# Statistical mechanics of point vortices

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Thermodynamical functions and probability distributions are found in an explicit form for a multicomponent vortex gas. A key point in the derivation is the interpretation of integration in phase space as an average, with respect to some "complex measure." The exact formulas for the probability distribution of a vortex gas are used to obtain the averaged equations for two-dimensional fluid motion. The exact formulas for thermodynamical functions help to clear up the relation between various definitions of entropy and temperature, which, in contrast to the classical models of statistical mechanics, are not equivalent for a vortex gas.

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## I. INTRODUCTION

It seems natural to try to develop a theory of turbulence using only ergodicity of motion, the standard postulate of statistical mechanics. Ergodicity seems plausible, at least for developed turbulent flows when molecular viscosity is negligible and the fluid is a Hamiltonian system moving chaotically. Two obvious obstacles are seen in applications of the methods of statistical mechanics. First, fluids have an infinite number of degrees of freedom. At present it is not known how to define such notions as "ergodicity" for infinite-dimensional systems. Second, in contrast to the usual models of statistical mechanics, fluids have an infinite number of integrals of motion (circulation of velocity over closed fluid contours). Trajectories in infinite-dimensional phase space lie in the cross section of an infinite number of surfaces. Intuition suggests that all integrals of motion affect the long-term behavior of fluids.

A possible way to overcome these difficulties is to construct a truncation of Euler's equations, which respect the integrals of fluid motion and have a finite number of degrees of freedom $N$. Then, assuming ergodicity, one can find all necessary statistical characteristics of the flow and consider the limit of these characteristics for $N \rightarrow \infty$. In the two-dimensional (2D) case, a very interesting candidate for truncation is a point vortex approximation of the vorticity field (see the reviews [1,2]). It respects all integrals of fluid motion and converges for $N \rightarrow \infty$ to the solution of Euler's equations [3]. Another interesting truncation has been discussed in [4–6]. In this paper, point vortex truncation is studied. The basic relations of statistical mechanics of a multicomponent point vortex gas are developed. Thermodynamic functions and probability distributions are found in an explicit form. Equations governing the behavior of a multicomponent vortex gas are used to obtain the limit relations for an infinite number of components. The major difficulty in developing statistical mechanics of a vortex gas is the calculation of the phase volume. This difficulty is overcome by means of an interpretation of the integration in phase space as averaging with respect to some "complex measure." It reduces the calculation of the phase volume to an application of the steepest descent method.

Statistical mechanics of point vortices started with a paper by Onsager [7]. Considering motion of $N$ point vortices in a closed domain $V$, he found that there is some critical value of energy $E_0$ at which the temperature of vortex motion $T$ changes the sign. At high values of energy, the temperature $T$ is negative. An attempt to calculate the critical energy $E_0$ was undertaken by Taylor [8] for the neutral case (there is an equal number of vortices with positive and negative equal intensities). It was corrected by Joyce and Montgomery [9,10], who got $E_0=0$. They also derived the equation for the averaged stream function $\bar{\psi}$ of a two-component vortex gas

$$\Delta\bar{\psi}=-k\frac{e^{-\beta\bar{\psi}}}{\int e^{-\beta\bar{\psi}(r')}d^2r'}+k\frac{e^{\beta\bar{\psi}}}{\int e^{\beta\bar{\psi}(r')}d^2r'} , \tag{1.1}$$

where $k,\beta$ are constants and $\beta$ is determined by the initial value of energy. Equation (1.1) was justified by a more rigorous consideration given by Pointin and Lundgren [11,12]. They derived (1.1) from the assumption of the mean field theory: the positions of any two vortices are statistically independent, i.e., the probability density of the positions of any two vortices $f(r_1,r_2)$ is given by

$$f(r_1,r_2)=f(r_1)f(r_2) , \tag{1.2}$$

where $f(r)$ is the probability density of one vortex. Recently, (1.2) has been proven in [13,14]. These and other aspects of the statistical mechanics of vortices are discussed also in [15–19].

We start with a reminder of the basic equations of a vortex gas (Sec. II) and statistical mechanics (Sec. III). Then, in Sec. IV we discuss the notion of a vortex temperature with emphasis on the physical sense of the sign of the temperature. In Sec. V we derive the relations of low energy dynamics of a one-component vortex gas. A generalization to a multicomponent vortex gas is presented in Sec. VI. High energy dynamics is discussed in Sec. VII. Relations between various definitions of entropy, which, in contrast to the classical models of statistical

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mechanics, are not equivalent for a vortex gas, are considered in Sec. VIII. Probability distributions are derived in Sec. IX. This is followed by the derivation of averaged equations of 2D hydrodynamics and concluding remarks.

## II. EQUATIONS OF VORTEX DYNAMICS

Consider a system of point vortices with positions $r_i$ and intensities $\gamma_i$, $i=1,\ldots,N$. Vortices move in a closed bounded domain $V$. The kinetic energy of fluid is equal to [20]

$$H=\tfrac{1}{2}\sum_{i\neq j}\gamma_i\gamma_j G(r_i,r_j)+\sum_i \gamma_i^2 g(r_i) . \tag{2.1}$$

Here $G(r,r')$ is the Green function of the Dirichlet boundary value problem

$$\Delta_r G(r,r')=-\delta(r-r') \text{ in } V, G(r,r')=0 \text{ if } r\in\partial V , \tag{2.2}$$

where $\Delta_r$ is Laplace's operator with respect to $r$ variables. The function $g(r)$ relates to the residual in the expansion of the Green function in the vicinity of the singular point

$$\begin{aligned} G(r,r')&=-\frac{1}{2\pi}\ln|r-r'|+g(r,r') ,\\ g(r)&\equiv\tfrac{1}{2}g(r,r) . \end{aligned} \tag{2.3}$$

In the expression for energy (2.1), an infinite self-energy of vortices has been dropped since it is constant in time.

Denote coordinates of the $i$th vortex by $x_i,y_i$, so $r_i$ means the couple $(x_i,y_i)$. The dynamic behavior of vortices is governed by the Hamiltonian system of equations [20,21]

$$\gamma_i\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i},\quad \gamma_i\frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i} . \tag{2.4}$$

For the purposes of the following discussion of the sign of the vortex temperature, note that Eqs. (2.4) are written in the right coordinate system, i.e., the rotation from the $x$ axis to the $y$ axis is assumed to be counterclockwise and the positive intensity of a vortex corresponds to a counterclockwise rotation of the surrounding fluid.

The first sum in the energy expression (2.1) describes a vortex-vortex interaction. The Green function $G(r,r')$ is symmetric and positive. The second sum in (2.1) is the energy of a vortex-wall interaction. The function $g(r)\to-\infty$ when the vortex approaches the wall (see, for example, [22]).

Equations (2.4) appear as a finite-dimensional truncation of equations of an ideal fluid. To obtain this truncation consider the motion of a 2D continuum in a bounded region $V$. Denote by $\xi=(\xi_1,\xi_2)$ Lagrangian coordinates of fluid particles, $\xi\in V$. Let $r(t,\xi)=(x(t,\xi),y(t,\xi))$ be the position vector of the particle $\xi, r\in V$. Consider the following functional of the position vector $r(t,\xi)$:

$$I(r)=\int_{t_0}^{t_1}dt\left[\int_V \mathring{\omega}(\xi)x(t,\xi)\dot{y}(t,\xi)d^2\xi -\tfrac{1}{2}\int_V\int_V G(r(t,\xi),r(t,\xi'))\times\mathring{\omega}(\xi)\mathring{\omega}(\xi')d^2\xi\, d^2\xi'\right] . \tag{2.5}$$

Here an overdot means the time derivative for fixed Lagrangian coordinates and $\mathring{\omega}(\xi)$ is the initial vorticity. It can be checked by inspection that the true motion of an ideal fluid is the stationary points of the functional (2.5).

An attractive point of this form of variational principle is that, in contrast to the usual form (see, for example, [23]), all integrals of fluid motion, corresponding to conservation of velocity circulation, are eliminated. Besides, only particles carrying a nonzero vorticity contribute to the action functional. It is convenient also that the admissible functions $r(t,\xi)$ are arbitrary and should not obey the incompressibility constraint $\|\partial r/\partial\xi\|=1$.

Point vortex truncation corresponds to dividing the set of Lagrangian coordinates in small subsets $V_i$, $i=1,\ldots,N$. The motion of each subset is characterized by position vector $r_i(t)$; $r_i(t)$ might be, for example, the centroid of position $V_i$ at moment $t$.

For small $V_i$ the action functional (2.5) takes the form

$$I(r_i)=\int_{t_0}^{t_i}\left[\sum_i\gamma_i x_i\dot{y}_i-H(r)\right]dt , \tag{2.6}$$

$$H(r)=\tfrac{1}{2}\sum_{i,j}G(r_i,r_j)\gamma_i\gamma_j . \tag{2.7}$$

Here

$$\gamma_i=\int_{V_i}\mathring{\omega}(\xi)d^2\xi .$$

In the expression $G(r_i,r_j)$ the leading (infinite) term can be dropped as it is independent of motion. Then (2.7) transforms into (2.1). The functional (2.6) is the action functional for Eq. (2.4).

## III. SUMMARY OF THE NECESSARY FACTS FROM STATISTICAL MECHANICS

Consider a Hamiltonian system with generalized coordinates $q=(q_1,\ldots,q_n)$, generalized momenta $p=(p_1,\ldots,p_n)$, and the Hamilton function $H$, which depends on some parameters $a=(a_1,\ldots,a_k)$ describing the influence of external factors. The Hamilton equations are

$$\dot{p}_i=-\frac{\partial H(p,q,a)}{\partial q_i},\quad \dot{q}_i=\frac{\partial H(p,q,a)}{\partial p_i} . \tag{3.1}$$

If the parameters $a$ are fixed, trajectories of the system belong to energy surfaces $H(p,q,a)=E=$const. It is assumed that the system is ergodic on energy surfaces and every surface bounds a finite volume of the phase space $\Gamma(E,a)$.

The following three statements express the laws of equilibrium thermodynamic and statistical mechanics. We present these statements in a form that is valid for any finite-dimensional Hamiltonian system following [24].

### A. Temperature

Denote by $\langle\ \rangle$ the time averaging operator along a trajectory. For any function $g(p,q)$ the quantity $\langle g \rangle$ does not depend on the trajectory chosen on an energy surface (up to some exceptional sets with zero measure). For any ergodic Hamiltonian system the equipartition law is valid

$$\left\langle p_1 \frac{\partial H}{\partial p_1} \right\rangle = \left\langle p_2 \frac{\partial H}{\partial p_2} \right\rangle = \cdots = \left\langle p_n \frac{\partial H}{\partial p_n} \right\rangle . \tag{3.2}$$

The common value of (3.2) is called, by definition, the absolute temperature $T$. This temperature is expressed in terms of the function $\Gamma(E,a)$ by a relation

$$T = \frac{\Gamma(E,a)}{\partial\Gamma(E,a)/\partial E} . \tag{3.3}$$

### B. Entropy

Allow slow variations of the external parameters $a$. Then the energy of the system is also changed. The quantity $\Gamma(E,a)$ is an adiabatic invariant; i.e., it could be considered as a constant (in some sense) in the course of the variation in $E,a$ (the Hertz-Kasuge theorem [25,26]). Any adiabatic invariant is a function of $\Gamma(E,a)$ (Kasuge's theorem [26]). It is natural to introduce the entropy of a finite-dimensional system in such a way that (i) entropy is an adiabatic invariant and (ii) the energy equation is true: $dE = dA + TdS$, where $dA$ is the work done by external forces in order to change the parameters $a$.

For Hamiltonian systems the work done by external forces is

$$dA = \left\langle \frac{\partial H}{\partial t} \right\rangle dt = \left\langle \frac{\partial H}{\partial a_i} \right\rangle da_i .$$

The mean value of $\partial H/\partial a_i$ can be calculated for the ergodic system in terms of the function $\Gamma(E,a)$. Therefore the work done by external forces can also be expressed in terms of $\Gamma(E,a)$:

$$dA = -\frac{1}{\partial\Gamma(E,a)/\partial E} \frac{\partial\Gamma(E,a)}{\partial a_i} da_i .$$

Hence the energy equation takes the form

$$dE = -\frac{1}{\partial\Gamma/\partial E} \frac{\partial\Gamma}{\partial a_i} da_i + T\, dS . \tag{3.4}$$

It is easy to find that the only quantity satisfying requirements (i) and (ii) and Eq. (3.4) is

$$S(E,a) = \ln\Gamma(E,a) + \text{const} . \tag{3.5}$$

This function $S(E,a)$ is linked to the temperature $T$ by the relation

$$\frac{1}{T} = \frac{\partial S(E,a)}{\partial E} . \tag{3.6}$$

The work of external forces is found in terms of $S(E,a)$ as

$$dA = -\frac{\partial S(E,a)}{\partial a_i} da_i . \tag{3.7}$$

### C. Entropy and probability

Consider some set of characteristics $w_1(q,p), \ldots, w_m(q,p)$ and denote by $f(z_1,\ldots,z_m)$ the probability density function of these characteristics. The probability density function $f(z_1,\ldots,z_m)$ can be expressed in terms of the entropy $S(E,z)$ of some auxiliary Hamiltonian system. This system is obtained from the original one by setting the kinematic constraints $w_1(q,p)=z_1, \ldots, w_m(q,p)=z_m$. The relation between $f(z)$ and $S(E,z)$ has the form (we do not mention here explicitly the dependence on the external parameters $a$)

$$f(z) = \frac{1}{\partial\Gamma(E)/\partial E} \frac{\partial}{\partial E} e^{S(E,z)} . \tag{3.8}$$

Formula (3.8), derived in [24], is an exact relation valid for finite-dimensional systems with finite fluctuations of characteristics. It generalizes the Einstein formula

$$f(z) = \text{const} \times e^{S(E,z)} \tag{3.9}$$

derived for small fluctuations near the equilibrium state: (3.8) reduces to (3.9) in the limit $N \to \infty$.

Note that the Gibbs distribution $f(x) = Z^{-1}\exp[-\beta H(x)]$ can be derived from the Einstein formula (3.9); therefore, the Einstein formula could be considered as the basic relation of equilibrium statistical mechanics. Formula (3.8) plays a similar role in the statistical mechanics of systems with a finite number of degrees of freedom.

