Schrödinger problem, Lévy processes, and noise in relativistic quantum mechanics

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(Received 14 November 1994)

The main purpose of the paper is an essentially probabilistic analysis of relativistic quantum mechanics. It is based on the assumption that whenever probability distributions arise, there exists a stochastic process that is either responsible for the temporal evolution of a given measure or preserves the measure in the stationary case. Our departure point is the so-called Schrödinger problem of probabilistic evolution, which provides for a unique Markov stochastic interpolation between any given pair of boundary probability densities for a process covering a fixed, finite duration of time, provided we have decided *a priori* what kind of primordial dynamical semigroup transition mechanism is involved. In the nonrelativistic theory, including quantum mechanics, Feyman-Kac–like kernels are the building blocks for suitable transition probability densities of the process. In the standard "free" case (Feynman-Kac potential equal to zero) the familiar Wiener noise is recovered. In the framework of the Schrödinger problem, the "free noise" can also be extended to any infinitely divisible probability law, as covered by the Lévy-Khintchine formula. Since the relativistic Hamiltonians $|\nabla|$ and $\sqrt{-\triangle + m^2} - m$ are known to generate such laws, we focus on them for the analysis of probabilistic phenomena, which are shown to be associated with the relativistic wave (D'Alembert) and matter-wave (Klein-Gordon) equations, respectively. We show that such stochastic processes exist and are spatial jump processes. In general, in the presence of external potentials, they do not share the Markov property, except for stationary situations. A concrete example of the pseudodifferential Cauchy-Schrödinger evolution is analyzed in detail. The relativistic covariance of related wave equations is exploited to demonstrate how the associated stochastic jump processes comply with the principles of special relativity.

PACS number(s): 05.40.+j, 02.50.−r, 03.65.Pm

I. THE ANALYTIC CONTINUATION IN TIME OF HOLOMORPHIC SEMIGROUPS AS A MAPPING BETWEEN TWO FAMILIES OF STOCHASTIC PROCESSES

A. Gaussian exercises

The Schrödinger equation and the generalized heat equation are connected by analytic continuation in time. The link (casually viewed as a kind of analogy or correspondence) can be implemented by a rotation in the complex time plane, taking the Feynman-Kac kernel into the Green function of the corresponding quantum mechanical problem, which is an exploitation of properties of holomorphic semigroups generated by Laplacians and their sums with appropriate potentials.

For $V = V(x), x \in R$, bounded from below, the generator $H = -2mD^2\triangle + V$ is essentially self-adjoint on a natural dense subset of L^2, and the kernel $k(x,s,y,t) = \{\exp[-(t-s)H]\}(x,y)$ of the related dynamical semigroup is strictly positive. The quantum unitary dynamics $\exp(-iHt)$ is a final result of the analytic continuation.

As repeatedly emphasized [1–3], any temporal evolution that is analyzable in terms of a probability measure may be interpreted as a stochastic process. In view of the Born statistical interpretation postulate for quantum mechanics, the analytic continuation in time discussed above induces a class of probability measures, namely, consider $\rho(x,t) = |\psi(x,t)|^2$ as the density of a probability measure associated with a given solution $\psi(x,t)$ of the Schrödinger equation. Then, it is possible to address the problem of that stochastic dynamics that would be either (i) measure preserving or (ii) induce the time evolution of the measure proper. Keep in mind that the Schrödinger equation itself *is not* a genuine partial differential equation of probability theory; rather it is the Born postulate that embeds the unitary evolution problem into the probabilistic framework.

A simple illustration of the analytic continuation in time is provided by considering the force-free propagation, where the formal recipe gives rise to the equations of motion (one should be aware that to execute a mapping for concrete solutions, the proper adjustment of the time interval boundaries is indispensable):

$$i\partial_t\psi = -D\triangle\psi \longrightarrow \partial_t\theta_* = D\triangle\theta_*,$$

$$i\partial_t\overline{\psi} = D\triangle\overline{\psi} \longrightarrow \partial_t\theta = -D\triangle\theta, \tag{1}$$

$$it \rightarrow t.$$

Here $\overline{\psi}$ denotes the complex conjugate of ψ, while the notation θ_* and θ refers to real functions which solve the time adjoint parabolic equations. Then

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$$\psi(x,t) = [\rho^{1/2}\exp(iS)](x,t) = \int dx' G(x-x',t)\psi(x',0),$$

$$G(x-x',t) = (4\pi i Dt)^{-1/2} \exp\left[-\frac{(x-x')^2}{4iDt}\right], \tag{2}$$

$$\theta_*(x,t) = \int dx' k(x-x',t)\theta_*(x',0),$$

$$k(x-x',t) = (4\pi Dt)^{1/2} \exp\left[-\frac{(x-x')^2}{4Dt}\right],$$

where the imaginary time substitution

$$k(x-x',it) = G(x-x',t)$$

$$k(x-x',t) = G(x-x',-it) \tag{3}$$

seems to persuasively suggest the notion of "evolution in imaginary time," which in the usual interpretation relates quantum theory to an "imaginary time diffusion." Here we shall emphasize a different viewpoint in which the quantum dynamics and the so-called Euclidean dynamics [4] (dealing with the Wiener process and conditional Brownian motions, for example) may both be seen as real time diffusions.

At this point let us observe that given the initial data

$$\psi(x,0) = (\pi\alpha^2)^{-1/4} \exp\left(-\frac{x^2}{2\alpha^2}\right) \tag{4}$$

the free Schrödinger evolution $i\partial_t\psi = -D\triangle\psi$ implies that

$$\psi(x,t) = \left(\frac{\alpha^2}{\pi}\right)(\alpha^2 + 2iDt)^{-1/2} \exp\left[-\frac{x^2}{2(\alpha^2+2iDt)}\right] \tag{5}$$

with

$$\rho(x,t) = |\psi(x,t)|^2 = \frac{\alpha}{[\pi(\alpha^4+4D^2t^2)]^{1/2}} \times \exp\left(-\frac{x^2\alpha^2}{\alpha^4+4D^2t^2}\right) = \int p(y,0,x,t)\rho(y,0)dy \tag{6}$$

$$p(y,0,x,t) = (4\pi Dt)^{-1/2} \exp\left[-\frac{(x-y-\frac{2D}{\alpha^2}yt)^2}{4Dt}\right],$$

where $p(y,0,x,t)$ is the (distorted Brownian) transition probability density for Nelson's diffusion [2,3]. The transition density $p(y,0,x,t)$ is not uniquely specified by (6), but the nonuniqueness problem for the diffusion process involved may be resolved when we consider transition densities for arbitrary intermediate times.

For example [2], if for convenience we set $\alpha^2 = 2$, $D = 1$, the transition density for two arbitrary times, $t > s$, reads

$$p(y,s,x,t) = [4\pi(t-s)]^{-1/2} \exp\left[-\frac{(x-cy)^2}{4(t-s)}\right],$$

$$c = c(s,t) = \left[\frac{(1-t)^2+2s}{1+s^2}\right]^{1/2}, \tag{7}$$

$$\rho(x,t) = \int dy p(y,s,x,t)\rho(y,s).$$

Notice that the $s \downarrow 0$ limit of the transition density (7) does not coincide with the rescaled form of (6), $|1-t|$ instead of $(1-t)$ appears in the exponent. It reflects the nonuniqueness of the definition of the transition probability density as long as we do not insist on having defined all intermediate densities as well [2]. Anyway, while integrated with $\rho(x,0)$ they give the same output at time $t > 0$:

$$\rho_0(x) = (2\pi)^{-1/2}\exp\left[-\frac{x^2}{2}\right] \longrightarrow \rho(x,t) = [2\pi(1+t^2)]^{-1/2}\exp\left[-\frac{x^2}{2(1+t^2)}\right]. \tag{8}$$

Clearly $\rho(x,t)$ admits a factorization $\rho(x,t) = |\psi(x,t)|^2 = \Theta(x,t)\Theta_*(x,t)$. The standard Madelung exponents $R(x,t), S(x,t)$ such that $\psi(x,t) = \exp[R(x,t) + iS(x,t)]$ are given by

$$R(x,t) = -\tfrac{1}{4}\ln[2\pi(1+t^2)] - \frac{x^2}{4(1+t^2)}, \tag{9}$$

$$S(x,t) = \frac{x^2}{4}\frac{t}{1+t^2} - \tfrac{1}{2}\arctan(t),$$

and allow us to define the real functions $\Theta(x,t) = \exp[R(x,t) + S(x,t)]$, $\Theta_*(x,t) = \exp[R(x,t) - S(x,t)]$ which solve the pair of time adjoint generalized diffusion equations

$$\partial_t\Theta = -\triangle\Theta + Q\Theta,$$

$$\partial_t\Theta_* = \triangle\Theta_* - Q\Theta_*, \tag{10}$$

$$Q(x,t) = 2\frac{\triangle\rho^{1/2}}{\rho^{1/2}}.$$

The diffusion governed by the pair of adjoint equations (10) belongs to the category of "Nelson's diffusions" [3], but only in the framework presented here can it be singled out uniquely. It is exactly due to the "Schrödinger problem" uniqueness theorem [5]. It is really amazing that Schrödinger originated the problem of a stochastic interpolation between the prescribed input-output statistics data long before the modern probability theory was created.

