)$$

In the far zone of the Schwarzschild geometry $U^t \propto r$, in this case one has

$$U^r \propto (g_{\theta\theta}g_{\phi\phi})^{1/4}/(g_{tt}g_{rr})^{1/2} \propto r,$$

$$\rho \propto (g_{\theta\theta}g_{\phi\phi})^{-3/4} \propto r^{-3}, \quad (77)$$

$$p_\perp \propto p_\| \propto B^2 \propto 1/(g_{\theta\theta}g_{\phi\phi}) \propto r^{-4}. \quad (78)$$

One can see that the pressure decreases rapidly with the increase of r, so that one can expect that the plasma temperature quickly becomes nonrelativistic in the plasma rest frame.

VI. LINEAR WAVES

The fully covariant formulation should be especially useful for the analysis of the small amplitude perturbations and comparison with the nonrelativistic results, since this analysis is usually carried out in the plasma rest frame (where B is defined). Such an analysis of the waves in the relativistic plasma in the framework of the (specially) relativistic anisotropic MHD has been done in [15]. The generalization of the analysis onto the generally relativistic case requires separate investigation since it is significantly complicated by (a) problems of the equilibrium state determination, (b) inhomogeneity due to $\Gamma^\mu_{\nu\alpha} \neq 0$, and (c) perturbations of the metric. In the present paper we consider the case when the plasma and magnetic field energy density are not large, so that one can neglect the metric perturbations. The difficulties (a) and (b) are avoided by consideration of the waves in the limit of the geometrical optics.

Namely, let the plasma variables be disturbed by $\delta\rho$, δB, δU^μ, and δn^μ. We assume that the coordinate dependence of these variables is $\propto \exp i\eta^{-1}\Theta(x^\mu)$, where $\eta \ll 1$. In this case

$$[\exp i\eta^{-1}\Theta(x^\mu)]_{;\nu} = \left(\frac{ik_\nu}{\eta} + \Gamma^\sigma_{\sigma\nu}\right)\exp i\eta^{-1}\Theta(x^\mu)$$
$$\approx \frac{ik_\nu}{\eta}\exp i\eta^{-1}\Theta(x^\mu),$$

where $k_\mu = \Theta_{,\mu}$ and we assume $\Gamma^\mu_{\nu\sigma} \sim O(1)$. It is easy to see that in this short wavelength approximation all covariant derivatives should be substituted by ik_μ and the equations for perturbations take the same form as in the specially relativistic case. The derivation is straightforward and we refer the reader to [15] for the details. The resulting dispersion relation will take the following form:

$$v^2 = v_A^2\cos^2\theta \quad (79)$$

for the intermediate (Alfven) wave, and

$$v^4 - v^2[(v_s^2 + v_A^2)\cos^2\theta + v_F^2\sin^2\theta] + \cos^2\theta[v_s^2(v_A^2\cos^2\theta + v_F^2\sin^2\theta) - v_t^4(1 - v_A^2)\sin^2\theta] = 0 \quad (80)$$

for the fast and slow magnetosonic waves. Here the following notation is used:

$$v = \omega/k, \quad \omega = k_\mu U^\mu, \quad k^2 = \omega^2 - k_\mu k^\mu,$$

$$k\cos\theta = k_\mu n^\mu, \quad (81)$$

$$v_s^2 = \frac{\partial p_\|/\partial\ln\rho}{\epsilon + p_\|}, \quad (82)$$

$$v_t^2 = \frac{\partial p_\perp/\partial\ln\rho}{\epsilon + p_\|}, \quad (83)$$

$$v_A^2 = \frac{p_\perp - p_\| + (B^2/4\pi)}{\epsilon + p_\perp + (B^2/4\pi)}, \quad (84)$$

$$v_F^2 = \frac{(\partial p_\perp/\partial\ln B) + (\partial p_\perp/\partial\ln\rho) + (B^2/4\pi)}{\epsilon + p_\perp + (B^2/4\pi)}. \quad (85)$$

It is worthwhile to notice that the intermediate solution (79) is unstable (firehose instability), when $p_\| - p_\perp > B^2/4\pi$. As we have seen above in the parallel pressure dominated wind the ratio $p_\|/B^2 \propto r^2$ monotonically increases and becomes large. Therefore, the parallel pressure dominated wind always achieves the point of firehose instability, where the wind should be efficiently isotropized.

A more detailed analysis of the dispersion relations, as well as consideration of the deviations from geometrical optics and/or metric perturbations, are beyond the scope of the present paper.

VII. CONCLUSION

In the present paper we have derived generally covariant hydrodynamical equations and the corresponding state equations for an anisotropic plasma in a strong magnetic field. We assumed that the plasma can be described in the framework of one-fluid hydrodynamics, and that it is magnetized. The last assumption implies that the electric field in the plasma rest frame should be absent. We have shown that generalization of hydrodynamical equations onto the generally relativistic case can be obtained from the corresponding equations of special relativity [7] by direct substitution of ordinary derivatives to covariant derivatives. All fluid and thermodynamical variables, such as density, pressure, and internal energy density, are defined in the plasma local rest frame in the invariant way. Most general state equations are derived which are a generalization of the CGL state equations onto the generally relativistic case. The exact CGL form of equations is recovered when the fluid temperature in the fluid rest frame is nonrelativistic.

We applied the obtained HD equations to the relativistic wind description in the monopole geometry and found the wind density, velocity, and pressure behavior in two ultrarelativistic limits. The MHD dispersion relations in the limit of geometrical optics are also presented.

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