## IV. TEMPERATURE OF VORTEX MOTION

The equipartition law (3.2) yields a clear physical meaning of the temperature of vortex motion [17]. In accordance with (2.4), $y$ coordinates of vortices play the role of generalized momenta, while $x$ coordinates are the generalized coordinates of the Hamiltonian system. The additional factors $\gamma_i$ on the left-hand sides of (2.4) do not affect the equipartition law in the form (3.2). This can be checked by an inspection of the proof of the equipartition law. Substituting the expressions for $\partial H/\partial y_i$ (2.4) into (3.2), we obtain the relation

$$\gamma_1\langle y_1\dot{x}_1\rangle = \gamma_2\langle y_2\dot{x}_2\rangle = \cdots = \gamma_N\langle y_N\dot{x}_N\rangle = T . \tag{4.1}$$

The quantity $\langle y_1\dot{x}_1\rangle$ denotes the average area bounded by the trajectory of the first vortex per unit time. If the motion of a vortex is periodic and bounds some area $A$, then $\langle y\dot{x}\rangle$ is equal to $A/\tau$, where $\tau$ is the period of motion. The averaged area is positive if the vortex moves (in average) clockwise and negative in the opposite case. The equipartition law (4.1) says that for ergodic motion the products of the vortex intensity and the averaged area bounded by the vortex trajectory per unit time should be the same for all vortices. The sign of the temperature controls the direction of the rotation of the vortices.

The relation between the temperature and the phase volume (3.3) is not valid in the case of Hamilton's function (2.1). Formula (3.3) assumes that $H$ obeys the following condition: region $H(r_i) \le E$ has the boundary $H(r_i) = E$. This is not the case for $H$ (2.1). The positions of the vortices $r_i$ satisfy the additional constraint $r_i \in V$. The vicinity of the boundary $\partial(V_N)$ belongs to the region $H(r_i) \le E$ because $g(r_i)$ (and hence $H$) tends to $-\infty$ if the $i$th vortex approaches the wall while the positions of all other vortices remain unchanged. Thus the boundary of the region $H(r_i) \le E$ consists of two pieces: the surface $H=E$ and the surface $\partial(V^N)$. Modification of (3.3) for this case is [27]

$$T = -\frac{|V|^N - \Gamma(E)}{d\Gamma(E)/dE} . \tag{4.2}$$

One can derive (4.2) by considering the Hamiltonian system with Hamilton's function $H_1 = -H$. The equations of vortex motion (2.4) take the form

$$\gamma_i \frac{dx_i}{dt} = -\frac{\partial H_1}{\partial y_i}, \quad \gamma_i \frac{dy_i}{dt} = \frac{\partial H_1}{\partial x_i} . \tag{4.3}$$

Hamilton's function $H_1$ has been used in a number of pioneering papers on vortex motion [21,28,20]. In Eqs. (4.3), $x$ coordinates of vortices play the role of generalized momenta. Note that the trajectories of the systems (2.4) and (4.3) are identical.

Consider the region $H_1 \le E_1$. If a vortex approaches the boundary while the positions of all other vortices are fixed, $H_1 \to +\infty$. The function $H_1$ might be finite if simultaneously one of the other vortices is moving toward a vortex of the opposite sign. This means that the boundary of the region $H_1 \le E_1$ consists of the $(N-1)$-dimensional surface $H_1 = E_1$ and maybe some submanifolds of $\partial(V^N)$. Since only $(N-1)$-dimensional pieces of the boundary contribute to the Stokes theorem used in the proof of (3.3), we can apply (3.3) for the Hamiltonian system (4.3). We have, from (3.2) and (4.3),

$$\gamma_1 \langle x_1 \dot{y}_1 \rangle = \gamma_2 \langle x_2 \dot{y}_2 \rangle = \cdots = \gamma_N \langle x_N \dot{y}_N \rangle = T_1 \tag{4.4}$$

and the usual relation (3.3) between the temperature and the phase volume

$$T_1 = \frac{\Gamma_1(E_1)}{\partial \Gamma_1(E_1)/\partial E_1} , \tag{4.5}$$

where $\Gamma_1(E_1)$ is the volume of the region $H_1 \le E_1$. Note that

$$T_1(E_1) = -T(E) \quad \text{if } E_1 = -E \tag{4.6}$$

because surfaces $H_1 = -E$ and $H = E$ coincide and $\langle x_i \dot{y}_i \rangle = -\langle y_i \dot{x}_i \rangle$ [$x\dot{y} = d(xy)/dt - y\dot{x}$ and the time average of the derivative of a bounded function is zero].

Regions $H_1 = -E$ and $H = E_1$ are the complementary ones; hence

$$\Gamma_1(-E) + \Gamma(E) = |V|^N , \tag{4.7}$$

where $|V|$ is the area of the container. Differentiating (4.7) with respect to $E$, we have

$$-\left.\frac{d\Gamma_1}{dE_1}\right|_{E_1=-E} + \frac{d\Gamma(E)}{dE} = 0 . \tag{4.8}$$

Formula (4.2) follows from (4.6)–(4.8).

The function $\Gamma_1(E_1)$ is an increasing function of $E_1$ because the increase of $E_1$ corresponds to an expansion of the region $H_1 \le E_1$. Therefore, $d\Gamma_1/dE_1 > 0$ and, as follows from (4.5), $T_1 > 0$ for all values of energy. Hence the temperature $T$ is always negative for all values of energy. To make sure that the negative sign of the temperature $T$ is the correct one, we can consider a motion of just one vortex in a closed domain. One vortex moves periodically along a closed curve. This motion is obviously ergodic and all thermodynamical relations are valid. Equation (4.1) takes the form

$$\gamma \langle y\dot{x} \rangle = T . \tag{4.9}$$

Let us put the vortex very close to the wall. Then the boundary acts as if it were flat. Vortex velocity is generated by the vortex image, which has the opposite sign. Thus a positive vortex moves along the wall counterclockwise, while a negative one moves clockwise. This corresponds to negative $T$ in (4.9).

Both of Hamilton's functions $H$ and $H_1$ can be used; this is a matter of convention. If Hamilton's function $H_1$ is used (as in [17]), then the corresponding temperature $T_1$ is positive. The Hamiltonian $H_1$ has the advantage of being similar to standard Hamiltonians of statistical mechanics: the phase volume increases if the value of the Hamiltonian increases. On the other hand, the Hamiltonian $H$ has the sense of "desingularized" kinetic energy of fluid motion. In the following consideration we use the Hamiltonian $H$.

From (4.2), (3.6), and the assumption that entropy is a function of $\Gamma$ (i.e., that entropy is an adiabatic invariant), we find the expression for the entropy of a vortex gas

$$S(E) = \ln[\,|V|^N - \Gamma(E)] + \text{const} . \tag{4.10}$$

The phase volume $\Gamma(E)$ has the following limit behavior:

$$\Gamma(E) \to 0 \quad \text{if } E \to -\infty , \qquad \Gamma(E) \to |V|^N \quad \text{if } E \to +\infty . \tag{4.11}$$

Therefore, the entropy approaches a constant if $E \to +\infty$, and $-\infty$ if $E \to -\infty$.

In statistical mechanics, entropy is usually defined by the relation

$$S = \ln \frac{d\Gamma}{dE} + \text{const} \tag{4.12}$$

while the link between temperature and entropy (3.6) stays the same. The corresponding expression of temperature in terms of the phase volume has the form

$$T = \frac{d\Gamma(E)/dE}{d\Gamma^2(E)/dE^2} . \tag{4.13}$$

For the classical models of statistical mechanics, the expressions for entropies (3.5) and (4.12) and for temperatures (3.3) and (4.13) coincide in the limit $N \to \infty$. If $N$ is finite, these expressions are not equivalent. For a vortex gas, they are not equivalent even in the limit $N \to \infty$. It is seen from (4.11) and (4.13) that temperature (4.13) changes sign at some value of energy $E_0$ [7], while temperature (4.1) has the same sign for all values of energy.

In this connection, we need to introduce some additional terminology. We keep the terms entropy and temperature for quantities (4.12) and (4.13), while quantities (4.10) and (4.2) will be called thermodynamic entropy $S_{\text{th}}$ and equipartition temperature $T_{\text{eq}}$, correspondingly.

## V. THERMODYNAMICS OF A ONE-COMPONENT VORTEX GAS AT LOW ENERGIES

To describe the method of calculation of thermodynamical functions, we start from the simplest case of a gas of vortices with equal intensities $\gamma_1=\cdots=\gamma_N=\gamma$. We assume that the total vorticity $N\gamma\equiv\sigma$ stays constant when $N\rightarrow\infty$. Therefore $\gamma\sim N^{-1}$.

The energy of fluid motion takes the form

$$H=\frac{\sigma^2}{N^2}\left[\sum_{i<j}G(r_i,r_j)+\sum_i g(r_i)\right] . \tag{5.1}$$

We have to find the phase volume

$$\Gamma(E)=\int_{H(r)\le E}d^{2N}r . \tag{5.2}$$

Here $d^{2N}r=dx_1dy_1\cdots dx_Ndy_N$ and it is assumed that $r_i\in V$.

We will use the probabilistic interpretation of the integral (5.2): $\Gamma(E)/|V|^N$ is equal to the probability of the event $H(r_i)\le E$ under the condition that all "random" variables $r_i$ are distributed homogeneously:

$$\Gamma(E)/|V|^N=\text{Prob}\{H(r_i)\le E\} . \tag{5.3}$$

We must find the function (5.3) in the limit $N\rightarrow\infty$.

This problem is reminiscent of the problem solved by the central limit theorem [29], which states that

$$\text{Prob}\left\{\frac{1}{\sqrt{N}}\sum_{i=1}^N w(r_i)\le\lambda\right\}\xrightarrow[N\rightarrow\infty]{}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda/b}e^{-x^2/2}dx\equiv\Phi\left[\frac{\lambda}{b}\right] . \tag{5.4}$$

Here $r_i$ are independent random variables, which, for our purposes, are assumed to be homogeneously distributed over $V$ and

$$\int_V w(r)d^2r=0,\quad\left[\frac{1}{|V|}\int_V w^2(r)dr\right]^{1/2}=b . \tag{5.5}$$

The main difference between (5.4) and (5.3) is that the members of the sum (5.1) are dependent random variables. Nevertheless, the problem of calculation (5.3) can be reduced to the central limit theorem in the case of low energies. More precisely, one can extract from the sum (5.1) a sum $\xi$ of independent random variables and the remainder $\eta$ in such a way that $\eta$ is small compared to $\xi$. To do this, we present $G(r,r')$ in the form

$$G(r,r')=\tilde{G}(r,r')+w(r)+w(r')+h , \tag{5.6}$$

where functions $\tilde{G}(r,r')$ and $w(r)$ have zero "mathematical expectations"

$$\int_V\tilde{G}(r,r')d^2r=0 ,\quad \int_V\tilde{G}(r,r')d^2r'=0 ,\quad \int_V w(r)d^2r=0 \tag{5.7}$$

and $h$ is a constant. Conditions (5.7) determine the function $w(r)$ and the constant $h$ uniquely. In fact, integrating (5.6) over $r'$ we get

$$w(r)+h=\frac{1}{|V|}\int_V G(r,r')d^2r' . \tag{5.8}$$

Integrating (5.8) over $r$, we find the constant $h$

$$h=\frac{1}{|V|^2}\int_V\int_V G(r,r')d^2r\,d^2r' . \tag{5.9}$$

Hence

$$w(r)=\frac{1}{|V|}\int_V G(r,r')d^2r'-\frac{1}{|V|^2}\int_V\int_V G(r,r')d^2r\,d^2r' . \tag{5.10}$$

In accordance with (5.8), the function $w(r)$ satisfies the boundary-value problem

$$\Delta w(r)=-\frac{1}{|V|}\ \text{ in } V,\quad w(r)=-h\ \text{ at } \partial V . \tag{5.11}$$

Since the function $w(r)$ and the constant $h$ are determined uniquely, Eq. (5.6) might be considered as the definition of function $\tilde{G}(r,r')$.

Let us substitute (5.6) into the energy expression (5.1). We have

$$H=\frac{\sigma^2}{N^2}\left[\sum_{i<j}\tilde{G}(r_i,r_j)+(N-1)\sum_{i=1}^N w(r_i)+\frac{N(N-1)}{2}h+\sum_{i=1}^N g(r_i)\right] .$$

Therefore, (5.3) can be rewritten as

$$\Gamma(E)/|V|^N=\text{Prob}\left\{\frac{1}{\sqrt{N}}\sum_{i=1}^N w(r_i)+\eta\le\frac{E-\frac{1}{2}h\sigma^2}{\sigma^2}\sqrt{N}\right\} , \tag{5.12}$$

where

$$\eta=\frac{1}{\sqrt{N}}\left[\frac{1}{N}\sum_{i<j}\tilde{G}(r_i,r_j)+\frac{1}{N}\sum g(r_i)\right] . \tag{5.13}$$

Here $N-1$ is replaced by $N$ because we consider the limit relations for $N\rightarrow\infty$.

To evaluate the order of $\eta$, note that $(1/N)\sum_{i<j}\tilde{G}(r_i,r_j)$ has a finite probability distribution if $N\rightarrow\infty$ [18]. In accordance with the central limit theorem for the second sum, we have

$$\frac{1}{N}\sum g(r_i)\cong\frac{1}{V}\int_V g\,d^2r+\left[\text{random variable of order}\frac{1}{\sqrt{N}}\right] .$$

Therefore, the variance of $\eta$ is of order $1/\sqrt{N}$.

If small deviations of the energy $E$ from the critical value

$$E_0=\tfrac{1}{2}h\sigma^2 \tag{5.14}$$

are of order $1/\sqrt{N}$, then $(1/\sqrt{N})\sum w(r_i)$ is of order unity, $\eta$ in (5.12) can be neglected, and the phase volume is given by the central limit theorem

$$\Gamma(E)/|V|^N \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{[(E-E_0/\sigma^2 b)]\sqrt{N}} e^{-x^2/2}dx = \Phi\left[\sqrt{N}\frac{E-E_0}{\sigma^2 b}\right], \tag{5.15}$$

$$b=\left[\frac{1}{|V|}\int_V w^2(r)d^2r\right]^{1/2}.$$

Note that expression (5.15) is valid only if $(E-E_0)\sqrt{N}$ is finite, i.e., in a small vicinity of $E_0$. This vicinity is of order $1/\sqrt{N}$. As follows from (5.12), in the case of finite deviations of the initial energy from $E_0$ (high energy case), the calculation of probability distributions relates to the problem of large deviations of the sum of independent variables. Random variable $\eta$ contributes to the result. This problem will be considered in Sec. VII, while here we discuss the consequences of formula (5.15).