An interesting observation is that we can give the transition density (7) another form [2,4]

$$p(y,s,x,t) = k(y,s,x,t)\frac{\Theta(x,t)}{\Theta(y,s)}, \tag{11}$$

$$\lim_{\triangle s \downarrow 0} \frac{1}{\triangle s}\left[1 - \int k(y,s,x,s+\triangle s)dx\right] = Q(y,s).$$

On the other hand, coming back to the previous notation and (5), we can straightforwardly pass to

$$\begin{aligned}\theta_*(x,t) :&= \psi(x,-it) \\ &= \left(\frac{\alpha^2}{\pi}\right)^{1/4}(\alpha^2+2Dt)^{-1/2} \\ &\quad\times \exp\left[-\frac{x^2}{2(\alpha^2+2Dt)}\right].\end{aligned} \tag{12}$$

Let us confine t to the time interval $[-T/2, T/2]$ with $DT < \alpha^2$. Then we arrive at

$$\partial_t \theta_* = D\triangle\theta_*,$$

$$\partial_t \theta = -D\triangle\theta,$$

$$-\frac{T}{2} \le t \le \frac{T}{2}, \tag{13}$$

$$\begin{aligned}\theta(x,t) &= \left(\frac{\alpha^2}{\pi}\right)^{1/4}(\alpha^2-2Dt)^{-1/2} \\ &\quad\times \exp\left[-\frac{x^2}{2(\alpha^2-2Dt)}\right],\end{aligned}$$

where [we use an overbar to distinguish between the probability densities (14) and (6) or (7), respectively; notice also that θ replaces Θ]

$$\begin{aligned}\overline{\rho}(x,t) &= \theta(x,t)\theta_*(x,t) \\ &= \left[\frac{\alpha^2}{\pi(\alpha^4-4D^2t^2)}\right]^{1/2}\exp\left[-\frac{\alpha^2x^2}{\alpha^4-4D^2t^2}\right]\end{aligned} \tag{14}$$

with the interesting and certainly unexpected—if one follows traditional Brownian intuitions—outcome that

$$\overline{\rho}(x,-t) = \overline{\rho}(x,t) \tag{15}$$

for all $|t| \le T/2$. The density (14) refers to a conditional Brownian motion, and the interpolating probability density can be represented as the conditional probability density (identifiable as the Bernstein density [4])

$$\overline{\rho}(x,t) = \frac{\overline{k}(0,-\alpha_0,x,t)\overline{k}(x,t,0,\alpha_0)}{\overline{k}(0,-\alpha_0,0,\alpha_0)},$$

$$\overline{k}(y,s,x,t) := [4\pi D(t-s)]^{-1/2}\exp\left[-\frac{(x-y)^2}{4D(t-s)}\right], \tag{16}$$

$$\alpha_0 = \frac{\alpha^2}{2D}.$$

Since $\overline{\rho}(x,t)$ trivially factorizes into a product of solutions of time adjoint heat equations (set $\theta(x,t) = [\overline{k}(0,-\alpha_0,0,\alpha_0)]^{-1/2}\overline{k}(x,t,0,\alpha_0)$), we have in hand the microscopic transport recipe [4]

$$p(y,s,x,t) = \overline{k}(y,s,x,t)\frac{\theta(x,t)}{\theta(y,s)} \tag{17}$$

with the heat kernel $\overline{k}(y,s,x,t)$, (16), and the property [to be compared with (11)]

$$\lim_{\triangle s \downarrow 0} \frac{1}{\triangle s}\left[1 - \int \overline{k}(y,s,x,s+\triangle s)dx\right] = 0. \tag{18}$$

The resemblance of formulas (11) and (17), (18) is not accidental, and suggests that any given Feynman-Kac kernel can be used to generate a probability measure. In addition, we should keep in mind that the two levels of probabilistic description, i.e., (11) and (17), are *indirectly* linked by the analytic continuation in time of a holomorphic semigroup with the Laplacian as its generator.

Remark 1. The time development of a density $\rho(x,t)$ ($\overline{\rho}$, respectively) is dictated by the Fokker-Planck (second Kolmogorov in the probabilistic lore) equation. It is instructive to notice that $i\partial_t\psi = -\triangle\psi$ upon setting $v = 2\mathrm{Re}\frac{\nabla\psi}{\psi}$ and $u = 2\mathrm{Im}\frac{\nabla\psi}{\psi}$ gives rise to $\partial_t\rho = -\nabla(v\rho)$, which may be rewritten as $\partial_t\rho = \triangle\rho - \nabla(b\rho)$ with $b = u + v$. Proceeding analogously with $\overline{\rho} = k(x_1,t_1,x,t)k(x,t,x_2,t_2)/k(x_1,t_1,x_2,t_2)$, where $k(y,s,x,t)$ is the heat kernel, and $s < t, t_1 < t < t_2$, while t_1, x_1, t_2, x_2 are fixed, we immediately recover $\partial_t\overline{\rho} = \triangle\overline{\rho} - \nabla(b\overline{\rho})$ with $b = b(x,t) = 2\nabla k(x,t,x_2,t_2)/k(x,t,x_2,t_2)$.

Furthermore to this remark, let us emphasize that the emergence of the nonvanishing drift field $b(x,t)$ is not connected with any external force. It is a traditional assumption when studying the Brownian motion in a conservative force field to define the drift as being proportional to the force itself (Stokes law); these Smoluchowski diffusions form a subclass of problems we are considering [1], however the previous drift is an exclusive effect of the conditioning.

B. The Schrödinger problem: from Feynman-Kac kernels to probability measures

The previous examples are very particular solutions of what we call [1,2] *the Schrödinger problem* of deducing the probabilistic interpolation (stochastic process) consistent with a given pair of boundary measure data at fixed initial and terminal time instants $t_1 < t_2$. Originated by Schrödinger himself [6], the problem was solved much later [4,5] by invoking the machinery of Bernstein stochastic processes; see also Ref. [7]. For our purposes the relevant information is that [5], if the interpolating process is to display the Markov property, then it has to be specified by the joint probability measure (A and B are Borel sets in R)

$$m(A,B)=\int_A dx\int_B dy\, m(x,y),$$

$$\int_R m(x,y)dy=\rho(x,t_1), \qquad (19)$$

$$\int_R m(x,y)dx=\rho(y,t_2),$$

where we assign densities to all measures to be dealt with, and the density $m(x,y)$ is given in the functional form

$$m(x,y)=f(x)k(x,t_1,y,t_2)g(y) \qquad (20)$$

involving two unknown functions $f(x)$ and $g(y)$ which are of the same sign and nonzero, while $k(x,s,y,t)$ is any bounded strictly positive (dynamical semigroup) kernel defined for all times $t_1\le s<t\le t_2$ continuous in variables. The integral equations (19) determine functions $f(x), g(y)$ uniquely (up to constant factors) in this case [5].

By denoting $\theta_*(x,t) = \int f(z)k(t_1,z,x,t)dz$ and $\theta(x,t)=\int k(x,t,z,t_2)g(z)dz$ it follows [1,2,4,5] that

$$\overline{\rho}(x,t)=\theta(x,t)\theta_*(x,t)=\int p(y,s,x,t)\overline{\rho}(y,s)dy,$$

$$p(y,s,x,t)=k(y,s,x,t)\frac{\theta(x,t)}{\theta(y,s)},$$

$$t_1\le s<t\le t_2. \qquad (21)$$

Hence the transition densities are intrinsically entangled with the dynamical semigroup kernels in the solution of the Schrödinger stochastic interpolation problem. The crucial step in the construction of any explicit propagation consistent with the boundary measure data is to decide what is the appropriate dynamical semigroup.

We shall address the issue in its full generality. Strictly positive semigroup kernels generated by Laplacians plus suitable potentials are very special examples in a surprisingly rich encompassing family. First of all, the concept of the "free noise," normally characterized by a Gaussian probability distribution appropriate to a Wiener process, can be extended to all infinitely divisible probability distributions via the Lévy-Khintchine formula [8,9]. It expands our framework from continuous diffusion processes to jump or combined diffusion-jump propagation scenarios. All such (Lévy) processes are associated with strictly positive dynamical semigroup kernels, and all of them give rise to Markov solutions of the Schrödinger stochastic interpolation problem (19)–(21).

Remark 2. Apart from the wealth of physical phenomena described in terms of Gaussian stochastic processes, there is a number of physical problems where the Gaussian toolbox proves to be insufficient to provide satisfactory probabilistic explanations. Non-Gaussian Lévy processes naturally appear in the study of transient random walks when long-tailed distributions arise [10–12]. They are also found necessary to analyze fractal random walks [13], intermittency phenomena, anomalous diffusions, and turbulence at high Reynolds numbers [10,14,15].

Let us consider Hamiltonians of the form $H=F(\hat{p})$, where $\hat{p}=-i\nabla$ stands for the momentum operator and for $-\infty<k<+\infty$, $F=F(k)$ is a real valued, bounded from below, locally integrable function. Then, $\exp(-tH)=\int_{-\infty}^{+\infty}\exp[-tF(k)]dE(k)$, $t\ge 0$, where $dE(k)$ is the spectral measure of $\hat{p}$.

Most of our discussion will pertain to processes in one spatial dimension, and let us specialize the issue accordingly. Because $[(E(k)f](x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{k}\exp(ipx)\hat{f}(p)dp$, where $\hat{f}$ is the Fourier transform of f, we learn that

$$\begin{aligned}[\exp(-tH)]f(x)&=\left\{\int_{-\infty}^{+\infty}\exp[-tF(k)]dE(k)f\right\}(x)\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp[-tF(k)]\frac{d}{dk}\left[\int_{-\infty}^{k}\exp(ipx)\hat{f}(p)dp\right]dk\\&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp[-tF(k)]\exp(ikx)\hat{f}(k)dk=\{\exp[-tF(p)]\hat{f}(p)\}^{\vee}(x),\end{aligned} \qquad (22)$$

where the superscript $\vee$ denotes the inverse Fourier transform.

Let us set $k_t=\frac{1}{\sqrt{2\pi}}[\exp(-tF(p)]^{\vee}$, then the action of $\exp(-tH)$ can be given in terms of a convolution: $\exp(-tH)f=f*k_t$, where $(f*g)(x):=\int_R g(x-z)f(z)dz$.

We shall restrict consideration only to those $F(p)$ that give rise to positivity preserving semigroups: if $F(p)$ satisfies the celebrated Lévy-Khintchine formula, then k_t is a positive measure for all $t\ge 0$. The most general case refers to a contribution from three types of processes: deterministic, Gaussian, and an exclusively jump process. We shall concentrate on the integral part of the Lévy-Khintchine formula, which is responsible for arbitrary stochastic jump features,

$$F(p)=-\int_{-\infty}^{+\infty}\left[\exp(ipy)-1-\frac{ipy}{1+y^2}\right]\nu(dy), \qquad (23)$$

where $\nu(dy)$ stands for the so-called Lévy measure [8,16].

The disregarded Gaussian contribution would read $F(p)=p^2/2$; cf. Refs. [1–4,7] for an exhaustive discussion of related topics. In this case we know in detail how the analytic continuation in time of the Laplacian generated holomorpic semigroup induces a mapping to a quantum mechanical (since the Schrödinger equation is involved) diffusion processes.

