Efficient switching between controlled unstable periodic orbits in higher dimensional chaotic systems

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We develop an efficient targeting technique and demonstrate that when used with an unstable periodic orbit stabilization method, fast and efficient switching between controlled periodic orbits is possible. This technique is particularly relevant to cases of higher attractor dimension. We present a numerical example and report an improvement of up to four orders of magnitude in the switching time over the case with no targeting.

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The presence of chaos, far from being an inconvenience, allows for a large range of useful behaviors not present in nonchaotic systems. For example, unstable periodic orbits (UPO's) that naturally occur in a chaotic attractor can be stabilized by applying only *small* perturbations to available system parameters [1–3]. This technique has been experimentally applied to a wide variety of different systems including mechanical systems [4], lasers [5], circuits [6], chemical reactions [7], biological systems [8], etc. Furthermore, it is possible to switch from one periodic orbit to another at will [2,4]. In contrast, control of nonchaotic systems typically requires large control signals to exhibit similar behavior.

This switching, however, may not take place quickly: one may have to wait for a considerable time before a particular UPO is stabilized. This is because the method of Ref. [1] relies on ergodic wandering of the orbit to bring it close to the desired state before the controls are applied. For such a process, the time required to approach a small region is typically ϵ^{-D}, where ϵ is the linear dimension of the region, and D is the pointwise dimension at the periodic point [9]. For low values of D this time can be acceptably small. For systems of higher dimension, however, these transient times may be prohibitively long.

In order to reduce these transient times, we employ a targeting method. Targeting refers to a method by which a chaotic orbit can be rapidly steered to a desired part of the attractor. Several such methods have been proposed [10]. Recently, a *tree targeting* method has been developed that is computationally efficient for higher dimensional systems [11]. In this paper, we extend this approach and demonstrate that when coupled with a UPO stabilization method very dramatic improvements in switching times between different controlled UPO's can be realized. Before describing the method we report the result of its application on a simple prototypical example, in which we obtain improvements of up to four orders of magnitude in the switching times.

We use the kicked double rotor system, illustrated in Fig. 1, for our numerical tests. This is an idealized phys-

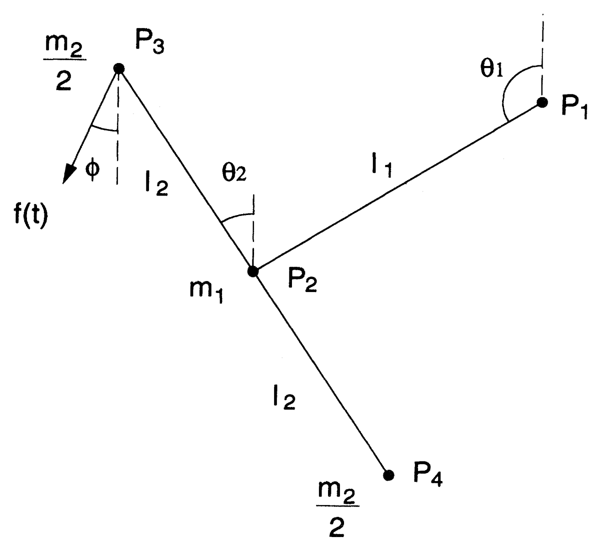

FIG. 1. The kicked double rotor. A massless rod of length l_1 pivots about the stationary point P_1. A second massless rod of length $2l_2$ is mounted on pivot P_2, which in turn is mounted at the end of the first rod. Periodic impulsive kicks $f(t) = \sum_{n=0}^{\infty} \rho_n \delta(t-n)$ are applied at an angle ϕ as shown. The state of the system immediately after the $(n+1)$th kick is given by a four dimensional map of the form $\mathbf{X}_{n+1} = \mathbf{M}\mathbf{Y}_n + \mathbf{X}_n$ and $\mathbf{Y}_{n+1} = \mathbf{L}\mathbf{Y}_n + \mathbf{G}(\mathbf{X}_{n+1})$, where $\mathbf{X} = (\theta_1, \theta_2)^T$ are the two angular position coordinates, $\mathbf{Y} = (\dot\theta_1, \dot\theta_2)^T$ are the corresponding angular velocities, and $\mathbf{G}(\mathbf{X})$ is a nonlinear function. $\mathbf{M}$ and $\mathbf{L}$ are both constant matrices which involve the coefficients of friction at the two pivots and the moments of inertia of the rotor. Gravity is absent. Control parameters at time n are $\rho_n = 9.0 + \Delta\rho_n$ and $\phi_n = 0.0 + \Delta\phi_n$, with $|\Delta\rho|/\rho_0 \leq 0.1$ and $|\Delta\phi| \leq 0.5$. We take $l_2 = 1/\sqrt{2}$, and set all other parameters to 1. For further details, see Refs. [2,12].