Calculating derivatives of $\Gamma(E)$, we obtain

$$\frac{d\Gamma}{dE}=|V|^N\frac{\sqrt{N}}{\sqrt{2\pi}\sigma^2 b}\exp\left[-\frac{1}{2}\left[\sqrt{N}\frac{E-E_0}{\sigma^2 b}\right]^2\right],$$

$$\frac{d^2\Gamma}{dE^2}=|V|^N\frac{1}{\sqrt{2\pi}}\left[\frac{\sqrt{N}}{\sigma^2 b}\right]^2\left[-\sqrt{N}\frac{E-E_0}{\sigma^2 b}\right]\times\exp\left[-\frac{1}{2}\left[\sqrt{N}\frac{E-E_0}{\sigma^2 b}\right]^2\right].$$

Therefore, the temperature $T$ is given by

$$T=\frac{d\Gamma(E)/dE}{d\Gamma^2(E)/dE^2}=-\frac{(\sigma^2 b)^2}{(E-E_0)N}. \tag{5.16}$$

We see that temperature $T$ really changes sign at $E=E_0$ and $T<0$ if $E>E_0$. The temperature $T$ approaches infinity if $E\to E_0$. If $E-E_0\sim 1/\sqrt{N}$, $T$ is also of order $1/\sqrt{N}$.

The entropy $S=\ln d\Gamma/dE+\text{const}$ in the vicinity of the critical energy is equal to

$$S=\text{const}-\frac{1}{2}\left[\sqrt{N}\frac{E-E_0}{\sigma^2 b}\right]^2. \tag{5.17}$$

The entropy reaches its maximum value at the critical energy. This property was established for a one-component vortex gas from other reasonings by Eyink and Spohn [13].

In the vicinity of the critical energy, the equipartition temperature $T_{\text{eq}}$ is given by

$$T_{\text{eq}}=-\frac{|V|^N-\Gamma}{\partial\Gamma(E)/\partial E} = -\frac{\sqrt{2\pi}\sigma^2 b}{\sqrt{N}}\left[1-\Phi\left[\sqrt{N}\frac{E-E_0}{\sigma^2 b}\right]\right]\times\exp\left[\frac{1}{2}\left[\sqrt{N}\frac{E-E_0}{\sigma^2 b}\right]^2\right]. \tag{5.18}$$

The equipartition temperature does not have a singularity at $E_0$ and does not change the sign. In the vicinity of $E_0$, the equipartition temperature is of order $1/\sqrt{N}$.

The thermodynamic entropy $S_{\text{th}}$ is equal to

$$S_{\text{th}}=\ln\left[1-\Phi\left[\sqrt{N}\frac{E-E_0}{\sigma^2 b}\right]\right]+\text{const}. \tag{5.19}$$

Formulas (5.16)–(5.19) leave no doubt that the two expressions for entropy are really different, in contrast to most models of statistical mechanics.

The plot of the dimensionless normalized temperature $T^*=\sqrt{N}T/\sigma^2 b$ and the equipartition temperature $T^*_{\text{eq}}=\sqrt{N}T_{\text{eq}}/\sigma^2 b$ versus the dimensionless normalized energy deviation from the critical energy $\Delta E^*=\sqrt{N}(E-E_0)/\sigma^2 b$ is shown in Fig. 1. It is seen that for large energies $E>E_0$, the equipartition temperature approaches the temperature $T$. They become practically equal if $E\geq E_0+2\sigma^2 b/\sqrt{N}$. The coincidence of $T$ and $T_{\text{eq}}$ for the high energy limit will be confirmed in Sec. VIII. In the vicinity of the critical energy, $T$ and $T_{\text{eq}}$ are different.

The critical energy $E_0$ has a simple physical sense. If one puts vortices in the container randomly and independently, then $H(r_i)$ is a random variable. Its mathematical expectation is $E_0$, as follows from (5.14), (5.9), and (5.1). If the initial positions of the vortices are chosen independently and randomly, then the initial energy $E$ deviates from $E_0$ for the value of order $1/\sqrt{N}$. High energy cases correspond to "special choices" of the initial positions of vortices; for example, a finite deviation of the initial energy from $E_0$ is obtained if all vortices are put together in a small "vortex cloud." One consequence of the basic relation (5.15) is worth noting here. Two values of the phase volume $\Gamma(E)$ follow from its definition:

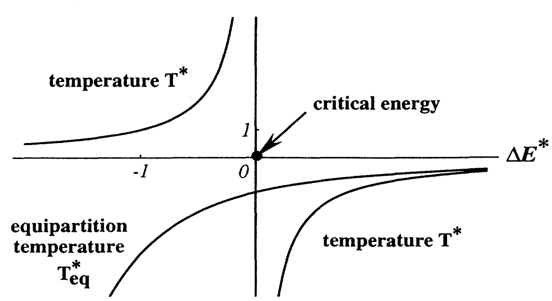


FIG. 1. Graph of temperature and the equipartition temperature in the vicinity of the critical energy.

$$\Gamma(-\infty)=0, \quad \Gamma(+\infty)=|V|^N .$$

Equation (5.15) provides one more exact value of $\Gamma(E)$

$$\Gamma(E_0)=\tfrac{1}{2}|V|^N . \tag{5.20}$$

So the energy for which the phase volume is exactly between its two limit values is the critical energy.

## VI. THERMODYNAMICS OF A MULTICOMPONENT VORTEX GAS AT LOW ENERGIES

Let the vortex intensities be not necessarily equal. We assume that $\gamma_i \sim N^{-1}$ and $\sigma_i=\gamma_i N$ stay constant when $N\rightarrow\infty$. Applying the same transformation (5.6), we arrive at the energy expression

$$H=\frac{\sum\gamma_i}{N}\sum_i \sigma_i w(r_i)+\frac{1}{2N^2}\sum_{i\neq j}\sigma_i\sigma_j\tilde{G}(r_i,r_j)+\tfrac{1}{2}h\left[\sum\gamma_i\right]^2+\frac{1}{N^2}\sum_i\sigma_i^2\left[g(r_i)-w(r_i)-\frac{h}{2}\right] . \tag{6.1}$$

The first term is supposed to be the leading one. We see that two cases should be distinguished: a neutral gas ($\sum\gamma_i=0$) and a rotating gas ($\sum\gamma_i\neq 0$). Let first $\sum\gamma_i\neq 0$. Then $\Gamma(E)$ can be written in the form, generalizing (5.12),

$$\Gamma(E)/|V|^N=\text{Prob}\left\{\frac{\sum\gamma_i}{\sqrt{N}}\sum_i\sigma_i w(r_i)+\eta\le(E-E_0)\sqrt{N}\right\} , \tag{6.2}$$

where

$$E_0=\tfrac{1}{2}h(\sum\gamma_i)^2 , \tag{6.3}$$

$$\eta=\frac{1}{\sqrt{N}}\left[\frac{1}{2N}\sum_{i\neq j}\sigma_i\sigma_j\tilde{G}(r_i,r_j)+\frac{1}{N}\sum_i\sigma_i^2\left[g(r_i)-w(r_i)-\frac{h}{2}\right]\right] . \tag{6.4}$$

The mathematical expectation of $\eta$ has the order $1/\sqrt{N}$. The variance of the first sum in (6.4) also has the order $1/\sqrt{N}$ (see Appendix A). The variance of the second sum in (6.4) has the order $1/\sqrt{N}$. Therefore, $\eta$ can be dropped in (6.2) if $(E-E_0)\sqrt{N}$ has the order unity. From the central limit theorem we obtain

$$\Gamma(E)/|V|^N\rightarrow\Phi\left[\sqrt{N}\frac{E-E_0}{\left[\sum\gamma_i\right]\sigma b}\right] ,$$
$$\sigma^2=\lim_{N\rightarrow\infty}\frac{1}{N}(\sigma_1^2+\cdots+\sigma_N^2) . \tag{6.5}$$

To determine the sense of the reference energy $E_0$, consider the expression for energy (2.1). Assume that the positions of all vortices are independent random vectors homogeneously distributed over $V$. Then the average value of energy is

$$\bar{H}=\frac{1}{2}\sum_{i\neq j}\frac{1}{|V|^2}\int G(r,r')dr\,dr'\gamma_i\gamma_j+\sum_i\gamma_i^2\frac{1}{|V|}\int g(r)dr .$$

When $N\rightarrow\infty$ the last sum vanishes; therefore

$$\bar{H}\rightarrow\tfrac{1}{2}h\left[\sum_i\gamma_i\right]^2=E_0 . \tag{6.6}$$

So the reference energy is the averaged energy of the vortex system if all vortices are distributed over $V$ independently and homogeneously.

In the case of equal vortices, the total vorticity $\sum\gamma_i$ is equal to $\sigma$ and expression (6.6) reduces to (5.14). Modifications of the expressions for entropies and temperatures (5.16)–(5.19) are evident.

The case of a neutral vortex gas is different. In this case, the first term in (6.1) is equal to zero and the second term takes the leading role:

$$\Gamma(E)/|V|^N=\text{Prob}\left\{\frac{1}{2N}\sum\sigma_i\sigma_j\tilde{G}(r_i,r_j)+\eta\le(E-E_0)N\right\} , \tag{6.7}$$

where

$$E_0=\frac{e_0}{N},\quad e_0=\sigma^2\left[\frac{1}{|V|}\int_V g(r)d^2r-h\right] ,$$

$$\eta=\frac{1}{N}\sum_i\sigma_i^2[g'(r_i)-w(r_i)] ,$$

$$g'(r)=g(r)-\frac{1}{|V|}\int_V g(r)d^2r .$$

The sum in (6.7) has some finite probability distribution $K(\xi)$ when $N\rightarrow\infty$. The random variable $\eta$ has variances of the order $1/\sqrt{N}$. Therefore, it can be neglected and

$$\Gamma(E)/|V|^N\underset{N\rightarrow\infty}{\sim}K(EN-e_0) . \tag{6.8}$$

Note that, although the reference energy $E_0$ tends to zero for $N\rightarrow\infty$, there is some finite shift $e_0$ in the probability distribution for small energies $E$ of the order $1/N$. The function $dK/d\xi$ has been studied numerically in [16] for periodic flows.

## VII. PHASE VOLUME OF A MULTICOMPONENT VORTEX GAS AT HIGH ENERGIES

By high energy we mean energy deviating from the reference value $E_0$ for a finite value that does not depend on $N$ if $N\rightarrow\infty$. To find the phase volume in this case, we prepare first a convenient form of the expression for $\Gamma(E)$. The definition of $\Gamma(E)$ can be written as

$$\Gamma(E)=\int_{V^N}\theta(E-H(r_i))d^{2N}r ,$$

where $\theta(E)$ is the step function. Differentiating this equation by $E$, we have

$$\frac{d\Gamma(E)}{dE}=\int_{V^N}\delta(E-H(r_i))d^{2N}r \ , \qquad (7.1)$$

where $\delta(E)$ is the $\delta$ function. Using in (7.1) the Fourier presentation for the $\delta$ function

$$\delta(E)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ikE}dk \ , \qquad (7.2)$$

we have

$$\frac{d\Gamma(E)}{dE}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dk\left[e^{ikE}\int_{V^N}e^{-ikH(r_i)}d^{2N}r\right] .$$

It is convenient to make a change of variables $k\rightarrow z$: $ik=Nz$. Then

$$\frac{d\Gamma(E)}{dE}=\frac{N}{2\pi i}\int_{-i\infty}^{+i\infty}e^{NEz}A(z,N)dz \ ,$$
$$A(z,N)=\int_{V^N}e^{-zNH(r_i)}d^{2N}r \ . \qquad (7.3)$$

If we find the asymptotics of $A(z,N)$ for $N\rightarrow\infty$, we can use the steepest descent method to calculate $d\Gamma(E)/dE$. To obtain the asymptotics, we make the transformation of energy similar to (5.6): for each particle $i$ we introduce some function $w_i(r)$, which will be specified later, and a "modified" particle-particle interaction $G_{ij}$

$$G_{ij}(r_i,r_j)\equiv G(r_i,r_j)-w_i(r_j)-w_j(r_i)-h_{ij} \ , \qquad (7.4)$$

where $h_{ij}$ are some constants. The difference from (5.6) is that $w$ functions should be taken differently for different vortices in the high energy case. After substituting (7.4) into the energy expression (2.1), the function $A(z,N)$ takes the form

$$A(z,N)=\exp\left[-\frac{z}{2N}\sum_{i\neq j}\sigma_i\sigma_j h_{ij}\right] \times\int_{V^N}\exp\left[-\frac{z}{2N}\sum\sigma_i\sigma_j G_{ij}(r_i,r_j) -z\sum_i\sigma_i F_i(r_i)\right]d^{2N}r \ , \qquad (7.5)$$

$$F_i(r)=\frac{1}{N}\sum_{j\neq i}\sigma_j w_j(r)+\frac{1}{N}\sigma_i g(r) \ . \qquad (7.6)$$

Let us consider now $N$ independent variables $r_i$, taking the values in the region $V$, which have complex "probability density functions"

$$e^{-z\sigma_i F_i(r)}\Big/\int_V e^{-z\sigma_i F_i(r')}d^2r' \ . \qquad (7.7)$$

Denoting the "mathematical expectation" with respect to this measure by $M$, we may rewrite (7.5) as

$$A(z,N)=\exp\left[-\frac{z}{2N}\sum_{i\neq j}\sigma_i\sigma_j h_{ij}\right] \times\prod_i\int_V e^{-z\sigma_i F_i(r)}d^2r \times M\exp\left[-\frac{z}{2N}\sum\sigma_i\sigma_j G_{ij}(r_i,r_j)\right] . \qquad (7.8)$$

If $M$ in (7.8) were a usual mathematical expectation, if $z$ were real, and if the mathematical expectations of $G_{ij}(r_i,r_j)$ in both variables $r_i,r_j$ were zero, then, as in Sec. V, the last term in (7.8) converges for $N\rightarrow\infty$ to a function of $z$. A similar statement for "complex measures" is discussed in Appendix A.