Our further attention will focus on two selected choices for the characteristic exponent $F(p)$, namely: $F_0(p)=|p|$

and $F_m(p) = \sqrt{p^2 + m^2} - m, m > 0$, where we have chosen suitable units so as to eliminate inessential parameters. (The relativistic Hamiltonian is better known in the form $\sqrt{m^2c^4 + c^2p^2} - mc^2$ where c is the velocity of light.)

The respective Hamiltonians (semigroup generators) H_0, H_m are pseudodifferential operators. The semigroup kernels k_t^0, k_t^m in view of the "free noise" restriction (no potentials, will be defined in below) are transition densities of the jump (Lévy) processes regulated by the corresponding Lévy measures $\nu_0(dy)$, $\nu_m(dy)$. The affiliated Markov processes solving the Schrödinger problem (19)–(21) immediately follow. It is instructive to notice that as in the case of Gaussian derivations (1), it is [1] the case $\theta(x,t) \equiv 1$, $\theta_*(x,t) := \overline{\rho}(x,t)$ for which the pseudodifferential analog of the Fokker-Planck equation, as a consequence of $[\exp(-tH)\overline{\rho}](x) = \overline{\rho}(x,t)$ and in view of the identification $F(p \to -i\nabla) := H$ takes the fundamental form

$$F_0(p) \Longrightarrow \partial_t\overline{\rho}(x,t) = -|\nabla|\overline{\rho}(x,t) \tag{24}$$

or

$$F_m(p) \Longrightarrow \partial_t\overline{\rho}(x,t) = -\left[\sqrt{-\triangle + m^2} - m\right]\overline{\rho}(x,t), \tag{25}$$

respectively. Let us emphasize that the existence and uniqueness of solutions proof for the Schrödinger problem extends to all cases governed by the infinitely divisible probability laws, and has nothing to do with the "nonrelativistic" or "relativistic" options. The particular choice of semigroup generators, which are called "relativistic Hamiltonians" links the standard Schrödinger problem discussion with relativistic dynamics. But *only* after an analytic continuation in time, unless only stationary problems are studied (see the forthcoming discussion).

Although the pseudodifferential generator of the semigroup implies that the Fokker-Planck equation is no longer exclusively differential but an integro-differential equation, each solution $\overline{\rho}(x,t)$ in the present case is nevertheless a solution of a partial differential equation of higher order. Specifically, the respective partial differential equations are of the second order,

$$F_0(p) \Longrightarrow \square_E\overline{\rho}(x,t) = (\triangle_t + \triangle)\overline{\rho}(x,t) = 0. \tag{26}$$

Alternatively, if we set $\overline{\rho}(x,t) := \tilde{\rho}(x,t)\exp(mt)$ in (25) then

$$F_m(p) \Longrightarrow (\triangle_t + \triangle)\tilde{\rho}(x,t) = m^2\tilde{\rho}(x,t)$$

$$\Downarrow \tag{27}$$

$$(-\square_E + m^2)\tilde{\rho}(x,t) = 0,$$

where $\partial_t\tilde{\rho} = -\sqrt{-\triangle + m^2}\,\tilde{\rho}$ holds true instead of (25).

Our two semigroups are holomorphic [17], hence we can replace the time parameter t by a complex one $\sigma = t + is$, $t > 0$ so that $\exp(-\sigma H) = \int_R \exp[-\sigma F(k)]\, dE(k)$. Its action is defined by

$$[\exp(-\sigma H)]f = [(\hat{f}\exp(-\sigma F)]^{\vee} = f * k_\sigma \tag{28}$$

to be compared with (22). Here, the kernel reads $k_\sigma = \frac{1}{\sqrt{2\pi}}[\exp(-\sigma F)]^{\vee}$. Since H is self-adjoint, the limit $t \downarrow 0$ leaves us with the unitary group $\exp(-isH)$, acting in the same way: $[\exp(-isH)]f = [\hat{f}\exp(-isF)]^{\vee}$, except that now $k_{is} := \frac{1}{\sqrt{2\pi}}[\exp(-isF)]^{\vee}$ in general is *not* a measure. In view of unitarity, the unit ball in L^2 is an invariant of the dynamics. Hence density measures can be associated with solutions of the Schrödinger pseudodifferential equations,

$$F_0(p) \Longrightarrow i\partial_t\psi(x,t) = |\nabla|\psi(x,t) \tag{29}$$

or

$$F_m(p) \Longrightarrow i\partial_t\psi(x,t) = \left[\sqrt{-\triangle + m^2} - m\right]\psi(x,t) \tag{30}$$

provided with the appropriate initial data functions $\psi(x,0)$.

An obvious consequence of (29) and (30) is that the partial differential equation of the second order (26) takes on a familiar *relativistic* form

$$F_0(p) \Longrightarrow \square\psi(x,t) := (-\triangle + \triangle_t)\psi(x,t) = 0 \tag{31}$$

while after setting $\psi(x,t) = \tilde{\psi}(x,t)\exp(imt)$, we arrive at the Klein-Gordon equation,

$$F_m(p) \Longrightarrow (\square + m^2)\tilde{\psi}(x,t) = 0, \tag{32}$$

where the D'Alembert operator $\square = -\triangle + \triangle_t$ replaces its Euclidean counterpart $-\square_E$ in (27).

We have thus reached a point, at which the main questions addressed in the present paper can be precisely stated.

(i) What are the stochastic processes consistent with the probability measure dynamics $\rho(x,t) = |\psi(x,t)|^2$ determined by pseudodifferential Eqs. (29) and (30)?

(ii) Can we extend the Schrödinger problem idea to the special relativistic domain and be able to reproduce the interpolating stochastic process from the given input $\rho(x,t_1)$ and output $\rho(x,t_2)$, $t_1 < t_2$ statistics data, just as in the nonrelativistic (Laplacian generated motion) case?

(iii) To what extent can we attribute a definite probabilistic meaning to solutions of the relativistic wave equations (31) and (32)?

II. CAN WE ASSOCIATE FEYNMAN-KAC KERNELS WITH THE PSEUDODIFFERENTIAL-SCHRÖDINGER DYNAMICS ?

Given the Schrödinger equations (29),(30), to set them in the *Schrödinger problem* framework of Sec. I B we need to choose any normalized solution and then take the associated probability density $\overline{\rho}(x,t) := |\psi(x,t)|^2$ as the boundary data at times $t_1 < t_2$. However, as stated before some additional requirements must be met, specifically the Markov property is necessary [4,5]. One should keep in mind that if we do not insist on the Markov property for the interpolating process, then a solution of the problem involves the general Bernstein processes [5] for which a reformulation in terms of a pair of time adjoint generalized diffusion equations no longer exists.

We have chosen two rather special pseudodifferential counterparts of the Laplacian guided by two reasons: (i) their similarity on analytic grounds (the same criteria [18] for the existence of the bound state spectrum if summed with suitable potentials, which we shall need in the sequel), (ii) the claim of Ref. [19] that the pertinent stochastic process in the mass $m > 0$ case actually displays the Markov property.

If the Markov property would hold true for the relativistic Hamiltonian generated dynamics, we would be able to repeat almost all steps of the previous Schrödinger problem analysis [1,2,4,7]. However, the situation is not that simple, and the subsequent argument *excludes* the Markov property, in all nonstationary situations, in a flat contradiction with general statements by De Angelis [19].

Before embarking on this issue, let us introduce some probabilistic notions, which will tell us how to work with pseudodifferential operators. We shall notice that for explicit computational purposes, the Cauchy generator $|\nabla|$ is much more suited than the $m > 0$ relativistic Hamiltonian. It is a real disadvantage when dealing with Lévy processes that rather limited number of concrete examples is available, in contrast to the wealth of the general theory.

The Lévy-Khintchine formula (23) tells us that the action of the Hamiltonian $H = F(-i\nabla)$ on a function in its domain can be represented as follows [8,16]:

$$(H f)(x) = -\int_R \left[f(x+y) - f(x) - \frac{y \nabla f(x)}{1+y^2} \right] \nu(dy). \tag{33}$$

It is important to observe that for the free noise processes whose semigroup generators are $|\nabla|$ and $\sqrt{-\triangle + m^2} - m$ we do know explicitly their kernels (transition probability densities) and the involved Lévy measures, as well as about the extension of the Feyman-Kac *path integral* construction of the semigroup kernels to these particular Lévy processes [18,20,21], in case of arbitrary space dimensions. Therefore, we feel free to use the Feynman-Kac kernel notion instead of the semigroup kernel.

For the Cauchy process, whose generator is $|\nabla|$, we deal with a probabilistic classics [8,16]:

$$\begin{aligned} \overline{\rho}(x,t) &= \frac{1}{\pi} \frac{t}{t^2+x^2} \Longrightarrow k^0(y,s,x,t) \\ &= \frac{1}{\pi} \frac{t-s}{(t-s)^2+(x-y)^2}, \end{aligned} \tag{34}$$

$$0 < s < t,$$

$$\begin{aligned} \langle \exp[ipX(t)] \rangle : &= \int_R \exp(ipx) \overline{\rho}(x,t)\, dx \\ &= \exp[-tF_0(p)] = \exp(-|p|t). \end{aligned}$$

The characteristic function of $k^0(y,s,x,t)$ for y,s fixed, reads $\exp[ipy - |p|(t-s)]$, and the Lévy measure needed to evaluate the Lévy-Khintchine integral reads [20,22,23]:

$$\nu_0(dy) := \lim_{t \downarrow 0} \left[\frac{1}{t} k^0(0,0,y,t) \right] dy = \frac{dy}{\pi y^2}. \tag{35}$$

In the case of the relativistic generator $\sqrt{-\triangle + m^2} - m$, formulas determining the stochastic jump process are much less appealing [20,21],

$$\begin{aligned} \langle \exp[ipX(t)] \rangle : &= \exp[-tF_m(p)] \\ &= \exp\left[-t \left(\sqrt{p^2+m^2} - m \right) \right], \end{aligned}$$

$$\overline{\rho}(x,t) = \frac{m}{\pi} \frac{t \exp(mt)}{\sqrt{x^2+t^2}} K_1\left(m\sqrt{x^2+t^2} \right), \tag{36}$$

$$\begin{aligned} &\{\exp[-(t-s)F_m(-i\nabla)]\}(x-y) \\ &\qquad = k^m(y,s,x,t) := \overline{\rho}(x-y,t-s), \end{aligned}$$

$$\nu_m(dy) = \frac{m}{\pi |y|} K_1(m|y|) dy,$$

where $K_1(z)$ is the modified Bessel function of the third kind of order one.