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ical system of two connected rods subject to periodic impulsive kicks. The time evolution, sampled immediately after each kick, is given by a four dimensional map [2,12]. We take our control parameters to be the strength of the kick ρ and the angle ϕ at which the kick is applied. Small perturbations ($|\Delta\rho|/\rho_0 \leq 0.1; |\Delta\phi| \leq 0.5$) are applied around nominal values ($\rho_0 = 9.0, \phi_0 = 0.0$) at which the map exhibits 36 fixed points within a chaotic attractor of Lyapunov dimension 2.8 (see figure caption for further details). This dimension is to be contrasted with most experimental and numerical studies where the dimension was typically between 1 and 2, and often close to 1. Figure 2 illustrates the results of our numerical tests. We sought to stabilize a sequence of five fixed points in succession. The figure displays the θ_1 component of the state versus iteration number. In Fig. 2(a) we rely on ergodicity to bring the orbit close to the desired fixed point before it is stabilized. In Fig. 2(b) our tree targeting method is used. Very large improvements in the switching times are evident; note the great difference in the scales on the two horizontal axes. More precisely, in (a) the switching times t follow an exponential distribution $\langle t\rangle^{-1}\exp(t/\langle t\rangle)$ (this is typical of chaotic systems [13]), and we find that $\langle t\rangle$ ranges from $12\,000 \pm 80$ iterations to $252\,000 \pm 3000$ iterations, depending on the fixed point. In sharp contrast, our method permits the target to be attained in 17–19 iterations using one parameter control (ρ), and 13–15 iterations using two parameter control (ρ and ϕ). Thus, we achieve improvements of 3–4 orders of magnitude in the switching times.

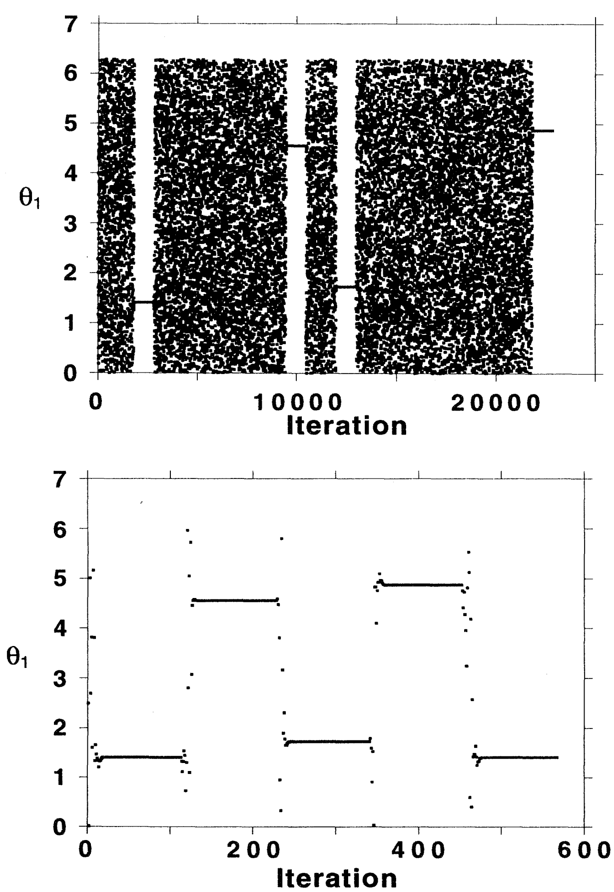

FIG. 2. Graphs illustrating switching between five different fixed points. The θ_1 coordinate of the state is plotted versus iteration. In (a) we rely on ergodicity to bring the orbit close to the desired UPO. The fifth fixed point required 153 485 iterations to be stabilized, and is not shown. In (b), tree targeting is employed.