Before studying the asymptotics of $A(z,N)$, let us establish that the functions $w_i$ are determined by the condition of zero mathematical expectations of $G_{ij}$

$$\int_V G_{ij}(r,r_j)e^{-z\sigma_i F_i(r)}d^2r=0 \ ,$$
$$\int_V G_{ij}(r_i,r)e^{-z\sigma_j F_j(r)}d^2r=0 \ . \qquad (7.9)$$

Here $i,j=1,\ldots,N$, and there is no summation over repeated indices. Substituting (7.4) in (7.9), we get

$$\int_V G(r,r_j)e^{-z\sigma_i F_i(r)}d^2r -[w_i(r_j)+h_{ij}]\int_V e^{-z\sigma_i F_i(r)}d^2r -\int_V w_j(r)e^{-z\sigma_i F_i(r)}d^2r=0 \ , \qquad (7.10)$$

$$\int_V G(r_1,r)e^{-z\sigma_j F_j(r)}d^2r -[w_j(r_i)+h_{ij}]\int_V e^{-z\sigma_j F_j(r)}d^2r -\int w_i(r)e^{-z\sigma_j F_j(r)}d^2r=0 \ . \qquad (7.11)$$

Equations (7.10) and (7.11) form a system of equations for $w_i(r)$ [it is assumed that $F_i(r)$ are expressed in terms of $w_i$ by means of (7.6)]. This system is overdetermined: it contains $2N^2$ equations for $n$ functions $w_i(r)$ and $N^2$ constants $h_{ij}$. Nevertheless, the system is consistent. It is seen from (7.10) and (7.11) that the functions $w_i(r)$ are constant at $\partial V$. We put an additional constraint on $w_i(r)$: $w_i(r)=0$ at $\partial V$. Then, from (7.10), we obtain the expression for $h_{ij}$

$$h_{ij}=-\int_V w_j(r)e^{-z\sigma_i F_i(r)}d^2r\Big/\int_V e^{-z\sigma_i F_i(r)}d^2r \ , \qquad (7.12)$$

while from (7.11)

$$h_{ij}=-\int_V w_i(r)e^{-z\sigma_j F_j(r)}d^2r\Big/\int_V e^{-z\sigma_j F_j(r)}d^2r \ . \qquad (7.13)$$

Applying the Laplace operator to (7.10) and (7.11), we obtain the equations for $w_i$

$$\Delta w_i = -\frac{e^{-z\sigma_i F_i(r)}}{\int_V e^{-z\sigma_i F_i(r')} d^2 r'} \quad \text{in } V ,$$
$$w_i = 0 \quad \text{at } \partial V . \tag{7.14}$$

Equations (7.14), along with the expressions for $F_i$ in terms of $w_i$ (7.6), form a closed system of $n$ equations for $n$ required functions $w_i$.

Equations (7.14) show that two different expressions for the constants $h_{ij}$ (7.12) and (7.13) are consistent: multiplying (7.14) by $w_j$ and integrating over $V$, we have

$$\int_V w_j \Delta w_i d^2 r = -\int_V w_j e^{-z\sigma_i F_i(r)} d^2 r \Big/ \int_V e^{-z\sigma_i F_i(r)} d^2 r$$
$$= h_{ij} = -\int_V \nabla w_j \nabla w_i d^2 r = \int_V w_i \Delta w_j d^2 = -\int_V w_i e^{-z\sigma_j F_j(r)} d^2 r \Big/ \int_V e^{-z\sigma_j F_j(r)} d^2 r .$$

Hence, the right-hand sides of (7.12) and (7.13) are equal due to (7.14). So if $w_i$ and $h_{ij}$ obey (7.10) and (7.11), then $w_i$ is a solution of (7.14) and the constants $h_{ij}$ are determined by this solution from (7.12). It can be checked by inspection that the inverse statement is also true: (7.10) and (7.11) follow from (7.14) and (7.12).

The system (7.14) takes a simple form in the limit $N \to \infty$. Note that the functions $F_i$ can be presented as

$$F_i(r) = u(r) + \frac{1}{N}\sigma_i (g(r) - w_i(r)) , \tag{7.15}$$

where

$$u(r) = \frac{1}{N}\sum_i \sigma_i w_i(r) . \tag{7.16}$$

In the limit $N \to \infty$ the last term in (7.15) can be dropped, functions $F_i(r)$ for all $i$ are equal to $u(r)$, and we get the equations

$$\Delta w_i = -\frac{e^{-z\sigma_i u(r)}}{\int_V e^{-z\sigma_i u(r')} d^2 r'} \quad \text{in } V ,$$
$$w_i = 0 \quad \text{at } \partial V \tag{7.17}$$

and the equation for $u$ which follows from (7.16) and (7.17)

$$\Delta u = -\frac{1}{N}\sum \sigma_i \frac{e^{-z\sigma_i u(r)}}{\int_V e^{-z\sigma_i u(r')} d^2 r'} \quad \text{in } V ,$$
$$u = 0 \quad \text{at } \partial V . \tag{7.18}$$

This mathematical problem is well posed at least for some $z, \sigma$. Note that if we make another transformation instead of (7.4),

$$G_{ij}(r_i, r_j) = G(r_i, r_j) - w_i(r_i) - w_j(r_j) - h_{ij} ,$$

we would arrive at the equations for $w_i$ which do not have a solution.

Let the vortex gas have $s$ components. This means that vortex intensities take $s$ values $\gamma_l, \ldots, \gamma_s$ and each value $\gamma_\alpha$ is carried by $N_\alpha$ vortices $\sum_\alpha N_\alpha = N$ (greek indices number the gas components). It is assumed that $\sigma_\alpha = \gamma_\alpha N$ and $c_\alpha = N_\alpha / N$ stay constant when $N \to \infty$.

Equation (7.18) for a multicomponent vortex gas takes the form

$$\Delta u = -\sum_\alpha c_\alpha \sigma_\alpha \frac{e^{-z\sigma_\alpha u(r)}}{\int_V e^{-z\sigma_\alpha u(r')} d^2 r'} \quad \text{in } V ,$$
$$u = 0 \quad \text{at } \partial V . \tag{7.19}$$

The sum $\sum \sigma_i \sigma_j h_{ij}$, which is contained in the expression for $A(z,N)$, can be expressed in terms of the solution of Eq. (7.19): in accordance with (7.12), (7.16), and (7.18)

$$\frac{1}{2N}\sum_{i \neq j} \sigma_i \sigma_j h_{ij} = -\frac{1}{2N}\sum_{i,j} \sigma_i \sigma_j \int_V w_j(r) e^{-z\sigma_i u(r)} d^2 r \Big/ \int_V e^{-z\sigma_i u(r)} d^2 r$$
$$= -\frac{1}{2}\sum_i \int_V u(r) \sigma_i e^{-z\sigma_i u(r)} d^2 r \Big/ \int_V e^{-z\sigma_i u(r)} d^2 r$$
$$= \frac{N}{2}\int_V u \Delta u \, d^2 r = -\frac{N}{2}\int_V (\nabla u)^2 d^2 r . \tag{7.20}$$

Now, everything is prepared to put $A(z,N)$ in the final form

$$A(z,N) = |V|^N \exp[NB(z,u)] C(z,N) , \tag{7.21}$$

$$B(z,u) = \frac{z}{2}\int_V (\nabla u)^2 d^2 r + \sum_\alpha c_\alpha \ln \frac{1}{|V|}\int_V e^{-z\sigma_\alpha u(r)} d^2 r , \tag{7.22}$$

$$C(z,N) = M \exp\left[ -\frac{z}{N}\sum_{i \neq j} \sigma_i \sigma_j G_{ij}(r_i r_j) \right] . \tag{7.23}$$

The function $u$ in (7.22) is the solution of Eq. (7.19). It is more attractive, however, to consider $B(z,u)$ as a functional of function $u$. Then, it turns out that the stationary points of the functional $B(z,u)$ are the solutions of the boundary-value problem (7.19). We will widely use this fact in the following. One immediate consequence is worth noting right now. For a real positive $z$ functional, $B(z,u)$ is strictly convex. Therefore, it has the only minimizing element. Hence, the boundary-value problem (7.19) makes sense and has the unique solution at least for real positive $z$.

Collecting (7.3) and (7.21)–(7.23), we obtain for $d\Gamma/dE$ the relation

$$\frac{d\Gamma(E)}{dE}=\frac{N}{2\pi i}|V|^N\int_{-i\infty}^{+i\infty}e^{N[Ez+B(z,u)]}C(z,N)dz\ , \tag{7.24}$$

where $B(z,u)$ is the functional (7.22). Since $C(z,N)$ converges for $N\to\infty$ to some function $C_0(z)$, one can use the steepest descent method to find the asymptotics of $d\Gamma/dE$. The asymptotics is determined by the stationary points of the function $Ez+B(z,u)$. To simplify the dependence of the functional $B$ on $z$ it is worth changing the variables $u\to v$ by setting $v=zu$. Then the last term in the expression for $B$ (7.22) becomes independent of $z$ and $B$ takes the form

$$B(z,v)=\frac{1}{2z}\int_V(\nabla v)^2d^2r+\sum_\alpha c_\alpha\ln\frac{1}{|V|}\int_V e^{-\sigma_\alpha v}d^2r\ . \tag{7.25}$$

For a given $z$, the stationary function of the functional $B(z,v)$ is denoted by $\hat{v}(z)$. It obeys the boundary-value problem

$$\Delta v_0=-z\sum c_\alpha\sigma_\alpha\frac{e^{-\sigma_\alpha v_0}}{\int_V e^{-\sigma_\alpha v_0(r')}d^2r'}\quad\text{in } V\ , \tag{7.26}$$
$$v_0=0 \text{ at } \partial V\ .$$

The stationary point of the functional $B(z,v_0(z))$ with respect to $z$ is denoted by $\beta$. The stationary point $\beta$ is determined by the equation

$$\frac{1}{2\beta^2}\int_V(\nabla v_0)^2d^2r=E\ . \tag{7.27}$$

It is shown in Appendix B that the functional $B(z,v)$ is convex with respect to $z,v$ for all real, positive $z$. In this case, $Ez+B$ has the only stationary point and this stationary point is the point of minimum. For real negative $z$, (7.26) can be interpreted as Euler's equation in maximization of the functions $B(z,v)$ with respect to $v$. It is assumed that for some values of energy $E$ the function $Ez+B(z,v_0(z))$ might have a minimum with respect to $z$ at some negative value $\beta$.

For real positive $z$, the solution $v_0$ of (7.26) is real; therefore, as seen from (7.27), $E>0$. For $E<0$ there are no real stationary points. We will confine ourselves to the case of $E>0$.

To get the asymptotics of $d\Gamma/dE$, we move the line of integration in such a way that is passes the point $z=\beta$:

$$\frac{d\Gamma(E)}{dE}=\frac{N}{2\pi i}|V|^N\int_{\beta-i\infty}^{\beta+i\infty}e^{N[Ez+B(z,v)]}C(z,N)dz\ . \tag{7.28}$$

Since $Ez+B(z,v_0(z))$ has the minimum value at $z=\beta$, $\mathrm{Re}[Ez+B(z,v(z))]$ has the local maximum at $z=\beta$ along the line of integration. It is shown in Appendix D that this maximum is, in fact, the global maximum of $\mathrm{Re}[Ez+B(z,v(z))]$ along the line of integration. Hence, in accordance with the steepest descent method, in the first approximation

$$\frac{d\Gamma(E)}{dE}=|V|^N\frac{\sqrt{N}}{\sqrt{2\pi B''}}C_0(\beta)e^{N[E\beta+B(\beta,v_0(\beta))]}\ , \tag{7.29}$$

where $\beta$ and $v_0(\beta)$ are the solutions of the system of equations (7.26) and (7.27) (they depend on $E$) and $B''$ is the second derivative of the function $B(z,v_0(z))$ with respect to $z$ at the point $z=\beta$.

## VIII. THERMODYNAMICS OF A VORTEX GAS

The relations obtained in the preceding section allow us to analyze the thermodynamical functions of vortex gas. First, we find from (7.29) the expression for the entropy $S$

$$N^{-1}S(E)=E\beta+B(\beta,v_0(\beta))\ . \tag{8.1}$$

Here $\beta$ is a function of energy determined by the relation

$$E+\frac{d}{d\beta}B(\beta,v_0(\beta))=0\ . \tag{8.2}$$

Small terms of order $N^{-1}\ln N$ and $N^{-1}$ are dropped on the right-hand side of (8.1). Entropy can be presented as the stationary value $V^{\rm st}_{z,v(x)}$ of the functional $Ez+B$:

$$N^{-1}S(E)=V^{\rm st}_{z,v(x)}\left[Ez+\frac{1}{2z}\int_V(\nabla v)^2d^2r+\sum_\alpha c_\alpha\ln\frac{1}{|V|}\int_V e^{-\sigma_\alpha v}d^2r\right]\ . \tag{8.3}$$

The stationary value with respect to $v$ is, in fact, the minimum value for positive $z$ and the maximum value for negative $z$.

If the energy $E$ is close to $E_0$, the variational problem (8.3) yields the expression for entropy in the low energy limit (5.17) (see Appendix E). Differentiating $S(E)$ with respect to $E$, we find the temperature

$$NT=\frac{N}{dS/dE}=\frac{1}{\beta(E)}\ . \tag{8.4}$$

For the motion of a one-component vortex gas in a circular domain, the variational problem (8.3) can be solved exactly (see Appendix F). Functions $\beta(E)$ and $S(E)$ are shown in Figs. 2 and 3, where $E^*$ and $\beta^*$ are dimensionless $E$ and $\beta$. The characteristic features of these functions are (i) that the entropy $S$ has its maximum value at $E=E_0$, (ii) at this value of energy, the temperature $T$ changes sign; (iii) the negative values of temperature correspond to $E>E_0$; (iv) $S(E)\to-\infty$ if $E$ tends to zero or infinity; and (v) the admissible values of $\beta$ are bounded below. The following analysis pertains to the cases when these properties take place.