We are interested in acting with the pseudodifferential generators $H = F(-i\nabla)$ on functions in the exponential form (recall the familiar Madelung procedure in the Gaussian case) $f(x,t) = \exp \Phi(x,t)$:

$$\begin{aligned} (H \exp \Phi)(x) &= -\int_R \left[\exp\Phi(x+y) - \exp\Phi(x) \right. \\ &\qquad \left. - \frac{y\Phi'(x)\exp\Phi(x)}{1+y^2} \right] \nu(dy) \\ &= \exp\Phi(x) \int_R \left\{ \exp[\Phi(x+y) - \Phi(x)] \right. \\ &\qquad \left. -1 - \frac{y\Phi'(x)}{1+y^2} \right\} \nu(dy), \end{aligned} \tag{37}$$

where $\Phi'(x) = \nabla\Phi(x)$. Since $(H\Phi)(x) = -\int_R [\Phi(x+y) - \Phi(x) - y\Phi'(x)/(1+y^2)]\nu(dy)$, we can make a safe rearrangement of (37):

$$\begin{aligned} (H \exp\Phi)(x) = \exp\Phi(x) \Bigg[(H\Phi)(x) - \int_R (\exp\Phi_{xy} \\ -1 - \Phi_{xy})\nu(dy) \Bigg], \end{aligned} \tag{38}$$

$$\Phi_{xy} := \Phi(x+y) - \Phi(x).$$

In application to the pseudodifferential dynamics $i\partial_t\psi(x,t) = (H\psi)(x,t)$ with $\psi = \exp(R+iS)$, we shall investigate its implications for the real functions $\Theta = \exp(R+S)$ and $\Theta_* = \exp(R-S)$; our argument will admit a trivial extension from H to $H+V$ situations.

Remark 3. Experience [1,24] with the Gaussian (standard Laplacian generated) noise proves that the Madelung substitution $\psi(x,t) = \exp[R(x,t) + iS(x,t)]$ would associate with the Schrödinger equation a pair of time adjoint generalized diffusion equations where the Feynman-Kac potential (time dependent in the general case) equals $\frac{1}{2mD}[2Q(x,t) - V(x)]$. Here $Q(x,t) = 2mD^2 \frac{\triangle\rho^{1/2}}{\rho^{1/2}}(x,t)$ and $V(x)$ is taken as an external conservative force potential. Let us emphasize that $V(x)$ actually *was* the Feynman-Kac potential of the dynamical semigroup prior to the analytic continuation in time procedure. The mapping $V(x) \to 2Q(x,t) - V(x)$ is an effect of the analytic continuation in time, as manifested on the level of the associated Feynman-Kac kernels. Previous comments suggest that the nonrelativistic formalism can be viewed as a kind of probabilistic reinterpretation of Bohm's point of view [25]. Specifically, the function $-Q(x,t)$ is the familiar de Broglie–Bohm "quantum potential." The analogous connection is generally invalid in the context of the Klein-Gordon equation, as explained in Ref. [25].

To this remark, let us add that the analytic continuation in time as outlined in Sec. I is not a well-known procedure although it is repeatedly mentioned in the previous publications [1,2]. It is new as a regular method. Also, it is not a part of Nelson's "quantum fluctuations" [3] point of view, just as the Feynman-Kac kernels were not ingredients of Nelson's theory. The idea is developed on the basis of Zambrini's [4] and Carmona's [7] research. References [1,2,4,5,7] give a complete description of the state of the art in this respect. Therefore, we do not propose to indicate any further connections with Nelson's stochastic mechanics than by referring to Nelson's monograph [3]. The paper itself proposes a self-contained probabilistic analysis which, in addition to Zambrini's work is motivated by the paper due to De Angelis [19].

In view of (38), the pseudodifferential-Schrödinger equation $i\partial_t\psi(x,t) = H\psi(x,t)$ implies the following time evolution of the Madelung exponents:

$$\partial_t R = HS - \int_R [\exp(R_{xy}) \sin S_{xy} - S_{xy}] d\nu(y)$$

$$\partial_t S = -HR + \int_R [\exp(R_{xy}) \cos S_{xy} - 1 - R_{xy}] d\nu(y), \tag{39}$$

where $H = F(-i\nabla)$.

By employing (38) with respect to $\rho^{1/2} = \exp(R)$, we arrive at

$$Q := \frac{H\rho^{1/2}}{\rho^{1/2}} = HR - \int_R [\exp(R_{xy}) - 1 - R_{xy}] d\nu(y) \tag{40}$$

and hence

$$\partial_t S = -Q + \int_R \exp(R_{xy}) [\cos(S_{xy}) - 1] d\nu(y). \tag{41}$$

The same procedure can be repeated for $\Theta = \exp(R+S)$ and $\Theta_* = \exp(R-S)$, where Eqs. (39) and (41) imply

$$\partial_t\Theta = H\Theta + \Theta\left\{-2Q + \int_R \exp(R_{xy})[-\sin S_{xy} + \cos S_{xy} + \exp(S_{xy}) - 2] d\nu(y)\right\}, \tag{42}$$

$$\partial_t\Theta_* = -H\Theta_* + \Theta_*\left\{2Q - \int_R \exp(R_{xy})[\sin S_{xy} + \cos S_{xy} + \exp(-S_{xy}) - 2] d\nu(y)\right\}.$$

In contrast to the Gaussian case [1,24], Eqs. (42) do not take the form of a time adjoint pair, unless some additional restrictions are imposed on the Madelung exponent $S(x,t)$ (notice that we have restored time dependence, skipped before for convenience). An obvious demand is $S(x+y,t) = S(x,t)$ for all y,t, and any fixed x. But then, Eqs. (42) would manifestly refer to the *stationary* (measure preserving) random dynamics, governed by the pair of equations

$$\partial_t\Theta = H\Theta - 2Q\Theta,$$

$$\partial_t\Theta_* = -H\Theta_* + 2Q\Theta_* \tag{43}$$

which are mutually time adjoint. Hence they would fall into the Schrödinger problem framework, with a trivial implication that the measure preserving process is Markovian. This, however, cannot be a property of the "free" dynamics since we need external potentials to secure stationarity. Let us, therefore, make an essential amelioration by performing the previous analysis for the case $i\partial_t\psi = (H+V)\psi$ with $V = V(x)$. Then, the stationary system of Eqs. (43) would take the form

$$\partial_t\Theta = H\Theta - (2Q+V)\Theta, \tag{44}$$

$$\partial_t\Theta_* = -H\Theta_* + (2Q+V)\Theta_*,$$

which upon substituting $S(x,t) = Et$, where E is a constant, yields a pseudodifferential version of the Sturm-Liouville problem

$$H\rho^{1/2}(x) - \left[2\frac{H\rho^{1/2}}{\rho^{1/2}} + V(x) - E\right]\rho^{1/2}(x) = 0 \tag{45}$$

$$\Downarrow$$

$$V(x) - E = -\frac{H\rho^{1/2}(x)}{\rho^{1/2}(x)}$$

to be solved (for a chosen value of E) with respect to the square root of the probability density $\rho(x)$, once the external force potential $V(x)$ is selected.

This problem has its Gaussian counterpart in the study of the measure preserving dynamics [1,26], and in the present context it can be solved by invoking those potentials for the original pseudodifferential Schrödinger equation, for which the bound states (i.e., stationary solutions) have granted the existence status. The relevant analysis has been carried out in the studies of the relativistic stability of matter [18,27–29]. In addition we know [18,21] that in the stationary case, the Feynman-Kac path integral generalization to Lévy semigroup kernels is available.

However, the Markov property cannot automatically be attributed to the nonstationary dynamics, as described by (42). Below, we shall make a careful analysis of the Cauchy-Schrödinger ($H = |\nabla|$) dynamics, to produce a definite counterexample, for which the unrestricted equations (42) would hold true, but the associated random dynamics would be non-Markovian.

III. THE CAUCHY-SCHRÖDINGER DYNAMICS

A. Construction of an explicit nonstationary solution

While it is clear that $\exp(-t|\nabla|)$ and $\exp(-it|\nabla|)$ have a common, identity operator limit as $t \downarrow 0$, an analytic continuation of the Cauchy kernel by means of (28) gives rise to

$$k_t(x) = \frac{1}{\pi}\frac{t}{x^2+t^2} \longrightarrow \tag{46}$$

$$k_{is} = \tfrac{1}{2}[\delta(x-s)+\delta(x+s)] + \frac{1}{\pi}\mathrm{P}\frac{is}{x^2-s^2}.$$

Here, we use the usual notation for the Dirac δ functionals, and the new time label s is a remnant of the limiting procedure $t \downarrow 0$ in $\sigma = t + is$. The function denoted by $is/\pi(x^2-s^2)$ comes from the inverse Fourier transform of $-\frac{i}{\sqrt{2\pi}}\mathrm{sgn}(p)\sin(sp)$. Because of

$$[\mathrm{sgn}(p)]^{\vee} = i\sqrt{\frac{2}{\pi}}\mathrm{P}\left(\frac{1}{x}\right), \tag{47}$$

where $\mathrm{P}(\frac{1}{x})$ stands for the functional defined in terms of a principal value of the integral. Using the notation $\delta_{\pm s}$ for the Dirac δ functional $\delta(x \mp s)$:

$$[\sin(sp)]^{\vee} = i\sqrt{\frac{\pi}{2}}(\delta_s - \delta_{-s}) \tag{48}$$

we realize that

$$\frac{1}{\pi}\frac{is}{x^2-s^2} = \frac{i}{2\pi}(\delta_s - \delta_{-s}) * \mathrm{P}\left(\frac{1}{x}\right) \tag{49}$$

is given in terms of the implicit convolution of two generalized functions.

Let $\psi(x,0) = f(x) := \sqrt{\frac{2}{\pi}}\,\frac{1}{1+x^2}$ be a L^2 normed function, which we take as the initial data for the Cauchy-Schrödinger evolution.