We now outline the general method that was used to obtain these results. First, the desired UPO's $\mathbf{p}_1$, $\mathbf{p}_2$, ... are identified (for simplicity, we take these to be fixed points, i.e., periodic orbits of period 1). For each such point we construct targeting trees which will function as "road maps" of the attractor. To stabilize $\mathbf{p}_1$, a chaotic orbit is directed along the corresponding tree into the vicinity of $\mathbf{p}_1$. The UPO stabilization method is then applied to stabilize the orbit. To switch to $\mathbf{p}_2$, one abandons $\mathbf{p}_1$, follows the tree leading to $\mathbf{p}_2$, and subsequently stabilizes the orbit there.

Each targeting tree is constructed by first choosing the target, say the fixed point $\mathbf{p}_1$. The map is then iterated from a random initial condition while keeping in memory a short history of the iterates encountered (for example, 10 consecutive points), until the orbit lands within a tolerance distance of the target [14]. This point, together with the recorded preiterates, comprise the *trunk path* of the tree, and are stored in memory. The map is then iterated again, still keeping track of a brief iterate history, until the orbit lands near any one of the points already in the tree. When this happens, we add the new path as a *branch.* Continuing in this way, we build a tree with a hierarchy of branches: the trunk path is level 1; level 2 branches are those that are rooted at some point in the trunk path; level 3 branches are rooted at a level 2 branch, and so on. The objective is to build a tree with enough branches such that a typical uncontrolled chaotic orbit lands near a point in the tree after a small number of iterations.

The basic targeting procedure is illustrated in Fig. 3. Assume that a target point $\mathbf{t} = \mathbf{x}_{10}$ on the attractor has been selected, and that the trunk path consisting of points $\mathbf{x}_9$, $\mathbf{x}_8$, ..., $\mathbf{x}_0$ has been recorded. Let $\mathbf{y}_0$ be a point near $\mathbf{x}_0$. Without targeting, the orbit $\mathbf{y}_1$, $\mathbf{y}_2$, ...

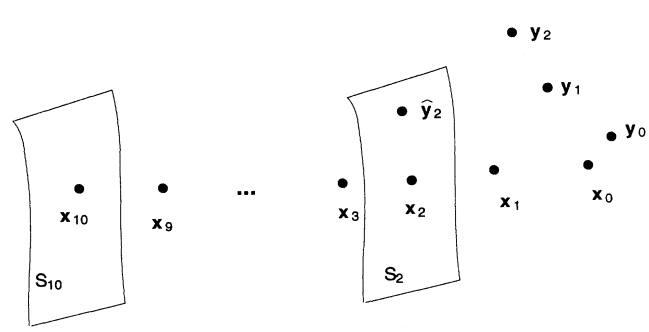

FIG. 3. Schematic diagram of the targeting procedure. Two successive perturbations of the kick are applied at y_0 to steer it onto the stable manifold associated with the point x_2.

quickly diverges from the path. We seek a series of small perturbations to available control parameters such that the perturbed orbit $\hat{\mathbf{y}}_1$, $\hat{\mathbf{y}}_2$, ... lands on the stable manifold of a subsequent point $\mathbf{x}_i$ in the path. If this can be accomplished for a small value of i, then the orbit will quickly approach the path, and $\hat{\mathbf{y}}_{10}$ will be very close to the target $\mathbf{x}_{10}$.

For specificity, we consider the case of our example (Fig. 1), i.e., a four dimensional map $\mathbf{F}$ with two positive and two negative Lyapunov exponents. Let S_2 represent the (typically two dimensional) stable manifold associated with the point $\mathbf{x}_2$. For simplicity we assume that only one parameter ρ is available for control. Recall that two two-planes generically intersect at a single point in $\mathcal{R}^4$. Hence, the vectors $\mathbf{g}_1 = \partial\mathbf{F}(\mathbf{F}(\mathbf{y}_0, \rho_1), \rho_2)/\partial\rho_1$ and $\mathbf{g}_2 = \partial\mathbf{F}(\mathbf{F}(\mathbf{y}_0, \rho_1), \rho_2)/\partial\rho_2$ typically span a two-plane through $\mathbf{y}_2$ that intersects the two dimensional stable manifold S_2 at a unique point $\hat{\mathbf{y}}_2$. Therefore, we look for two successive parameter perturbations ρ_1 and ρ_2 such that $\hat{\mathbf{y}}_2$ lies on S_2.