Let us now find the thermodynamic entropy. To do so

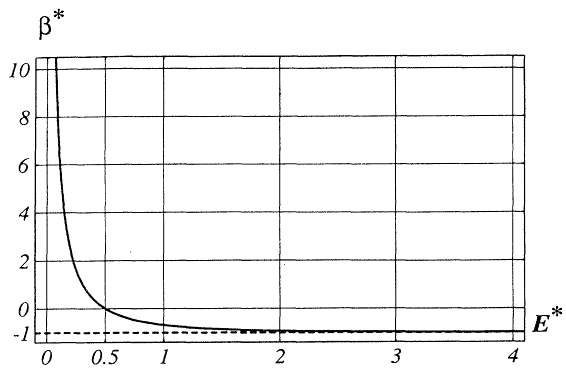


FIG. 2. Dependence of the inverse temperature on energy for a one-component vortex gas in the circular domain.

we must determine the phase volume $\Gamma(E)$. Integrating (7.29), we have

$$\Gamma(E)=\frac{|V|^N}{\sqrt{2\pi B''N}\,\beta(E)}C(\beta)e^{N(E\beta+B)}+\text{const} . \qquad (8.5)$$

Relation (8.5) can be verified by differentiation with respect to $E$. In this calculation one must take into account that the derivative of $E\beta+B(\beta,v_0(\beta))$ with respect to $\beta$ is equal to zero and the contributions of derivatives $B''$ and $\beta$ are negligibly small compared to the right-hand side of (7.29).

Note that expression (8.5) is true only out of the vicinity of the critical energy $E_0$ where $\beta(E_0)=0$. The constant in (8.5) can be found from the condition that $\Gamma(E)\to|V|^N$ if $E\to\infty$. Since $S(E)\to-\infty$ for $E\to\infty$, the first term in (8.5) tends to zero and we find that the constant is equal to $|V|^N$. Finally,

$$\Gamma(E)=|V|^N\left[1+\frac{C(\beta)}{\sqrt{2\pi B''N}\,\beta}e^{N(E\beta+B)}\right] . \qquad (8.6)$$

Note that expression (8.6) is valid only for negative temperatures because $\Gamma(E)$ should be less than $|V|^N$. The rest of this section deals only with the case of negative temperatures.

Substituting (8.6) into (4.10), we obtain that $S(E)$ and $S_{\text{th}}(E)$ coincide. Hence canonical and equipartition temperatures are also equal

$$T_{\text{eq}}=T=\frac{1}{N\beta} . \qquad (8.7)$$

Since $\beta$ does not depend on $N$, the temperature decays as $N^{-1}$. This fact has a simple physical meaning. In accordance with (4.1), the decay of $\gamma$ and $T_{\text{eq}}$ as $N^{-1}$ yields that the averaged area bounded by each vortex trajectory per unit of time stays finite. Substituting $\gamma_i=\sigma_i/N$ in (4.1) and using (8.7), we obtain a "finite" form of the equipartition law

$$\sigma_1\langle y_1\dot{x}_1\rangle=\cdots=\sigma_N\langle y_N\dot{x}_N\rangle=\frac{1}{\beta} . \qquad (8.8)$$

This relation determines the physical sense of parameter $\beta$.

The equality of the equipartition temperature and the canonical temperature holds only outside some vicinity of the critical energy $E_0$. The size of this vicinity depends on $N$. If energy decreases and enters a small vicinity of $E_0$ of order $1/\sqrt{N}$, then $T\neq T_{\text{eq}}$, as we have discussed in Sec. V (see Fig. 1). Moreover, $T$ changes sign when $E$ goes below $E_0$, while $T_{\text{eq}}$ keeps its sign in accordance with common sense: the direction of rotation should not be changed in the course of variation of energy. For $E<E_0$, $NT$ and $NT_{\text{eq}}$ are different.

For energies close to $E_0$, the order of magnitude of the equipartition temperature becomes $1/\sqrt{N}$. That is much larger than $1/N$, the order of magnitude of the equipartition temperature for finite $E-E_0$. This indicates that the averaged area bounded by the vortex trajectory per unit time becomes very large, of order $\sqrt{N}$, at the vicinity of critical energy. A possible explanation is that, at the critical energy, trajectories become much more curling.

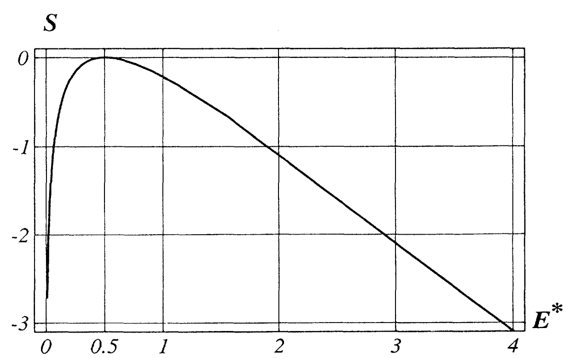


FIG. 3. Dependence of entropy on energy for a one-component vortex gas in the circular domain.

## IX. PROBABILITY DISTRIBUTIONS

One of the most interesting probabilistic characteristics of fluid motion is the probability density of vortex coordinates $f_i(r)$ for the $i$th vortex. In this section, it is shown that, in the first approximation,

$$f_i(r)=\frac{e^{-\beta\sigma_i u(r)}}{\int_V e^{-\beta\sigma_i u(r')}d^2r'} , \qquad (9.1)$$

where $u(r)$ is the solution of Eq. (7.19). Formula (9.1) gives the first term of an asymptotic series in $N^{-1}$. The next term is of order $N^{-1}$.

Let us derive (9.1) for the probability density of the first vortex $f_1(r)$. As follows from (3.8) and (3.5) (see also [24,30]), the probability density can be written in the form

$$f_i(r)=\frac{\partial\Gamma(r,E)}{\partial E}\bigg/\frac{\partial\Gamma(E)}{\partial E} , \qquad (9.2)$$

where $\Gamma(r,E)$ is the phase volume of the $[2(N-1)]$-dimensional region, extracted by the inequalities $H(r,r_2,\ldots,r_N)\le E$, $r_i\in V$. The derivative $\partial\Gamma(E)/\partial E$ has been found in Sec. VII. Determining the derivative $\partial\Gamma(r,E)/\partial E$ is quite similar; the only difference is the dependence of the Hamiltonian on the parameter $r$, the position of the first vortex. Following the line of Sec. VII, we start from the formula for $\partial\Gamma(r,E)/\partial E$ analogous to (7.3):

$$\frac{d\Gamma(r,E)}{dE}=\frac{N}{2\pi i}\int_{-i\infty}^{i\infty}e^{NEz}A(z,r,N)dz\ ,$$
$$A(z,r,N)=\int_{V^{N-1}}e^{-zNH(r,r_2,\ldots,r_N)}d^2r_2\cdots d^2r_N\ . \tag{9.3}$$

The Hamiltonian $H(r,r_2,\ldots,r_N)$ can be written as

$$H=\frac{1}{N^2}\left[\frac{1}{2}\sum_{\substack{i\neq j\\ i,j\geq 2}}\sigma_i\sigma_jG(r_i,r_j)+\sum_{i=2}^{N}\sigma_i(\sigma_1G(r,r_i)+\sigma_ig(r_i))+\sigma_1^2g(r)\right]\ . \tag{9.4}$$

Now we make the transformation (7.4) for $i,j\geq 2$. The functions $w_i$ and $G_{ij}$ in (7.4) depend on the parameter $r$. We obtain

$$A(z,r,N)=\exp\left[-\frac{z}{2N}\sum_{\substack{i\neq j\\ 2\leq i,j}}\sigma_i\sigma_jh_{ij}-\frac{z\sigma_1^2}{N}g(r)\right]\times\int_{V^{N-1}}\exp\left[-\frac{z}{2N}\sum_{\substack{i\neq j\\ 2\leq i,j}}\sigma_i\sigma_jG_{ij}-z\sum_{i=2}^{N}\sigma_iu_i(r_i,r)\right]\times d^2r_2\cdots d^2r_N\ , \tag{9.5}$$

$$u_i(r',r)=\frac{1}{N}\sum_{j\neq i}\sigma_jw_j(r',r)+\frac{\sigma_1}{N}G(r',r)+\frac{\sigma_1}{N}g(r')\ . \tag{9.6}$$

As before, we put the following constraints on the functions $G_{ij}(r_i,r_j)$: the average values of $G_{ij}$ with respect to the complex measure const $e^{-z\sigma_iu_i}$ are zeros,

$$\int_VG_{ij}(r',r_j)e^{-z\sigma_iu_i(r',r)}d^2r'=0\ ,$$
$$\int_VG_{ij}(r_i,r')e^{-z\sigma_ju_j(r',r)}d^2r'=0\ . \tag{9.7}$$

Here, for brevity of notation, we do not mention the dependence of $G_{ij}$ on parameter $r$. The constraints (9.7) yield the system of integral equations for $w_i$

$$\int_VG(r',r_j)e^{-z\sigma_iu_i(r',r)}d^2r'-[w_i(r_j,r)+h_{ij}]\int_Ve^{-z\sigma_ju_j(r',r)}d^2r'-\int_Vw_j(r',r)e^{-z\sigma_iu_i(r',r)}d^2r'=0\ ,$$
$$\int_VG(r_i,r')e^{-z\sigma_ju_j(r',r)}d^2r-\int_Vw_i(r',r)e^{-z\sigma_ju_j(r',r)}d^2r-[w_j(r_i,r)+h_{ij}]\int_Ve^{-z\sigma_ju_j(r',r)}d^2r'=0\ . \tag{9.8}$$

It follows from (9.8) that the functions $w_i(r',r)$ do not depend on $r'$ if $r'\in\partial V$. Therefore, we define the constant $h_{ij}$ by the additional condition

$$w_i(r',r)=0\quad\text{if } r'\in\partial V\ . \tag{9.9}$$

Tending $r'$ to $\partial V$ in (9.8) we arrive at the values of $h_{ij}$

$$h_{ij}(r)=-\int_Vw_j(r',r)e^{-z\sigma_iu_i(r',r)}\times d^2r'\Big/\int_Ve^{-z\sigma_iu_i(r',r)}d^2r'\ . \tag{9.10}$$

The differential equations for $w_i$ can be obtained by applying Laplace's operator to (9.8). We have

$$\Delta_{r'}w_i(r',r)=-\frac{e^{-z\sigma_iu_i(r',r)}}{\int_Ve^{-z\sigma_iu_i(r'',r)}d^2r''}\ . \tag{9.11}$$

Equations (9.11) and (9.9), along with the expressions for $u_i$ (9.6), form a closed system of equations for $w_i$.

It follows from (9.11) and (9.9) that $h_{ij}=h_{ji}$. Then, as is easy to check, $2(N-1)^2$ equations (9.8) are the consequences of $2(N-1)$ equations (9.11) and (9.6) and expressions for $h_{ij}$ (9.10).

Using (9.10) and (9.11), the sum $\sum\sigma_i\sigma_jh_{ij}$ in the expression for $A(z,r,N)$ can be presented in the form

$$\sum_{i\neq j}\sigma_i\sigma_jh_{ij}=-\sum_{i\neq j}\sigma_i\sigma_j\int_V\nabla w_i\nabla w_jd^2r'\ .$$

Finally,

$$A(z,r,N)=|V|^N\exp[NB(z,w_i,r)]C(z,r,N)\ ,$$
$$B(z,w_i,r)=\frac{z}{2N^2}\sum_{i\neq j}\sigma_i\sigma_j\int_V\nabla w_i\nabla w_jd^2r+\frac{1}{N}\sum_i\ln\frac{1}{|V|}\int_Ve^{-z\sigma_iu_i(r,r')}d^2r'-\frac{1}{N^2}\sigma_1^2g(r)\ , \tag{9.12}$$

$$C(z,r,N)=M\exp\left[-\frac{z}{2N}\sum_{i\neq j}\sigma_i\sigma_jG_{ij}\right]\ , \tag{9.13}$$

where $M$ denotes a mathematical expectation with respect to the complex measure with the "probability density" const $\exp[-z\sum_2^N\sigma_iu_i(r_i,r)]$.

It is assumed that $u_i$ in (9.12) is expressed in terms of $w_i$ in accordance with (9.6). Then $B$ becomes a functional of $w_i$. The solutions of the system of equations (9.11), (9.6), and (9.9) are the stationary points of functional $B$

(Appendix G).

In the case of a multicomponent vortex gas, we search for solutions for which functions $w_i$ are equal for the vortices of the same component. In this case functional $B$ takes the form

$$B(z,w_\alpha,r)=\frac{z}{2}\int_V\left[\nabla\sum_\alpha c_\alpha\sigma_\alpha w_\alpha\right]^2 d^2r' +\sum_\alpha c_\alpha \ln\frac{1}{|V|}\int_V e^{-z\sigma_\alpha u_\alpha(r,r')}d^2r' -\frac{1}{N^2}\sigma_1^2 g(r)-\frac{1}{N}\sum_\alpha c_\alpha\sigma_\alpha^2\int_V(\nabla w_\alpha)^2 d^2r\ , \tag{9.14}$$

where

$$u_\alpha(r',r)=\sum_\alpha c_\alpha\sigma_\alpha w_\alpha(r',r)+\frac{\sigma_1}{N}G(r,r') +\frac{\sigma_\alpha}{N}(g(r')-w_\alpha(r',r))\ . \tag{9.15}$$