With the unitary kernel k_{is} in hand, we can define the pertinent evolution in terms of a convolution $\psi(x,s) := f * k_{is} = \frac{i}{\sqrt{2\pi}} f * (\delta_s - \delta_{-s})] * \mathrm{P}(\frac{1}{x})$. Let us consider

$$\begin{aligned}\frac{i}{2\pi} f * \delta_s * \mathrm{P}\left(\frac{1}{x}\right) &= \frac{i}{2\pi}\mathrm{P}\int_{-\infty}^{+\infty}\frac{f(x-s-y)}{y}\,dy \\ &= \frac{i}{2\pi}\lim_{\epsilon\downarrow 0}\left[\int_{-\infty}^{-\epsilon}\frac{f(x-s-y)}{y}dy \right. \\ &\quad \left. + \int_{\epsilon}^{\infty}\frac{f(x-s-y)}{y}dy\right].\end{aligned} \tag{50}$$

Because of

$$\begin{aligned}\int\left[\frac{1}{1+(x-s-y)^2}\right]\frac{dy}{y} &= A\,\ln\left[\frac{|y|}{\sqrt{1+(x-s-y)^2}}\right] \\ &\quad - A\,(x-s) \\ &\quad \times \arctan(x-s-y),\end{aligned} \tag{51}$$

where

$$A = \frac{1}{1+(x-s)^2}, \tag{52}$$

the definite integrals in (50) read

$$\begin{aligned}\int_{-\infty}^{-\epsilon}\frac{f(x-s-y)}{y}\,dy &= \sqrt{\frac{2}{\pi}}A\left[\ln\frac{\epsilon}{\sqrt{1+(x-s+\epsilon)^2}}\right. \\ &\quad -(x-s)\arctan(x-s+\epsilon) \\ &\quad \left. + \frac{\pi}{2}(x-s)\right]\end{aligned} \tag{53}$$

$$\begin{aligned}\int_{\epsilon}^{\infty}\frac{f(x-s-y)}{y}\,dy &= \sqrt{\frac{2}{\pi}}A\left[-\ln\frac{\epsilon}{\sqrt{1+(x-s-\epsilon)^2}}\right. \\ &\quad + \frac{\pi}{2}(x-s) + (x-s) \\ &\quad \left. \times \arctan(x-s-\epsilon)\right]\end{aligned}$$

and, therefore,

$$\mathrm{P}\int_{-\infty}^{+\infty}\frac{f(x-s-y)}{y}\,dy = \frac{\sqrt{2\pi}(x-s)}{1+(x-s)^2}. \tag{54}$$

By proceeding analogously [with δ_{-s} replacing δ_s in (50)] we find

$$\mathrm{P}\int_{-\infty}^{+\infty}\frac{f(x+s-y)}{y}\,dy = \frac{\sqrt{2\pi}(x+s)}{1+(x+s)^2}. \tag{55}$$

All that finally implies [remembering that $f(x) = \sqrt{\frac{2}{\pi}}/(1+x^2) = \psi(x,0)$],

$$\psi(x,s) = [\exp(-isH)\,f](x) = \tfrac{1}{2}\,[f(x+s) + f(x-s)] + \frac{i}{2}\,[(x-s)f(x-s) - (x+s)f(x+s)] \quad (56)$$

with an interesting formula for the time development of the probability density,

$$\rho(x,s) := |\psi(x,s)|^2 = (1+s^2)\,\sqrt{\rho_0(x+s)\rho_0(x-s)} \quad (57)$$

$$\rho_0(x) = |\psi(x,0)|^2 = [f(x)]^2 = \frac{2}{\pi}\frac{1}{(1+x^2)^2}.$$

Notice that by a direct evaluation, we can check the normalization identity $\int_{-\infty}^{+\infty}\rho(x,s)dx = 1$.

Now, we can address the problem of whether the stochastic process, implying the propagation (57) of the probability density, is a Markov process.

B. The nonstationary Cauchy-Schrödinger stochastic process is not Markov

1. Candidates for the transition probability density

We are interested in representing the time evolution of $\rho(x,t)$, (57), in the integral (probabilistic transport rule-looking) form

$$\rho(x,t) = \int_R p(y,0,x,t)\,\rho_0(y)\,dy = \int_R p(y,s,x,t)\,\rho(y,s)\,dy \quad (58)$$

without bothering at the moment whether we can assign to $p(y,0,x,t)$ or $p(y,s,x,t)$, $s<t$, any true meaning of the transition probability density of a stochastic process.

Since $\rho(x,t)$ *is* a probability density, we can evaluate its characteristic function (Fourier transform [30]): $\phi(p,t) := \sqrt{2\pi}\hat{\rho}(p,t)$. Of course, $\phi(p,0) = \hat{f}(p) = \int_R[1/\pi(1+x^2)]\exp(ipx)\,dx = \exp(-|p|)$ is a characteristic function as well.

By observing that $[\psi(x,0) := \psi_0(x)]$

$$\tfrac{1}{2}\,[\psi_0(x+t)+\psi_0(x-t)] = \sqrt{\frac{2}{\pi}}\,\frac{(1+t^2)+x^2}{[1+(x+t)^2][1+(x-t)^2]} \quad (59)$$

while

$$\tfrac{1}{2}[(x-t)\psi_0(x-t) - (x+t)\psi_0(x+t)] = t\sqrt{\frac{2}{\pi}}\,\frac{-(1+t^2)+x^2}{[1+(x+t)^2][1+(x-t)^2]} \quad (60)$$

we arrive at [cf. (56) for the definition of $\psi(x,t)$]

$$\mathrm{Re}\,\psi(x,t) - \frac{1}{t}\,\mathrm{Im}\,\psi(x,t) = 2\sqrt{\frac{2}{\pi}}\,\frac{(1+t)^2}{[1+(x+t)^2][1+(x-t)^2]} = \sqrt{2\pi}\,\rho(x,t). \quad (61)$$

In view of (61), we have

$$\hat{\rho}(p,t) = \frac{1}{\sqrt{2\pi}}\int_R \exp(ipx)\,\rho(x,t)\,dx = \frac{1}{\sqrt{2\pi}}\int_R \exp(ipx)\left[\mathrm{Re}\,\psi(x,t) - \frac{1}{t}\,\mathrm{Im}\,\psi(x,t)\right]dx, \quad (62)$$

where

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_R \exp(\pm iqx - |q| - i|q|t)\,dq. \quad (63)$$

So that

$$\frac{1}{\sqrt{2\pi}}\phi(p,t) = \hat{\rho}(p,t) = \frac{1}{\sqrt{2\pi}}\exp(-|p|)\left[\cos(t|p|) + \frac{1}{t}\sin(t|p|)\right]. \quad (64)$$

Finally, in view of $\hat{\rho}_0(p) = (1/\sqrt{2\pi})(1+|p|)\exp(-|p|)$, we find $(t \geq 0)$

$$\hat{\rho}(p,t) = \hat{\rho}_0(p)\,\frac{\cos(t|p|) + \frac{1}{t}\sin(t|p|)}{1+|p|} \quad (65)$$

and $(0 < s < t)$

$$\hat{\rho}(p,t) = \hat{\rho}(p,s)\,\frac{\cos(t|p|) + \frac{1}{t}\sin(t|p|)}{\cos(s|p|) + \frac{1}{s}\sin(s|p|)}. \quad (66)$$

Ignoring the issues of existence and positive definiteness of Fourier transformed integrands, we can proceed in the standard way,

$$\rho(x,t) = \frac{1}{\sqrt{2\pi}}\int_R \exp(-ipx)\,\hat{\rho}(p,t)\,dp = \int_R p(y,s,x,t)\rho(y,s)dy = \int_R\left\{\frac{1}{2\pi}\int_R \exp[ip(y-x)] \times \frac{\cos(t|p|)+\frac{1}{t}\sin(t|p|)}{\cos(s|p|)+\frac{1}{s}\sin(s|p|)}\,dp\right\}\rho(y,s)\,dy, \quad (67)$$

where the formal, homogeneous in space "transition probability density" $p(y,s,x,t)$, $s<t$, trivially satisfies the formal Chapman-Kolmogorov identity: $p(y,s,x,t) = \int_R p(y,s,z,u)p(z,u,x,t)dz$ $s<u<t$.

To go beyond formal arguments,

(i) We need to prove that the function $p(y,0,x,t)$, $t \geq 0$ is a well defined transition probability density of the stochastic process transporting the initial (time 0) density into the terminal (time $t \geq 0$) one.

(ii) We need to demonstrate that the would-be transition density $p(y,s,x,t)$, $s < t$, is a well defined probability measure, and actually we shall prove that it is *not*, which excludes the Markov property for the stochastic process under consideration in agreement with our previous conclusions of Sec. II.

2. Existence of the probabilistic transport from time 0 to time $t \geq 0$

In agreement with (65)–(67), we can introduce an integral kernel $p(y,0,x,t)$ effecting the transport of $\rho(y,0)$ into $\rho(x,t)$, $t \geq 0$,

$$p(y,0,x,t) = \frac{1}{2\pi}\int_R \exp[ip(y-x)] \times \frac{\cos(t|p|) + \frac{1}{t}\sin(t|p|)}{1+|p|}dp. \qquad (68)$$

At the moment, its status as the transition probability density of a stochastic process is not settled. We must know whether its Fourier transformed integrand is positive-definite function and satisfies a normalization identity $\int_R p(y,0,x,t)\,dx = 1$ for all times $t \geq 0$.

In view of the homogeneity in space we observe that $p(y,0,x,t) = p(0,0,x-y,t)$ and so we pass to the notation

$$\sqrt{2\pi}\, p(x,t) = \left[\frac{\cos(tp)}{1+|p|} + \frac{|p|}{1+|p|}\frac{\sin(tp)}{tp}\right]^{\vee}. \qquad (69)$$

The function

$$g(x) := \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+|p|}\right)^{\vee}(x) = \frac{1}{2\pi}\int_R \exp(-ipx)\,\frac{dp}{1+|p|} \qquad (70)$$

will play a distinguished role in what follows. Indeed, because

$$[\cos(tp)]^{\vee} = \sqrt{\frac{\pi}{2}}(\delta_t + \delta_{-t}), \qquad (71)$$

$$\left(\frac{\sin(tp)}{tp}\right)^{\vee} = \frac{1}{t}\sqrt{\frac{\pi}{2}}\,\chi_{[-t,t]},$$

where $\chi_{[-t,t]} = 1$ if $|x| \leq t$, and $= 0$ if $|x| > t$, we can rewrite $p(x,t)$ as follows:

$$p(x,t) = \tfrac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2t}\chi_{[-t,t]}(x) - \frac{1}{2t}(g * \chi_{[-t,t]})(x). \qquad (72)$$

In view of (72), properties of $p(x,t)$ are completely determined by those of $g(x)$.