The intersection point $\hat{\mathbf{y}}_2$ may not be sufficiently close to $\mathbf{x}_2$ to justify using a linear approximation for estimating S_2. This is generally the case in our kicked double rotor example. To deal with this, we estimate the intersection point on S_2 by calculating the inverse images of points near subsequent points further down the path. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ denote vectors that span a plane at $\mathbf{x}_8$. A point $\mathbf{z} = \mathbf{x}_8 + \sigma_1\mathbf{s}_1 + \sigma_2\mathbf{s}_2$ is chosen with σ_1 and σ_2 small (typically of order 10^{-6}). The inverse images $\mathbf{F}^{-1}(\mathbf{z})$, $\mathbf{F}^{-2}(\mathbf{z})$, ... rapidly approach the stable manifolds S_7, S_6, ..., because under the inverse map S_8 is an expanding set, and components perpendicular to S_8 contract.

Thus, in the case of one parameter control, we calculate two successive parameter perturbations ρ_1 and ρ_2, together with values for σ_1 and σ_2, such that

$$\mathbf{F}^{-k}(\mathbf{x}_{k+2} + \sigma_1\mathbf{s}_1 + \sigma_2\mathbf{s}_2) = \mathbf{F}(\mathbf{F}(\mathbf{y}_0, \rho_1), \rho_2) = \hat{\mathbf{y}}_2. \quad (1)$$

In our example, we use $k = 6$. Equation (1) can be solved numerically using Newton's method. Once the prescribed kicks ρ_1 and ρ_2 are applied at $\mathbf{y}_0$ and $\hat{\mathbf{y}}_1$, the orbit lands on the stable manifold of $\mathbf{x}_2$ (at $\hat{\mathbf{y}}_2$), and subsequent iterations of the map approach the path exponentially.

In practice, values of k which yield numerically accurate results can be determined by performing numerical trials on the particular map being considered. In order to correct for these and other nonideal effects such as noise, state measurement error, and an imperfect determination of the system parameters, the method is reapplied at every iteration.

If two parameters ρ and ϕ are available for control, then only one perturbation step is necessary: typically there is a two-plane through $\mathbf{y}_1$ spanned by $\mathbf{g}_\rho = \partial\mathbf{F}(\mathbf{y}_0, \rho, \phi)/\partial\rho$ and $\mathbf{g}_\phi = \partial\mathbf{F}(\mathbf{y}_0, \rho, \phi)/\partial\phi$ that intersects the stable manifold S_1 of $\mathbf{x}_1$. The procedure outlined above can be similarly extended to different dimensions and different numbers of positive Lyapunov exponents.

Assume now that a three level targeting tree has been constructed. The map can be iterated until the trajectory lands at a point $\mathbf{y}$ near a point $\mathbf{x}$ in the tree. Suppose $\mathbf{x}$ is in a level 3 branch. The base of this branch is chosen as an interim target, and the orbit is directed there by the method described above. Next, we set the interim target to be the root of the adjoining level 2 branch. The orbit is steered to this new target, and the process is repeated until the final target is attained. The orbit is then stabilized at the fixed point by applying the UPO control procedure.

We now discuss a further improvement that can reduce the time necessary to land on a given tree. In the procedure described above, an initial condition $\mathbf{z}$ is iterated until the uncontrolled orbit encounters the tree. Another possibility is to generate a cloud of points by calculating the image of $\mathbf{z}$ under a small random parameter perturbation (applied to ρ), and repeat this many times. Thus, all points in the cloud are within one iteration of $\mathbf{z}$. This entire cloud can then be iterated forward a certain number of times, and each time a point in the cloud encounters the tree, its position is recorded. In this way, many different paths from the initial condition $\mathbf{z}$ onto the tree can be found, and from these we can select the path that ultimately reaches the target in the fewest number of iterations.

This completes our description of the targeting method used in Fig. 2(b). We emphasize that this technique is very general and that targeting is essential for fast and efficient UPO stabilization when the attractor is of moderate dimension and the parameter perturbations are limited to be small.

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[14] We use a distance of 0.05 relative to the maximum norm, where the distance between two points is taken to be the largest absolute difference between corresponding components.