The functional $B$ depends on the parameter $r$ and on the small parameter $N^{-1}$. The dependence on $r$ enters by means of the functions $G(r',r)$ in (9.15) and $g(r)$ in (9.14). They have small factors. In the first approximation, the dependence of $B$ on $r$ disappears. If all the small terms [the last two terms in (9.14) and the last three in (9.15)] are dropped, the functional $B$ becomes the functional (7.22) because, in accordance with (9.15), all $u_\alpha$ are equal to $u=\sum_\alpha c_\alpha\sigma_\alpha w_\alpha$. To find the dependence of $B$ on $r$ one must consider the correction of the first order. To do that, one can use the variational-asymptotic method [31,23]. In accordance with this method, we present the stationary point $w_\alpha$ in the form

$$w_\alpha=\mathring{w}_\alpha+\tilde{w}_\alpha\ , \tag{9.16}$$

where $\mathring{w}_\alpha$ is the first approximation to $w_\alpha$ and $\tilde{w}_\alpha$ are small compared to $\mathring{w}_\alpha$. After substituting (9.16) in (9.14) and (9.15) all linear terms containing $\tilde{w}_\alpha$ cancel out due to Euler's equations for $\mathring{w}_\alpha$. The terms of the next order form a quadratic functional. To determine $\tilde{w}_\alpha$ we must find the stationary points of the quadratic functional with respect to $\tilde{w}_\alpha$. The corresponding linear problem contains "excitations" of order $N^{-1}$. Therefore, $\tilde{w}_\alpha\sim N^{-1}$ and the quadratic functional with respect to $\tilde{w}_\alpha$ has the order $N^{-2}$. Since we are interested in the leading corrections and there are terms in the functional of order $N^{-1}$, quadratic corrections, related to $\tilde{w}_\alpha$, can be dropped. Therefore, to obtain the first-order corrections, we must calculate the functional $B$ on the functions $\mathring{w}_\alpha$ and keep the terms of order $N^{-1}$. We have

$$B=B_0-\beta\sum_\alpha c_\alpha\sigma_\alpha\frac{\sigma_1}{N}\frac{\int_V e^{-\beta\sigma_\alpha\mathring{u}(r')}G(r',r)d^2r'}{\int_V e^{-\beta\sigma_\alpha\mathring{u}(r')}d^2r'}\ , \tag{9.17}$$

where $B_0$ is the stationary value of the functional (7.22) and $u_0$ is the solution of the boundary-value problem (7.19). There are additional terms of order $N^{-1}$ in (9.17), but they are dropped because they do not depend on parameter $r$. So, in the first approximation, the parameter $r$ enters in $B$ by means of the Green's function $G(r',r)$. In accordance with the steepest descent method,

$$\frac{d\Gamma(r,E)}{dE}=\mathcal{F}(E,N)\exp\left[-\beta\sigma_1\sum_\alpha c_\alpha\sigma_\alpha\frac{\int_V e^{-\beta\sigma_\alpha\mathring{u}(r')}G(r',r)d^2r'}{\int_V e^{-\beta\sigma_\alpha\mathring{u}(r')}d^2r'}\right]\ , \tag{9.18}$$

where $\mathcal{F}$ is a function of $E$ and $N$. Equation (9.1) follows from (9.18), (9.17), (7.19), and (9.2). Equation (9.1) has been found by Pointin and Lundgren [11,12] from the assumption (1.2) for a two-component vortex gas. The considered derivation does not contain any other assumption except the ergodicity of motion. The hypothesis (1.2) can be proved in the same way as (9.1): one must find, in the first approximation, $d\Gamma(r_1,r_2,E)/dE$ ($\Gamma(r_1,r_2,E)$ is the phase volume of the region in $[2(N-2)]$-dimensional space $(r_3,\ldots,r_n)$, which is bounded by the energy surface $H(r_1,r_2,r_3,\ldots,r_n)=E$) and show that $d\Gamma(r_1,r_2,E)/dE$ is proportional to the product $d\Gamma(r_1,E)/dE\ d\Gamma(r_2,E)/dE$. This can be done by following the reasoning of this section.

## X. MEAN STREAM FUNCTION

The ergodicity of vortex motion and the way of decaying the vortex intensities ($\gamma_i\sim N^{-1}$) yield a very special behavior of hydrodynamic characteristics. Consider first the vorticity field $\omega(t,r)$. In the point vortex approximation

$$\omega(t,r)=\sum_i\gamma_i\delta(r-r_i(t))\ .$$

Discussing the limit behavior of $\omega(t,r)$ for $N\to\infty$, it is sensible to consider the weak convergence, i.e., the convergence of integrals

$$\int_V\omega(t,r)\rho(r)d^2r=\sum_i\gamma_i\rho(r_i(t))\ , \tag{10.1}$$

where $\rho(r)$ is a smooth function. In accordance with the central limit theorem, integrals (10.1) converge to the mathematical expectation of $\sum_i\gamma_i\rho(r_i)$, i.e., to

$$\lim_{N\to\infty}\frac{1}{N}\sum_i \sigma_i \frac{\int_V \rho(r)e^{-\beta\sigma_i u(r)}d^2r}{\int_V e^{-\beta\sigma_i u(r)}d^2r} = \sum c_\alpha\sigma_\alpha \frac{\int_V \rho(r)e^{-\beta\sigma_\alpha u(r)}d^2r}{\int_V e^{-\beta\sigma_\alpha u(r)}d^2r} .$$

This means that vorticity converges weakly to the function that does not depend on time

$$\omega(t,r)\to\overline{\omega}(r)=\sum_\alpha c_\alpha\sigma_\alpha \frac{e^{-\beta\sigma_\alpha u(r)}d^2r}{\int_V e^{-\beta\sigma_\alpha u(r')}d^2r'} . \quad (10.2)$$

At each point $r$, the true vorticity field $\omega(t,r)$ fluctuates around the average value (10.2). Fluctuations are of order $1/\sqrt{N}$. The temporal fluctuations disappear if $N\to\infty$. One might say that temporal chaos is transformed into spatial chaos. This is a characteristic feature of all theories of mean field type (see the discussion in [32]). The feasibility of a mean-field-type theory for hydrodynamical problems is discussed in [32,33].

Consider the stream function of fluid motion. It can be found from the equation

$$\Delta\psi=-\sum_i \gamma_i\delta(r-r_i(t)) . \quad (10.3)$$

The solution of Eq. (10.3) converges weakly to the solution $\overline{\psi}(r)$ of the "averaged" equation

$$\Delta\overline{\psi}=-\overline{\omega}=-\sum_i \gamma_i \frac{e^{-\beta\sigma_i u(r)}}{\int_V e^{-\beta\sigma_i u(r')}d^2r'} \quad \text{in } V , \qquad \overline{\psi}=0 \quad \text{at } \partial V \quad (10.4)$$

or, in the case of a multicomponent vortex gas,

$$\Delta\overline{\psi}=-\sum_\alpha c_\alpha\sigma_\alpha \frac{e^{-\beta\sigma_\alpha u(r)}}{\int_V e^{-\beta\sigma_\alpha u(r')}d^2r'} \quad \text{in } V , \qquad \overline{\psi}=0 \quad \text{at } \partial V . \quad (10.5)$$

Comparing (10.5) and (7.19), we conclude that $u(r)=\overline{\psi}(r)$. Finally, the averaged stream function is the solution of the following boundary-value problem:

$$\Delta\overline{\psi}=-\sum_\alpha c_\alpha\sigma_\alpha \frac{e^{-\beta\sigma_\alpha \overline{\psi}(r)}}{\int_V e^{-\beta\sigma_\alpha \overline{\psi}(r')}d^2r'} \quad \text{in } V , \qquad \overline{\psi}=0 \quad \text{at } \partial V . \quad (10.6)$$

For the particular case of a two-component vortex gas, this equation was derived in a different form by Joyce and Montgomery [9,10].

## XI. AVERAGED EQUATIONS OF 2D HYDRODYNAMICS

One of the central tasks of the theory of turbulence is to find the equations for averaged characteristics of fluid flow. In this section, the "exact" averaged equation is derived from the averaged equation (10.6) for a multicomponent vortex gas. The term "exact" is put in quotation marks because the assumption on ergodicity might not be true for fluid motion.

Let us take the initial continuous vorticity distribution $\mathring{\omega}(r)$. We model this vorticity distribution by some multicomponent vortex gas. To "prepare" this gas, we divide region $V$ into a large number of small blobs of equal area $\Delta$ and identify a point vortex with each blob. The intensity of the vortex with the initial position $r_i$ is $\gamma_i=\mathring{\omega}(r_i)\Delta$. If the initial vorticity field is approximated by some piecewise function with $s$ values $\omega_\alpha$ $(\alpha=1,\ldots,s)$, we obtain the $s$-component vortex gas with intensities of $\gamma_\alpha=\omega_\alpha\Delta$. If the subregion where $\mathring{\omega}(r)$ takes the value $\omega_\alpha$ has the area $|V_\alpha|$, then the number of vortices in the $\alpha$th component is $N_\alpha=|V_\alpha|/\Delta$. The assumption that all blobs have the same area $\Delta$ means in fact that the number of vortices in each component is proportional to the corresponding area $|V_\alpha|$. For such a gas, $\sigma_\alpha=\gamma_\alpha N=\omega_\alpha\Delta N=\omega_\alpha|V|$, $\Delta N=|V|$, $c_\alpha=N_\alpha/N=|V_\alpha|/|V|$, and $c_\alpha\sigma_\alpha=\omega_\alpha|V_\alpha|$. With the number of each component to infinity, we obtain, from (10.6), the equation

$$\Delta\overline{\psi}=-\int_V \omega_0(r_0)\frac{e^{-\beta\omega_0(r_0)|V|\overline{\psi}(r)}}{\int_V e^{-\beta\omega_0(r_0)|V|\overline{\psi}(r')}d^2r'}d^2r_0 \quad \text{in } V , \qquad \overline{\psi}=0 \quad \text{at } \partial V . \quad (11.1)$$

The right-hand side of (11.1) can be written in terms of the distribution function of initial vorticity $g(\omega)$

$$g(\omega)\Delta\omega=\frac{1}{|V|}\text{ measure }\{x:\ \omega\le\mathring{\omega}(r)\le\omega+\Delta\omega\} . \quad (11.2)$$

We have

$$\Delta\overline{\psi}=-|V|\int_{-\infty}^{+\infty} g(\omega)\omega\frac{e^{-\beta|V|\omega\overline{\psi}(r)}}{\int_V e^{-\beta|V|\omega\overline{\psi}(r')}d^2r'}d\omega \quad \text{in } V , \qquad \overline{\psi}=0 \quad \text{at } \partial V . \quad (11.3)$$

For each $\omega$, $g(\omega)$ is the integral of motion: if one substitutes in (11.2) the function $\omega(r,t)$ instead of $\mathring{\omega}(r)$, one obtains the same function $g(\omega)$ for all $t$. We see that an infinite number of integrals of fluid motion contributes to the averaged equation (11.3). If the number of values of the initial vorticity field $\mathring{\omega}(r)$ is finite, we return to the original equation (10.6). In [32–34] a system of equations was obtained for $\overline{\psi}$ that differs from (11.3). The differences will be discussed elsewhere.

A confusing feature of the averaged equation is its dependence on the way of "preparation" of the vortex gas. This dependence can be seen from the following thought experiment. Let region $V$ contain a spot of constant vorticity $\omega$. Consider two different approximations of the spot by point vortices. In the first approximation we divide the spot into $N$ pieces of the same area and obtain point vortices of equal intensities. In the second one we divide the spot into two parts $V_1$ and $V_2$ and take $N_1$ equal vortices in $V_1$ and $N_2$ equal vortices in $V_2$. The intensities of the vortices in this case take two values $\gamma_1=\omega V_1/N_1$ and $\gamma_2=\omega V_2/N_2$. Then we consider the limit $N=N_1+N_2\rightarrow\infty$; $N_1/N$ and $N_2/N$ stay finite. It is seen from (10.6) that the averaged equations for these two approximations of the vorticity field are different. That leaves us with two options: (a) only one approximation captures correctly the long-term dynamics or (b) fluid motion is not ergodic and the above consideration is not relevant. Intuition suggests that equal vortices should better approximate the dynamics of the spot of constant vorticity. In fact, this has been reflected in the approximation of the continuous vorticity field accepted above by dividing $V$ into equal parts. Another consequence of dividing $V$ into equal pieces is that, in each region $\omega\approx$const, the number of vortices is proportional to the area of this region. These assumptions extract the unique averaged equation. The convergence of the point vortex approximation on infinite time has been proven in [3] under the assumption that the initial vorticity field is divided into equal pieces. The dependence of the accuracy of the approximation of the vorticity field by point vortices on the method of preparation of the vortex gas has not yet, to my knowledge, been considered. If it develops that convergence of the point vortex approximation on infinite time does not depend on the method of preparation of the vortex gas, it would mean that fluid motion is not ergodic and the above consideration is not relevant.

## XII. CONCLUDING REMARKS

In this paper, the basic relations of statistical mechanics of point vortices were derived. Thermodynamical functions and probability density functions are found and analyzed. An equation for the averaged stream function of a flow with a continuous distribution of initial vorticity is obtained. Theory predicts decay of temporal fluctuations in the limit $N\rightarrow\infty$. This feature is common for all theories of mean field type. In real flows, temporal fluctuations are presented. This contradiction of theory and experiment requires an explanation. There are a number of reasons for temporal fluctuations to persist. First, real flows are not closed and disturbances at the inlet could be a trigger for temporal fluctuations. Second, decay of temporal fluctuations might be just a two-dimensional effect. Third, a certain role might be played by viscosity. Viscosity, along with external excitation, yields the finiteness of the attractor's dimension and possibly finiteness of the number of degrees of freedom in the approximation of fluid motion by motion of point vortices. For finite $N$, temporal fluctuations exist; they are of the order of $N^{-1}$. Temporal fluctuations are described by the next term in the asymptotical expression for the probability distribution (9.1). This term is responsible for Reynolds stresses in vortex gas and can be found by the steepest descent method. At present, there is no clear understanding as to which of these reasons dominate.

## APPENDIX A: COMPLEX PROBABILITIES

The term complex probabilities is used in physics in various senses. In this paper, the term complex probability is defined in a formal way, putting aside a discussion of possible physical meanings.

Many statements of the theory of probability have a purely analytical nature and might be considered as some statements about integrals. Consider, for example, the central limit theorem for independent random variables $r_i$, $i=1,\ldots,N$. Each variable takes values in some region $V$ and has the probability density $f(r)$. Let $\rho(r)$ be a real function of $r$ with zero mean value

$$\int_V \rho(r)f(r)dr=0 \ .$$

Then the central limit theorem states that

$$\text{Prob}\left\{\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\rho(r_i)\le E\right\}\underset{N\rightarrow\infty}{\sim}\Phi\left\{\frac{E}{\sigma}\right\}, \tag{A1}$$

where $\sigma$ is the variance of $\rho$

$$\sigma^2=\int_V \rho^2(r)f(r)dr \ .$$

The central limit theorem (A1) can be written as a statement on the limit behavior of the multidimensional integral

$$\int_{(1/\sqrt{N})\sum_i\rho(r_i)\le E} f(r_1)\cdots f(r_N)dr_1\cdots dr_N \underset{N\rightarrow\infty}{\sim}\Phi\left\{\frac{E}{\sigma}\right\},\quad r_i\in V \ . \tag{A2}$$

The statement (A2) admits the following generalization: the probability density $f(r)$ can be a complex valued function, which satisfies the normalization condition

$$\int_V f(r)dr=1 \ . \tag{A3}$$

Note that the variance $\sigma$ in (A2) becomes, in general, a complex number. The relation (A2) can be further generalized in order to admit complex values of $E$ and $\rho(r)$. To do that, one can rewrite (A2) in the form

$$\int_{V^N}\theta\left[E-\frac{1}{\sqrt{N}}\sum_i\rho(r_i)\right]\times f(r_1)\cdots f(r_N)dr_1\cdots dr_N\underset{N\rightarrow\infty}{\sim}\Phi\left\{\frac{E}{\sigma}\right\}. \tag{A4}$$

In the standard formulation of the central limit theorem, $\theta(x)$ is the step function. However, if we understand by $\theta(E)$ the continuation of the step function $\theta(x)$ in the complex plane [$\theta(z)=1$ if $\text{Re}\,z>0$ and $\theta(z)=0$ if $\text{Re}\,z<0$], then (A4) is valid for complex valued $E$ and $\rho(r)$. The proof of (A2)–(A4) does not differ from the standard one [29].