Equation (70) suggests that $g(x)$ itself might be a probability density. For this to hold true, $1/(1+|p|)$ must be a characteristic function, and we are in the framework covered by the classic Bochner theorem [8,31,32]. Our integrand $1/(1+|p|)$ is continuous on R and equals 1 when $|p| = 0$. It is well known [8] that such a function is a characteristic function if and only if it is positive definite. To be a positive definite function $h(p)$ must satisfy the inequality

$$\sum_{i,j,=1}^{n} h(p_i - p_j)\,\lambda_i\bar{\lambda}_j \geq 0 \qquad (73)$$

for any finite sequence of complex numbers $\lambda_1, \lambda_2, ..., \lambda_n$, any sequence of points $p_1, p_2, ..., p_n$ in R , and any $n = 1, 2, \ldots$.

The identity (73) is trivially satisfied by $h(p) = 1/(1+|p|)$, and as a consequence it is a characteristic function of the probability density $g(x)$, (70). Then $\int_R g(x)dx = 1$ follows.

The function $p = p(x,t)$ is real and even in x, which allows us to consider a nonnegative semiaxis $x \geq 0$ for the moment. Let $0 \leq x < t$. Because $\int_R g(x)dx = 1$, we find that

$$\int_{x-t}^{x+t} g(y)dy < 1 \Longrightarrow \frac{1}{2t} - \frac{1}{2t}(g * \chi_{[-t,t]})(x) > 0 \qquad (74)$$

which implies that

$$p(x,t) = \tfrac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2t} - \frac{1}{2t}(g * \chi_{[-t,t]})(x) > 0 . \qquad (75)$$

If $x > t$ it suffices to consider

$$\frac{1}{2t}\int_{x-t}^{x+t} g(y)dy < \frac{g(x+t)+g(x-t)}{2} \Longrightarrow \tfrac{1}{2}[g(x+t) + g(x-t)] - \frac{1}{2t}[g * \chi_{[-t,t]}(x)] > 0 \Longrightarrow p(x,t) > 0. \qquad (76)$$

Hence, $p(x,t)$, (72) is strictly positive on R, and moreover it has a unit normalization. This implies that $p(0,0,x-y,t)$ is a well defined probability density.

Let us add that $\lim_{|x|\to\infty} p(x,t) = 0$ and $\lim_{t\downarrow 0} p(x,t) = \delta(x)$. At the points $x = \pm t$, $p(x,t)$ develops singularities in view of $\lim_{x\downarrow 0} g(x) = \infty$. The location of the singularities is quite significant, since they make a clear distinction between the random propagation regimes with jumps of size less than t, and those of size greater than t, for each terminal time instant of the evolution of $\rho(x,t)$.

Notice also that as a suitable probability density $p(x,t)$ leads to several finite moments: $\int_R xp(x,t)dx = 0$, $\int_R x^2 p(x,t)dx = t^2$. On the other hand, the long-tailed Cauchy transition density (34) has no finite moments at all.

3. Violation of Markov property

In view of our previous discussion we have specified consistent distribution functions for the process, which in principle permits its construction ([8], Chap. 15). For the Markov process what needs to be specified [8,32,33] are the transition probabilities for all intermediate times of the considered evolution and the initial distribution (which is given in our case). We shall investigate the existence of the *consistent* transition probability densities for the process in question.

If the Cauchy-Schrödinger process is to be Markov, the integral kernel

$$p(y,s,x,t) = \frac{1}{2\pi}\int_R \exp[ip(y-x)] \times \frac{\cos(t|p|) + \frac{1}{t}\sin(t|p|)}{\cos(s|p|) + \frac{1}{s}\sin(s|p|)} \tag{77}$$

must be the Fourier inversion of a characteristic function. A necessary condition for

$$h(p,s,t) = \frac{\cos(t|p|) + \frac{1}{t}\sin(t|p|}{\cos(s|p|) + \frac{1}{s}\sin(s|p|)} \tag{78}$$

to be a characteristic function is the positive definiteness of $h(p,s,t)$ as a function of $p \in R$ for all intermediate time instants $0 < s < t$.

We shall prove that for each terminal time instant t we can single out earlier time instants s, such that $h(p,s,t)$ is *not* a positive-definite function. As a consequence, the intermediate propagation cannot be Markov.

To this end, let us notice that for a fixed t, and $s > 0$ we can rearrange the denominator in (78)

$$\cos(s|p|) + \frac{1}{s}\sin(s|p|) = \frac{1}{\cos\,\alpha_s}\cos(s|p| - \alpha_s),$$

$$\tan\,\alpha_s := \frac{1}{s}. \tag{79}$$

Notice that

$$|p| = \frac{1}{s}\left(\alpha_s + \frac{2N+1}{2}\pi\right) \Longrightarrow \cos(s|p| - \alpha_s) = 0 \tag{80}$$

for all integer N.

Our $h(p,s,t)$ should satisfy the condition (73). Let us choose the simplest case of $n = 2$ in this formula, and consider $\Sigma_{i,j=1}^{2} h(p_i - p_j)\lambda_i\overline{\lambda}_j \geq 0$. Because of (78), the two by two matrix $h_{ij} := h(p_i - p_j)$, $i,j, = 1,2$ has matrix elements $h_{11} = 1 = h_{22}$ and $h_{12} = h_{21} = M$ with

$$M = \cos(\alpha_s)\frac{\cos(t|p_1 - p_2|) + \frac{1}{t}\sin(t|p_1 - p_2|)}{\cos(s|p_1 - p_2| - \alpha_s)}. \tag{81}$$

To have $\det[h_{ij}] \geq 0$ we need $|M| \leq 1$. This condition can always be violated by choosing any pair p_1, p_2 for which, at a given time s, the numerical value of $|p_1 - p_2| = |p|$ is close to any of those introduced in the formula (80).

Therefore, the considered stochastic process [with the transition mechanism (67)] is not Markov, as anticipated on the basis of arguments of Sec. II.

IV. MEANING OF THE PSEUDODIFFERENTIAL STOCHASTIC PROPAGATION: AN INSIGHT INTO JUMP FEATURES OF THE PROCESS

A. Fokker-Planck equations

The probability density $\rho(x,t)$ [respectively, $\overline{\rho}(x,t)$] was a fundamental entity in all our previous considerations: either (i) providing the input-output statistics data for the Schrödinger-random dynamics reconstruction problem, or (ii) providing the time evolution of the probability measure for the whole time interval of interest, so that the transition probability densities could be sought for.

In the Gaussian case of Remark 1 we dealt with the temporal evolution of the probability density given in its traditional Fokker-Planck form appropriate for Markov diffusion processes. In connection with the pseudodifferential (free noise) dynamics, Eqs. (24) and (25) are an obvious extension of the previous notion to a class of jump processes. We shall extend the usage of the name Fokker-Planck equation to any first order in time differential equation determining the space-time properties of $\rho(x,t)$ or $\overline{\rho}(x,t)$.

Let us investigate the time development of $\overline{\rho}(x,t) = \theta(x,t)\theta_*(x,t)$, where $\theta(x,t)$, $\theta_*(x,t)$ come out as solutions of the temporally adjoint pair of equations of the form

$$\partial_t\theta = H\theta - V\theta, \tag{82}$$
$$\partial_t\theta_* = -H\theta_* + V\theta_*,$$

with the initial (terminal) data $f(x), g(y)$ of the Schrödinger problem (19),(20) and a Feynman-Kac potential V. Then, in view of (37) and $\theta = \exp(R+S)$, $\theta_* = \exp(R-S)$, we get an evolution equation for the probability density [the usage of the overbar was explained preceding (14)]

$$\partial_t\overline{\rho}(x,t) = \theta_*(x,t)\,(H\theta)(x,t) - \theta(x,t)\,(H\theta_*)(x,t)$$
$$= \int_R\Bigg[-\theta_*(x,t)\theta(x+y,t) + \theta(x,t)\theta_*(x+y,t) +2\overline{\rho}(x,t)\nabla S(x,t)\frac{y}{1+y^2}\Bigg]d\nu(y). \tag{83}$$

Following the traditional recipes when dealing with Lévy measures [8], let us consider an open neighborhood of the origin $|\epsilon| \ll 1$. Instead of integrating over all possible jump sizes, let us integrate over jumps of size $|y| > \epsilon > 0$. The removal of this lower bound as $\epsilon \to 0$ will eventually amount to evaluating the principal value of the integral. In case $\epsilon > 0$, we can safely remove the compensating term including $y/(1+y^2)$ from the integral, and restrict considerations to the contribution from the first two terms only.

Our purpose is to establish a connection between (83) and the conventional theory of jump stochastic processes, as developed in [33]. Integrating over a Borel set $A \subset R$, $x \in A$ we get

$$\int_A dx \int_{|y|>\epsilon} [-\theta_*(x,t)\theta(x+y,t) + \theta(x,t)\theta_*(x+y,t)] d\nu(y)$$

$$= \int_R dx \int_{|y|>\epsilon} \chi_A(x) \left[-\overline{\rho}(x,t)\, \frac{\theta(x+y,t)}{\theta(x,t)} + \overline{\rho}(x+y,t)\, \frac{\theta(x)}{\theta(x+y)} \right] d\nu(y)$$
$$= \int_R dx\, \overline{\rho}(x,t) \int_{|y|>\epsilon} \frac{\theta(x+y,t)}{\theta(x,t)} [\chi_A(x+y) - \chi_A(x)]\, d\nu(y), \tag{84}$$

where we interchanged the order of integrations, and made appropriate adjustments of integration variables ($x \to x-y$ and $y \to -y$), while exploiting the property $d\nu(-y) = -d\nu(y)$ of measures (35),(36); $\chi_A(x)$ is an indicator function of the Borel set $A \subset R$, equal to 1 when $x \in A$ and 0 otherwise.