In the calculation of phase volume, we need a complex-valued version of the following statement:

$$M\exp\left[-z\frac{1}{N}\sum_{i\neq z}G_{ij}(r_i,r_j)\right]\underset{N\rightarrow\infty}{\sim}(\text{function of } z) \tag{A5}$$

if the mathematical expectation of $G_{ij}(r_i,r_j)$ with respect to each argument is equal to zero:

$$M_{r_i}G_{ij}(r_i,r_j)=0,\quad M_{r_j}G_{ij}(r_i,r_j)=0 \ . \tag{A6}$$

This statement has been proven [18] for $G_{ij}(r_i,r_j)=g(r_i,r_j)$ under the following assumptions:

$$g(r,r')=g(r',r),\quad \sup_{r'}\left|\int_V g^2(r,r')dr\right|<+\infty\ ,$$

all mathematical expectations $Mg(r_{i_1},r_{j_1})\cdots g(r_{i_k},r_{j_k})$ exist. The similar statement (A5) can be proven for complex probability densities by following the reasoning of Ref. [18].

## APPENDIX B: CONVEXITY OF THE FUNCTIONAL $B$

Let us show that the functional

$$B(z,v)=Ez+\frac{1}{2z}\int_V(\nabla v)^2d^2r + \sum_\alpha c_\alpha \ln\frac{1}{|V|}\int_V e^{-\sigma_\alpha v}d^2r \qquad \text{(B1)}$$

is strictly convex with respect to two variables $z,v$ for $z>0$ and $c_\alpha>0$. The last term in $B(z,v)$ is a strictly convex functional because the functional

$$b(v)=\int_V e^{-\sigma_\alpha v}d^2r$$

is strictly convex, while the functions $c_\alpha \ln x$ are monotonically increasing . So, to prove the statement it is enough to show the convexity of the functional

$$\frac{1}{2z}\int_V(\nabla v)^2d^2r\ .$$

To this end we have to prove the inequality

$$\frac{1}{4(z_1+z_2)}\int_V(\nabla v_1+\nabla v_2)^2d^2r \le \frac{1}{4}\left[\frac{1}{z_1}\int_V(\nabla v_1)^2d^2r+\frac{1}{z_2}\int_V(\nabla v_2)^2d^2r\right]\ , \qquad \text{(B2)}$$

which should be valid for any positive $z_1,z_2$ and any $v_1,v_2$. This inequality can be written as

$$\int_V(\nabla v_1+\nabla v_2)^2d^2r \le \int_V[(\nabla v_1)^2+(\nabla v_2)^2]d^2r + t\int_V(\nabla v_1)^2d^2r+\frac{1}{t}\int_V(\nabla v_2)^2d^2r\ , \qquad \text{(B3)}$$

where $t=z_2/z_1$. Since

$$\min_t\left[t\int_V(\nabla v_1)^2d^2r+\frac{1}{t}\int_V(\nabla v_2)^2d^2r\right] =2\left[\int_V(\nabla v_1)^2d^2r\int_V(\nabla v_2)^2d^2r\right]^{1/2}\ , \qquad \text{(B4)}$$

inequality (B2) follows from (B3), (B4), and the Cauchy-Buniakovsky inequality

$$\int_V\nabla v_1\nabla v_2d^2r\le\left[\int_V(\nabla v_1)^2d^2r\int_V(\nabla v_2)^2d^2r\right]^{1/2}\ . \qquad \text{(B5)}$$

The equality in (B5) takes place only when $\nabla v_2=c^2\nabla v_1$ and $c=$const. The minimum in (B4) is reached for

$$t=z_2/z_1=\left[\int_V(\nabla v_2)^2d^2r\Big/\int_V(\nabla v_1)^2d^2r\right]^{1/2}\ .$$

The two sides of (B1) are equal if and only if $\nabla v_2=c^2\nabla v_1$ and $z_1=c^2z_1$. So the first two terms of (B1) are convex, but not strictly convex. Since the last term in (B1) is strictly convex, $B(z,v)$ is strictly convex.

## APPENDIX C: AN INEQUALITY FOR ENTROPY

Let us show that the entropy of vortex motion is always negative. In accordance with (8.3), the entropy is the stationary value of the functional $Ez+B$. Denote by $J(v)$ the functional

$$J(v)=\sum_\alpha c_\alpha\ln\frac{1}{|V|}\int_V e^{-\sigma_\alpha v}d^2r\ . \qquad \text{(C1)}$$

Note that

$$J(0)=0\ . \qquad \text{(C2)}$$

The stationary point of the functional $Ez+B$ satisfies the equations

$$E-\frac{1}{2z^2}\int_V(\nabla v)^2d^2r=0,\quad \frac{1}{z}\Delta v=\frac{\delta J}{\delta v}\ ,$$

$$\frac{\delta J}{\delta v}\equiv-\sum c_\alpha\sigma_\alpha\frac{e^{-\sigma_\alpha v}}{\int_V e^{-\sigma_\alpha v}d^2r}\ .$$

Therefore, at the minimizing point

$$\frac{1}{z}\int_V(\nabla v)^2d^2r=-\int_V v\frac{\delta J}{\delta v}d^2r\ ,$$

$$Ez=\frac{1}{2z}\int_V(\nabla v)^2d^2r$$

and the entropy has the value

$$N^{-1}S=Ez+B=J(v)-\int_V v\frac{\delta J}{\delta v}d^2r\ . \qquad \text{(C3)}$$

Consider the functional of $v(r)$

$$\int_V v\frac{\delta J}{\delta v}d^2r-[J(v)-J(0)]\ . \qquad \text{(C4)}$$

Let us show that it is positive. For any convex function of a finite number of variables $v_1,\dots,v_2$, the following inequality is valid [35]:

$$\sum_i v_i\frac{\delta J}{\delta v_i}\ge J(v)-J(0)\ . \qquad \text{(C5)}$$

If we discretize the functional (C1), we get some convex function of a finite number of variables that obeys the inequality (C5). Tending the size of discretization to zero, we obtain, on the left-hand side of (C5),

$$\int_V v\frac{\delta J}{\delta v}d^2r\ .$$

Therefore, the continuum version of (C5) means the positiveness of the functional (C4) and the negativeness of the entropy. Note that the proof does not use the positiveness of $z$: entropy is negative for negative $z$ as well.

## APPENDIX D: FUNCTIONAL $B$ IN THE COMPLEX PLANE

Let $v_0(z)$ be a stationary point of the functional $B(z,v)$ with respect to $v$ for a complex $z$. Consider $\mathrm{Re}[Ez+B(z,v_0(z))]$ on the line $z=\beta+it$ ($t$ is the real parameter along the line; $\beta>0$). We will show that this function reaches its maximum value at $t=0$.

Let $J(v)$ be some complex valued functional of a complex valued function $v$, $v=v_1+iv_2$, $J=J_1(v_1,v_2)+iJ_2(v_1,v_2)$; $v_1,v_2,J_1,J_2$ are real. The following Cauchy-Riemann relations take place:

$$\frac{\delta J_1}{\delta v_1}=\frac{\delta J_2}{\delta v_2},\quad \frac{\delta J_2}{\delta v_1}=-\frac{\delta J_1}{\delta v_2}, \tag{D1}$$

where $\delta J_\alpha/\delta v_\beta$ are the variational derivates. These relations follow from the identity

$$\frac{\delta J}{\delta v}\delta v=\frac{\delta J}{\delta v_1}\delta v_1+\frac{\delta J}{\delta v_2}\delta v_2 .$$

At a stationary point

$$\frac{\delta J}{\delta v_1}=\frac{\delta J_1}{\delta v_1}+i\frac{\delta J_2}{\delta v_1}=0,\quad \frac{\delta J}{\delta v_2}=\frac{\delta J_1}{\delta v_2}+i\frac{\delta J}{\delta v_2}=0 . \tag{D2}$$

Due to identities (D1) among four real equations (D2), only two are independent. For definiteness, we can take the equations

$$\frac{\delta J_1(v_1,v_2)}{\delta v_1}=0,\quad \frac{\delta J_1(v_1,v_2)}{\delta v_2}=0 . \tag{D3}$$

Denote by $v_1$ and $v_2$ the real and imaginary parts of $v_0(z)$ and by $B_1(t,v_1,v_2)$ the real part of functional $B$. We have

$$B_1(t,v_1,v_2)=\frac{1}{2(\beta^2+t^2)}\int\{\beta[(\nabla v_1)^2-(\nabla v_2)^2]+2t\nabla v_1\nabla v_2\}d^2r + \sum_\alpha c_\alpha \ln\left[\left[\int_V e^{-\sigma_\alpha v_1}\cos(\sigma_\alpha v_2)d^2r\right]^2+\left[\int_V e^{-\sigma_\alpha v_1}\sin(\sigma_\alpha v_2)d^2r\right]^2\right]^{1/2} . \tag{D4}$$

We have to show that the stationary values of $B_1$ for $t\neq 0$ are less than the stationary value for $t=0$. The problem of the determination of $v_1,v_2$ can be set up as the saddle point problem

$$\inf_{v_1}\sup_{v_2} B_1(t,v_1,v_2) . \tag{D5}$$

Denote the function of $t$ (D5) by $B^*(t)$. We have to show that

$$B^*(t)<B(\beta,v(\beta)) .$$

Let us majorize $B_1$, trying to eliminate $v_2$. Since

$$\left[\int_V e^{-\sigma_\alpha v_1}\cos(\sigma_\alpha v_2)d^2r\right]^2 \le \int_V e^{-\sigma_\alpha v_1}d^2r\int_V e^{-\sigma_\alpha v_1}\cos^2(\sigma_\alpha v_2)d^2r$$

and

$$\left[\int_V e^{-\sigma_\alpha v_1}\sin(\sigma_\alpha v_2)d^2r\right]^2 \le \int_V e^{-\sigma_\alpha v_2}d^2r\int_V e^{-\sigma_\alpha v_1}\sin^2(\sigma_\alpha v_2)d^2r ,$$

the last term in (D4) does not exceed

$$\sum_\alpha c_\alpha \ln\int_V e^{-\sigma_\alpha v_1}d^2r .$$

The first term can be majorized by means of the inequality

$$2t\int_V \nabla v_1\nabla v_2 d^2r\le |t|\int_V\left[(\nabla v_1)^2\frac{|t|}{\beta}+\frac{\beta}{|t|}(\nabla v_2)^2\right]d^2r .$$

We have

$$\frac{1}{2(\beta^2+t^2)}\int_V\{\beta[(\nabla v_1)^2-(\nabla v_2)^2]+2t\nabla v_1\nabla v_2\}d^2r \le \frac{1}{2(\beta^2+t^2)}\int_V\left[\beta+\frac{|t|^2}{\beta}\right](\nabla v_1)^2d^2r =\frac{1}{2\beta}\int_V(\nabla v_1)^2d^2r .$$

Hence

$$B^*(t)<\min_{v_1}\max_{v_2}B(\beta,v_1)=\min_{v_1}B(B,v_1)=B(\beta,v(\beta)) ,$$

as claimed.

## APPENDIX E: DERIVATION OF THE LOW ENERGY LIMIT FROM THE GENERAL RELATIONS

Let us show that the formulas for the low energy limit derived in Secs. IV and V follow also from the general expression (7.29). We confine ourselves to the case of a non-neutral gas ($\sum\gamma_i=\sum c_\alpha\sigma_\alpha\neq 0$).

We must solve Eq. (7.19) in the limit of small $z$. In the first approximation $u\approx u_0$, $u_0$ does not depend on $z$ and obeys the boundary-value problem

$$\Delta u_0=-\sum_\alpha c_\alpha\sigma_\alpha\frac{1}{|V|}\ \text{ in } V,\quad u_0=0\ \text{ at } \partial V . \tag{E1}$$

This problem has the solution

$$u_0(r)=\frac{1}{|V|}\sum_\alpha c_\alpha\sigma_\alpha\int_V G(r,r')d^2r' . \tag{E2}$$

Multiplying (E1) by $u_0(r)$ and integrating over $V$, we get the relation

$$\int_V (\nabla u_0)^2 d^2r = \frac{1}{|V|} \sum_\alpha c_\alpha \sigma_\alpha \int_V u_0 d^2r \ . \tag{E3}$$

Using (E2), (4.9), and (5.4), we find that

$$E_0 = \int_V (\nabla u_0)^2 d^2r = \tfrac{1}{2} h \left[ \sum_\alpha c_\alpha \sigma_\alpha \right]^2 \ . \tag{E4}$$

Let us calculate entropy $S$ by taking into account the small terms of order $z$ and $z^2$. Presenting $u(r)$ in the form

$$u = u_0 + u' \ , \tag{E5}$$

we see from (7.19) that $u'$ is of order $z$. Substituting (E5) into the functional $N^{-1}S$ and dropping all terms of orders $z^3$ and higher, we have

$$\begin{aligned} N^{-1}S &= Ez + \frac{z}{2} \int_V (\nabla u)^2 d^2r + \sum_\alpha c_\alpha \ln \frac{1}{|V|} \int_V e^{-z\sigma_\alpha u} d^2r \\ &= Ez + \tfrac{1}{2} z \left[ \int_V (\nabla u_0)^2 d^2r + 2 \int_V \nabla u_0 \nabla u' d^2r \right] \\ &\quad + \sum_\alpha c_\alpha [ -z\sigma_\alpha \langle u_0 \rangle + \tfrac{1}{2} (z\sigma_\alpha)^2 (\langle u_0^2 \rangle - \langle u_0 \rangle^2) - z\sigma_\alpha \langle u' \rangle ] \ . \end{aligned} \tag{E6}$$