In the present case, (82), we deal with a Markov process with transition probability densities given for arbitrary time instants: $\overline{\rho}(x,t) = \int_R p(y,s,x,t)\overline{\rho}(y,s)dy$, $s < t$. By invoking the standard wisdom about jump Markov processes [33], and exploiting $\lim_{t\downarrow s} p(y,s,A,t) = \chi_A(y)$, for any Borel set $A \subset R$ away from $(-\epsilon, +\epsilon)$, we can define the jump process running with jumps of size $|y| > \epsilon > 0$. It should be viewed as an approximation of the original stochastic process governed by (83), with the initial data $\overline{\rho}(x,0)$ common for both

$$\partial_t \overline{\rho}_\epsilon(A,t) = \int_R q(x,t,A)\, \overline{\rho}_\epsilon(x,t)\, dx + \langle v\rangle_A(t) \int_{|y|>\epsilon} \frac{y}{1+y^2} d\nu(y), \tag{85}$$

where

$$q(x,t,A) := \lim_{u\downarrow t} \frac{1}{u-t}\, [p(x,t,A,u) - \chi_A(x)] = \int_{|y|>\epsilon} \frac{\theta(x+y,t)}{\theta(x,t)} [\chi_A(x+y) - \chi_A(x)] d\nu(y),$$
$$\langle v\rangle_A(t) := \int_A \overline{\rho}(x,t)\, [2\nabla S(x,t)]\, dx. \tag{86}$$

Here $q(x,t,A) \geq 0$ for all x that are *not* in A, in agreement with [33]. We have also introduced a pseudodifferential counterpart of the current velocity field $v(x,t) = 2\nabla S(x,t)$, previously attributed to diffusion processes (cf. Remark 1), where the probability conservation law (a continuity equation in another lore) $\partial_t \rho = -\nabla(v\rho)$ plays the role of the Fokker-Planck equation.

Notice, that in the particular case of $\theta(x,t) \equiv 1$ for all x,t, and $V = 0$, Eq. (82) reduces to the free noise situation covered by the Fokker-Planck equations (24) and (25). Then, $q(t,x,A) = \int_{|y|>\epsilon}[\chi_A(x+y) - \chi_A(x)]d\nu(y)$, while $R = -S$, $\overline{\rho} = \exp(2R) = \theta_*$ implies $\langle v\rangle_A(t) = -\overline{\rho}(x,t)|_a^b$ where $[a,b] := A \subset R$.

Now, let us address the Fokker-Planck equation for the pseudodifferential-Schrödinger dynamics case, which we consider in the form analogous to (82); see also (1) for comparison

$$i\partial_t \psi = H\psi + V\psi, \tag{87}$$

$$i\partial_t \overline{\psi} = -H\overline{\psi} - V\overline{\psi}.$$

We reemphasize that to define the probability density $\rho(x,t) = |\psi(x,t)|^2$ one actually employs solutions of the time adjoint pair of Schrödinger equations.

In view of (87), the pseudodifferential continuity equation follows:

$$\partial_t \rho(x,t) = -i[\overline{\psi}(x,t)(H\psi)(x,t) - \psi(x,t)(H\overline{\psi})(x,t)]$$
$$= -i \int_R \left[-\overline{\psi}(x,t)\psi(x+y,t) + \psi(x,t)\overline{\psi}(x+y,t) + 2i\rho(x,t)\nabla S(x,t) \frac{y}{1+y^2} \right] d\nu(y). \tag{88}$$

Our next step is a repetition of the procedures behind (84), which implies

$$\partial_t \rho(x,t) = \int_R \left\{ 2\mathrm{Im}[\psi(x,t)\overline{\psi}(x+y,t)] + 2\rho(x,t)\nabla S(x,t) \frac{y}{1+y^2} \right\} d\nu(y)$$

$$\Downarrow$$

$$\int_A dx \int_{|y|>\epsilon} 2\mathrm{Im}[\psi(x,t)\overline{\psi}(x+y,t)]$$
$$= \int_R dx \int_{|y|>\epsilon} \chi_A(x) 2\rho^{1/2}(x,t)\rho^{1/2}(x+y,t)\,\sin[S(x,t)-S(x+y,t)]d\nu(y)$$
$$= \int_R \rho(x,t)dx \int_{|y|>\epsilon} \frac{\rho^{1/2}(x+y)}{\rho^{1/2}(x)} \sin[S(x+y,t)-S(x,t)][\chi_A(x+y)-\chi_A(x)]d\nu(y). \quad (89)$$

So, a counterpart of (85) reads

$$\partial_t \rho_\epsilon(A,t) = \int_R q(x,t,A)\rho_\epsilon(x,t)dx + \langle v\rangle_A(t) \int_{|y|>\epsilon} \frac{y}{1+y^2} d\nu(y), \quad (90)$$

where, however

$$q(x,t,A) := \int_{|y|>\epsilon} \mathrm{Im}\left[\frac{\psi(x+y,t)}{\psi(x,t)}\right] [\chi_A(x+y) - \chi_A(x)]d\nu(y) \quad (91)$$

no longer can be derived from transition probability densities of the process, as in the previous discussion (86), because in general our process is *not* Markovian. At least in the case of nonstationary dynamics, the only transition probability density that is at our disposal connects an initial instant of the evolution with any later one. In fact, we might even not be sure that $q(x,t,A)$ is a well defined probabilistic object, because of the presence of $\sin[S(x+y,t)-S(x,t)]$ in the integrand. At this point an observation of [19] helps. Namely, in view of the identity

$$\int_R dx \int_{|y|>\epsilon} |\psi(x+y,t)\psi(x,t)|[\chi_A(x+y) - \chi_A(x)]d\nu(y) = 0 \quad (92)$$

valid for Borel sets $A \subset R$, which are away from $(-\epsilon, +\epsilon)$, we can always pass from (89) to the rearranged form of (91)

$$q(x,t,A) = \int_{|y|>\epsilon} \left\{ \left|\frac{\psi(x+y,t)}{\psi(x,t)}\right| + \mathrm{Im}\left[\frac{\psi(x+y,t)}{\psi(x,t)}\right] \right\} \times[\chi_A(x+y) - \chi_A(x)]\, d\nu(y) \quad (93)$$

implying that $q(x,t,A)$ is positive for all x that are *not* in A, as should be the case [33].

B. The jump processes toolbox

To have a better insight into stochastic jump processes associated with the Fokker-Planck evolutions (83), (85) and (88), (90), respectively, some further knowledge of the general theory is necessary. We shall try to minimize the level of sophistication by invoking arguments based on the exploitation of the standard Poisson process.

A random variable X taking discrete values $0, y, 2y, 3y, \ldots$, with $y > 0$ is said to have Poisson distribution $\mathcal{P}(\lambda)$, $\lambda \geq 0$ with jump size y, if the probability of $X = ky$ is given by $P(X = ky) = \frac{\lambda^k}{k!}\exp(-\lambda)$. The characteristic function of $\mathcal{P}(\lambda)$ reads

$$E[\exp(ipX)] = \exp[\lambda(e^{ipy}-1)] = \sum_0^\infty e^{-\lambda}\frac{\lambda^k}{k!} e^{ik(py)} = \sum_0^\infty e^{ik(py)}\, P(X=ky) \quad (94)$$

and its first moment equals $E[X] = \lambda$. Notice that $P(X=0) = \exp(-\lambda)$, hence the numerical value of $\lambda \geq 0$ tells us what is the probability of a jump *not* to occur at all for a given Poisson process. For the Poisson random variable with values $b + ky, k = 0, 1, \ldots$, we would get

$$E[\exp(ipX)] = \exp[ibp + \lambda(e^{ipy}-1)]\,. \quad (95)$$

If we consider n independent random variables $X_j, 1 \leq j \leq n$ such that X_j has Poisson distribution $\mathcal{P}(\lambda_j)$ with jump size y_j, then a new process X can be introduced with the distribution of $X_1 + \cdots + X_n$ so that its characteristic function reads

$$E[\exp(ipX)] = \exp\left[\sum_{j=1}^n \lambda_j(e^{ipy_j}-1)\right]. \quad (96)$$

The exponent in (96) would include an additional term $ip\sum_1^n b_j$ if nonrandom shifts of each jump ky_j by b_j would be allowed.

We can admit not only jumps of fixed magnitudes $y_1, \ldots, y_n$ but also jumps covering an arbitrary range in R_+. Let the distribution function of the magnitude of the jump be $P(x<y) = \mu(y)$. A possible generalization of (96) to this case is

$$E[\exp(ipX)] = \exp\left[\int_{R_+} (e^{ipy}-1)d\mu(y)\right] \quad (97)$$

assuming that the integral in the exponent exists. Notice that (96) is recovered, if we set

$$d\mu(y) = \sum_{j=1}^n \lambda_j \delta(y-y_j)\, dy. \quad (98)$$

The convergence of the exponent in (97) may be jeopardized in cases when jumps of very small amplitude are allowed to occur very often, while we take for granted that jumps of very large size seldom happen. On the other hand [8], for any Borel set $A \subset R$ bounded away from the origin, the process X_A of jumps bounded by A, has the characteristic exponent $\int_A (e^{ipy} - 1) d\mu(y)$, and the expected number $E_A[X]$ of jumps of size A is equal to $\mu(A)$. We can say that the processes of jumps of different sizes proceed independently of one another, and the jump process of jumps of size $[y, y + \triangle y)$, $\triangle y \ll 1$ contributes a Poisson component with exponent function approximately equal to $(e^{ipy} - 1)\mu([y, y + \triangle y))$. At the moment the processes have only upward jumps, hence their sample paths are nondecreasing.

For a process with the characteristic exponent $-F(p)$, (23), we can consider its restriction to upward jumps of size $y > \epsilon > 0$

$$\phi_\epsilon^+(p) = \int_{y>\epsilon} \left[e^{ipy} - 1 - \frac{ipy}{1+y^2} \right] d\nu(y) = \int_{y>\epsilon} [e^{ipy} - 1] d\nu(y) - ipb_\epsilon^+, \quad (99)$$

$$b_\epsilon^+ = \int_{y>\epsilon} \frac{y}{1+y^2} d\nu(y).$$

Clearly, we deal here with a process of the type considered before, and might try to isolate contributions from jumps of size $[y, y + \triangle y)$ by considering a coarse graining of a Borel set A of interest. A formal substitution of (98) in (99), with $d\mu$ replacing $d\nu$, gives rise to

$$E[\exp(ipX)] = \exp\left\{ \sum_{j=1}^n \left[\lambda_j (e^{ipy_j} - 1) - ip\, \frac{\lambda_j y_j}{1+y_j^2} \right] \right\} \quad (100)$$

to be compared with (95).