Here $\langle \cdots \rangle$ means average value over region $V$: $\langle \cdots \rangle = \int_V \cdots d^2r / |V|$. Two terms containing $u'$ cancel out due to (E1). The sum of linear terms in $z$ which contain $u_0$ is equal to $-E_0 z$, as follows from (E3) and (E4). So

$$\begin{aligned} N^{-1}S &= (E - E_0) z + \tfrac{1}{2} z^2 \sigma^2 (\langle u_0^2 \rangle - \langle u_0 \rangle^2) \ . \\ \sigma^2 &= \sum_\alpha c_\alpha \sigma_\alpha^2 \ . \end{aligned} \tag{E7}$$

The coefficient $\langle u_0^2 \rangle - \langle u_0 \rangle^2$ can be expressed in terms of the constant $b$ [$b = \langle w^2 \rangle^{1/2}$; see (5.15)]. To do so we note that, in accordance with (5.10), (5.11), and (E1),

$$u_0 = [w(r) + h] \sum_\alpha c_\alpha \sigma_\alpha \ .$$

Since $\langle w(r) \rangle = 0$,

$$\langle u_0 \rangle = h \sum_\alpha c_\alpha \sigma_\alpha \ ,$$

$$\langle u_0^2 \rangle = \left[ \sum_\alpha c_\alpha \sigma_\alpha \right]^2 [\langle w^2 \rangle + h^2] \ .$$

Hence

$$\langle u_0^2 \rangle - \langle u_0 \rangle^2 = \langle w^2 \rangle \left[ \sum c_\alpha \sigma_\alpha \right]^2 = b \left[ \sum_\alpha c_\alpha \sigma_\alpha \right]^2 \ .$$

So

$$N^{-1}S = (E - E_0) z + \tfrac{1}{2} z^2 \sigma^2 b \left[ \sum_\alpha c_\alpha \sigma_\alpha \right]^2 \ . \tag{E8}$$

Minimization of (E8) with respect to $z$ gives

$$N^{-1}S(E) = -\tfrac{1}{2} \sigma^2 b^{-2} \left[ \sum_\alpha c_\alpha \sigma_\alpha \right]^{-2} (E - E_0)^2 \ .$$

Plugging this relation into (7.29), taking into account that $C(0) = 1$, and, as follows from (E8), $B'' = \sigma^2 b^2 (\sum_\alpha c_\alpha \sigma_\alpha)^2$, we obtain

$$\frac{d\Gamma(E)}{dE} = \frac{\sqrt{N}}{\sqrt{2\pi} \sigma b \left[ \sum_\alpha c_\alpha \sigma_\alpha \right]} \times \exp \left[ -\frac{N}{2} \frac{(E - E_0)^2}{\left[ \sum_\alpha c_\alpha \sigma_\alpha \right]^2 \sigma^2 b^2} \right] \ .$$

The same relation follows from the differentiation of (6.5).

## APPENDIX F: A ONE-COMPONENT VORTEX GAS IN A CIRCULAR DOMAIN

For motion of a one-component vortex gas in a circular domain, thermodynamic functions can be found in terms of elementary functions. Of course, there is the additional integral of motion for a circular domain and the previous consideration should be modified to be physically relevant. However, all the relations make sense formally and allow one to get a qualitative impression of the behavior of thermodynamic functions. It seems plausible that small disturbances of a circular domain, destroying the additional integral, yield small disturbances of thermodynamic functions found for the circular domain.

For a one-component vortex gas the equation for the stationary point of the function $B(z,v)$ (7.26) takes the form

$$\begin{aligned} \Delta v_0 &= -z\sigma \frac{e^{-\sigma v_0(r)}}{\int_V e^{-\sigma v_0(r')} d^2r} \quad \text{in } V \ , \\ v_0 &= 0 \quad \text{at } \partial V \ . \end{aligned} \tag{F1}$$

It can be checked by inspection that Eq. (E1) has the solution

$$v = \frac{2}{\sigma} \ln \left[ 1 + \frac{\sigma^2 z}{8\pi} \left[ 1 - \frac{r^2}{R^2} \right] \right] \ , \tag{F2}$$

where $r^2 = x^2 + y^2$ and $R$ is the radius of the circle. Note that this is the solution of (F1) for complex $z$.

Using (F2) we find $B(z)$,

$$B(z)=1-\left[1+\frac{8\pi}{z\sigma^2}\right]\ln\left[1+\frac{z\sigma^2}{8\pi}\right] .$$

Then the energy $E$ is linked to the stationary value $\beta$ by the relation

$$E=\frac{1}{\beta}\left[1-\frac{8\pi}{\beta\sigma^2}\ln\left[1+\frac{\beta\sigma^2}{8\pi}\right]\right] .$$

The entropy $S$ as a function of $\beta$ is

$$S=2-\left[1+\frac{16\pi}{\beta\sigma^2}\right]\ln\left[1+\frac{\beta\sigma^2}{8\pi}\right] .$$

Let us introduce dimensionless parameters $\beta^*=\beta\sigma^2/8\pi$ and $E^*=8\pi E/\sigma^2$. Then

$$E^*=\frac{1}{\beta^*}\left[1-\frac{1}{\beta^*}\ln(1+\beta^*)\right] , \tag{F3}$$

$$S=2-\left[1+\frac{2}{\beta^*}\right]\ln(1+\beta^*) . \tag{F4}$$

In the vicinity of critical energy $\beta^*\to 0$. Expanding $\ln(1+\beta^*)$ in a Taylor series we find

$$E^*=\tfrac{1}{2}-\tfrac{1}{3}\beta^* .$$

Hence the dimensionless critical value of energy is $\frac{1}{2}$. Let us compare it with (5.14). We need to find the constant $h$ (5.9). The function

$$\varphi(r)=\int_V G(r,r')d^2r'$$

is the solution of the boundary-value problem

$$\Delta\varphi=-1 \quad \text{in } V, \quad \varphi=0 \quad \text{at } \partial V . \tag{F5}$$

For the circle of radius $R$ we have, from (F5),

$$\varphi=\tfrac{1}{4}(R^2-x^2-y^2) .$$

Therefore

$$h=\frac{1}{|V|^2}\int_V\int_V G(r,r')d^2r\,d^2r'=\frac{1}{|V|^2}\int_V \varphi\,d^2r=\frac{1}{8\pi} . \tag{F6}$$

It follows from (5.14) and (F6) that the critical value found for the energy coincides with (5.14).

If energy increases then $\beta^*$ approaches exponentially the constant value $\beta^*=-1$. Presenting entropy in the form

$$S=-\frac{1}{\beta^*}\ln(1+\beta^*)+2-\frac{1+\beta^*}{\beta^*}\ln(1+\beta^*)$$

and taking into account that the last term goes to zero for $\beta^*\to -1$, we conclude that the entropy tends to $-\infty$ when energy increases.

In the limit $E^*\to 0$, $\beta^*\approx 1/E^*$, and entropy again tends to $-\infty$. Graphs of the functions $\beta^*(E^*)$ and $S(E^*)$ are shown in Figs. 2 and 3.

## APPENDIX G: EULER'S EQUATIONS OF THE FUNCTIONAL $B(z,w_i,r)$

Let us show that the stationary points $w_i$ of functional $B(z,w_i,r)$ (9.12) are the solutions of the system of equations (9.11), (9.9), and (9.6). By varying the functional $B(z,w_i,r)$ we get the Euler equations for $w_i$: for each $i$

$$\sum_{j\neq i}\sigma_j R_j=0 , \tag{G1}$$

where

$$R_j=\Delta_{r'}w_j+\frac{e^{-z\sigma_j u_j(r',r)}}{\int_V e^{-z\sigma_j u_j(r'',r)}d^2r''} .$$

Consider (G1) as a system of $(N-1)$ linear equations with respect to $(N-1)$ variables $R_j$ $(j=2,\dots,N)$. The determinant of this system is equal to $\text{const}\times\sigma_2\cdots\sigma_N$. Suppose that none of the numbers $\sigma_2,\dots,\sigma_N$ is equal to zero. Then the determinant is not zero and the only solution of (G1) is $R_j=0$, i.e., we arrive at Eqs. (9.11).

## APPENDIX H: EQUIVALENCE OF MICROCANONICAL AND CANONICAL DISTRIBUTIONS

In some papers (see, for example, [14,32]) statistical mechanics of point vortices in a closed domain is considered using a canonical ensemble with Gibbs's probability distribution

$$f(p,q)=\frac{1}{Z(\beta)}e^{-\beta H(p,q)} , \tag{H1}$$

where $Z(\beta)$ is a partition function

$$Z(\beta)=\int e^{-\beta H(p,q)}d^Np\,d^Nq \tag{H2}$$

and the parameter $\beta$ related to the averaged energy $\bar{E}$

$$\bar{E}=\int H(p,q)\frac{1}{Z}e^{-\beta H(p,q)}d^Np\,d^Nq . \tag{H3}$$

If the number of vortices is finite, Gibbs's distribution differs from the microcanonical distribution and is not relevant because the energy of fluid flow is conserved. However, in the limit $N\to\infty$, canonical and microcanonical distributions might coincide. Let us show that this is the case. To have coincidence for high energies, we must scale the parameter $\beta$ in (H1)–(H3) properly: in (H1)–(H3) we change $\beta$ to $N\beta$. The "new" $\beta$ turns out to be equal to the parameter $\beta$ used in this paper. First, let us find the probability density function of energy $f(E)$. We have, from (H1),

$$f(E)=\frac{1}{Z(\beta)}e^{-\beta NE}\frac{d\Gamma(E)}{dE} . \tag{H4}$$

If canonical and microcanonical distributions are equivalent, the probability density function (H4) should converge for $N\to\infty$ to $\delta(E-\bar{E})$, while $\bar{E}$ should coincide with the value of energy prescribed by initial conditions,

$$\frac{1}{Z(\beta)}e^{-\beta NE}\frac{d\Gamma(E)}{dE}\to\delta(E-\bar{E})\quad\text{for } N\to\infty\ . \tag{H5}$$

Relation (H5) is a necessary condition for the equivalence of microcanonical and canonical distributions. It can be shown that it is also a sufficient condition in the following sense: if (H5) holds, then for any smooth function $\varphi(p,q)$, the probability density functions of $\varphi$, found by means of microcanonical and canonical distributions, will coincide.

Relation (H5) can be checked if $Z(\beta)$ and $d\Gamma/dE$ are known. The derivative of phase volume $d\Gamma/dE$ has been found in Sec. VII [formula (7.29)]. Let us now find the partition function $Z(\beta)$. From (H2) we have

$$Z(\sigma)=\int_{-\infty}^{+\infty}e^{-\sigma EN}\frac{d\Gamma}{dE}dE\ . \tag{H6}$$

Plugging in (H6), the expression for $d\Gamma/dE$ from (7.29), we obtain

$$Z(\sigma)=\int_{-\infty}^{+\infty}e^{-\sigma EN}\frac{\sqrt{N}\,|V|^N}{\sqrt{2\pi B''(\beta)}}C(\beta)\times e^{N[E\beta+B(\beta,v_0(\beta))]}dE\ . \tag{H7}$$

Here $\beta$ and $v_0(\beta)$ are assumed to be functions of $E$ determined by Eqs. (7.26) and (7.27). Formula (H7) can be rewritten in terms of entropy per one degree of freedom $s(E)=N^{-1}S(E)$ (8.3),

$$Z(\sigma)=\int_{-\infty}^{+\infty}\frac{\sqrt{N}\,|V|^N}{\sqrt{2\pi B''(\beta)}}C(\beta)e^{-N[s(E)-\sigma E]}dE\ . \tag{H8}$$

To find the asymptotics of the integral in (H8), one might use Laplace's method. The maximum value of $s(E)-\sigma E$ is reached at the point $\hat{E}$, which is the solution of the equation

$$\frac{ds(E)}{dE}-\sigma=0\ . \tag{H9}$$

It follows from (8.3), (7.26), and (7.27) that

$$\beta(E)=\frac{ds(E)}{dE}\ . \tag{H10}$$

Therefore, at the point of maximum $s(E)-\sigma E$, we have

$$\beta=\sigma\ . \tag{H11}$$

In the vicinity of the point of maximum

$$s(E)-\sigma E=s(\hat{E})-\sigma\hat{E}+\frac{1}{2}\frac{d^2s}{dE^2}(E-\hat{E})^2\ . \tag{H12}$$

We need to relate $d^2s/dE^2$ and $B''$. Differentiating (H10), we have

$$\frac{d^2s(E)}{dE^2}=\frac{d\beta}{dE}\ . \tag{H13}$$

On the other hand, since from (8.3) and (7.25)

$$E+\frac{dB}{d\beta}=0\ , \tag{H14}$$

we obtain by differentiating (H14) with respect to $E$

$$1+B''\frac{d\beta}{dE}=0\ . \tag{H15}$$

In accordance with (H13) and (H15)

$$\frac{d^2s}{dE^2}=-\frac{1}{B''}\ . \tag{H16}$$

Applying Laplace's method we see from (H8), (H12), and (H16) that

$$Z(\sigma)=C(\sigma)e^{[s(\hat{E})-\sigma\hat{E}]}|V|^N\ . \tag{H17}$$

The expression for entropy (8.30) yields that $s(E)-\sigma E$ is equal to $B(\sigma,v_0(\sigma))$. Finally

$$Z(\sigma)=C(\sigma)e^{-NB(\sigma,v_0(\sigma))}|V|^N\ . \tag{H18}$$

Relation (H18) can be written in the form of the variational principle

$$\frac{1}{N}\ln\frac{Z(\sigma)}{|V|^N}=\mathcal{V}_v^{\rm st}(B(\sigma,v))\ , \tag{H19}$$

where $\mathcal{V}_v^{\rm st}$ denotes the stationary value with respect to $v$. To check the validity of (H5) we use (H18) and (7.29). We have

$$\frac{1}{Z(\sigma)}e^{-\sigma NE}\frac{d\Gamma}{dE}=\frac{C(\beta)}{C(\sigma)e^{NB(\sigma,v_0(\sigma))}}\times\frac{\sqrt{N}}{\sqrt{2\pi B''}}e^{N[s(E)-\sigma E]}\ . \tag{H20}$$

The convergence of function (H20) to a $\delta$ function follows from (H12) and (H11). So the microcanonical and canonical ensembles are equivalent for high energies in the limit $N\to\infty$.


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