Further specializing the problem to relativistic Hamiltonians, we notice that the corresponding Lévy measures $\nu_0(y)$ and $\nu_m(y)$, (35),(36) are even under space reflections, hence $d\nu(-y) = -d\nu(y)$ in these cases. Consequently, we can easily extend our discussion to jumps of all sorts in R, i.e., y can take values in both R_+ and R_-, with the only restriction to be observed that $|y| > \epsilon > 0$. Notice that we shall deal with two processes, which run separately with either positive or negative jumps, and there is no common jump point for them. This fact means that they are independent components of the more general process defined by

$$\phi_\epsilon(p) = \int_{|y|>\epsilon} [e^{ipy} - 1] d\nu(y) - ipb_\epsilon, \quad (101)$$

where in the case of the Lévy measures given in (35),(36) the deterministic term identically vanishes in view of

$$b_\epsilon = b_\epsilon^+ + b_\epsilon^- = \int_{y>\epsilon} \frac{y}{1+y^2} d\nu(y) + \int_{y<-\epsilon} \frac{y}{1+y^2} d\nu(y) \equiv 0 \quad (102)$$

All our steps (94)–(101) involved the fact that we deal with infinitely divisible probability laws. One additional important property about them is that [8] if $\exp \phi(p)$ is a characteristic function of a given probability distribution, then $[\exp \phi(p)]^t = \exp[t\phi(p)]$, $t > 0$ is likewise a characteristic function of an infinitely divisible probability law again. This feature readily extends our discussion to time-dependent stochastic processes (time homogeneous with independent increments, associated by us with the free noise). Obviously, for such processes $E[\exp\{ipX(t)\}] = \exp[t\phi(p)]$ while $E_A[X(t)] = t\nu(A)$, and our previous arguments retain their validity with respect to

$$E[\exp\{ipX(t)\}]_\epsilon = \exp[t\phi_\epsilon(p)] = \exp\left[t \int_{|y|>\epsilon} (e^{ipy} - 1) d\nu(y) \right] \quad (103)$$

$$\partial_t \overline{\rho}(x,t) = -(H\overline{\rho})(x,t) \Longrightarrow \partial_t \overline{\rho}_\epsilon(A,t) = \int_R dx \left\{ \int_{|y|>\epsilon} [\chi_A(x+y) - \chi_A(x)] d\nu(y) \right\} \overline{\rho}_\epsilon(x,t).$$

Remark 4. In fact, (102) means that the Fokker-Planck equations (85),(90), if specialized to Lévy measures (35),(36), involve exclusively the integral term on their right-hand side

$$\partial_t \overline{\rho}_\epsilon(A,t) = \int_R \overline{q}(x,t,A) \overline{\rho}_\epsilon(x,t) dx, \quad (104)$$

$$\partial_t \rho_\epsilon(A,t) = \int_R q(x,t,A) \rho_\epsilon(x,t) dx,$$

where an overbar distinguishes between probabilistic quantities characterizing different families of stochastic jump processes (86) and (91), respectively. Let us emphasize that the simplification (104) occurs only in the $|y| > \epsilon > 0$ jumping size regime. The real role of two terms in (102) is to compensate the divergent contributions from the Lévy measure when the principal value integral $\epsilon \to 0$ limit is considered; then the *standard* jump process theory (104) does not apply. Anyway, those two terms are irrelevant for any $\epsilon > 0$, irrespectively of how small ϵ is.

One might expect that an infinitesimal (jump size) surrounding of the origin gives a dominant jump contribution to the process. However, generally it is not the case: explicit solutions (34),(36) and (57),(72) indicate that for times $t > 0$ the leading contribution does not necessarily come from jumps of infinitesimal sizes.

V. RELATIVISTIC WAVE EQUATIONS AND ASSOCIATED STOCHASTIC PROCESSES

We mentioned before that solutions of our pseudodifferential-Schrödinger equations solve the relativistic wave (or matter wave in the Klein-Gordon case) equations as well; see (29)–(32). Since each particular solution has an undoubted probabilistic significance, we can reanalyze the old-fashioned problem [25,34] of a“single-particle interpretation" for free Klein-Gordon solutions and analyze the same problem for the D'Alembert equation solutions, from a different perspective; see also [19]. As well, we can benefit from relativistic covariance properties of wave equations to understand how the pseudodifferential-Schrödinger stochastic processes comply with the principles of special relativity.

To begin with, let us consider the Klein-Gordon equation for a particle of mass $m > 0$,

$$(\Box + m^2)\phi(\vec{x},t) = 0. \tag{105}$$

The space-time metric signature is $\mathrm{diag}(g_{\mu\nu}) = (1,-1,-1,-1,)$, and the system of units is $\hbar = c = 1$. In view of the polar (Madelung) decomposition of the complex wave function, $\phi(\vec{x},t) = \exp[R(\vec{x},t) + iS(\vec{x},t)]$, we can split (105) into two real equations,

$$(\partial_\mu S)(\partial^\mu S) = m^2 + \frac{\Box \rho^{1/2}}{\rho^{1/2}}, \tag{106}$$

$$\partial_\mu j^\mu = 0,$$

$$j^\mu := \frac{1}{2i}[\overline{\phi}\partial^\mu \phi - \phi \partial^\mu \overline{\phi}] = -\rho(\partial^\mu S),$$

where $\rho(\vec{x},t) = |\phi(\vec{x},t)|^2 = \exp[2R(\vec{x},t)]$.

We can handle the $m = 0$ case corresponding to the D'Alembert equation in the same way, and the only change in formulas (106) would be the absence of the m^2 contribution.

We have noticed before, (32), that if $\psi(\vec{x},t)$ is a solution of the pseudodifferential-Schrödinger equation $i\partial_t \psi = [\sqrt{-\triangle + m^2} - m]\psi$, then $\tilde{\psi}(\vec{x},t) = \psi(\vec{x},t)\exp(-imt)$ is a positive energy solution of the free Klein-Gordon equation $(\Box + m^2)\tilde{\psi}(\vec{x},t) = 0$, since we surely have $i\partial_t \tilde{\psi} = \sqrt{-\triangle + m^2}\tilde{\psi}$. It is clear that the time adjoint Schrödinger equation refers to negative energy solutions of the Klein-Gordon equation. Notice that we need both positive and negative energy solutions to create (upon normalization) a probability density $\rho(\vec{x},t) = \psi(\vec{x},t,)\overline{\psi}(\vec{x},t)$.

Remark 5. At this point it is useful to emphasize that the timelike component $j^0(\vec{x},t)$ of the current $j^\mu(\vec{x},t)$ is *not* a probability density itself; by wrongly [35,36] and per force assuming that it generally would be the case, all known paradoxes and difficulties underlying the refutation of the Klein-Gordon equation as the proper relativistic generalization of its nonrelativistic Schrödinger partner are revealed [25,34]. The positive energy spectrum is not just correlated with positive (negative) values of $j_0(\vec{x},t)$, although one can establish such a correlation for the total charge $e\int j_0(\vec{x},t)d^3x$ [25,37] by assuming that $ej_0(\vec{x},t)$ is interpreted as the charge density. Even then, a clean partition of the positive and negative energy spectra into sets associated, respectively, with particles and antiparticles distinguished by the sign of the charge density is impossible.

Remark 6. The subject of our considerations is essentially a probabilistic analysis of relativistic quantum mechanics [38]. In view of our previous discussion it is clear that $\rho(\vec{x},t) = |\psi(\vec{x},t)|^2$ is a probability density of a well defined stochastic process, which is non-Markovian in nonstationary situations. Then, for a general Borel set $A \subset R^3$ we have defined a measure $\rho(A,t)$ telling us what is the probability for a jump to have its size matching a point $\vec{y} \in A$, at time t. We are not inclined to think that a concrete jump refers to an actual "physical particle" that jumps in space. In nonrelativistic quantum mechanics, a standard interpretation of $\rho(\vec{x},t)(\triangle x)^3$ as a probability to locate a *physical* particle in a cube of volume $(\triangle x)^3$ seems to be consistent. On the contrary, in relativistic quantum mechanics, the notions of position and localizability and their relation to any experimental determination of the physical particle position have been a subject of vigorous disagreements and no general consensus has been reached. An acceptance of the Newton-Wigner localization in the configuration-space approach to relativistic quantum theory implies the general breakdown of causality [39–44] and inevitably implies superluminal effects (instantaneous spreading of a localized wave packet). It is by no means a surprise if one carefully looks into the jump process features as revealed in the present paper. Let us also mention that the Newton-Wigner localization of mass $m = 0$ particles is in general impossible (well-known exceptions are massless particles of spin 0 and massless spin 1/2 particles possessing two helicity states), while we know how to assign the probability density notion, and hence a probability measure $\rho(A,t)$ to a class of solutions of the D'Alembert equation, see e.g., the arguments of Sec. III, albeit possibly with no connection to any [45,46] "position operator" notion.

Each *scalar* positive energy solution $\phi(\vec{x},t)$ of the free Klein-Gordon equation (105) can be represented [34] in the manifestly Lorentz covariant form

$$\phi(\vec{x},t) = \frac{\sqrt{2}}{(2\pi)^{3/2}} \int d^4k\, e^{(-ik_\mu x^\mu)}\, \delta(k_\mu k^\mu - m^2) \times \Theta(k_0)\, \Phi(k_0,\vec{k}), \tag{107}$$

where $k := (k_0,\vec{k})$, $k_\mu k^\mu := k_0^2 - \vec{k}^2$, $\Phi(k)$ is a scalar and $\Theta(k_0)$ is the Heaviside function equal to 1 if $k_0 > 0$ and to 0 otherwise. The representation (107) extends to all solutions of $i\partial_t\tilde{\psi} = \sqrt{-\triangle + m^2}\tilde{\psi}$, and upon changing $k_0 \to -k_0$ in $\Theta(k_0)$ followed by a complex conjugation of (108), to solutions of the time adjoint equation as well. It implies that general solutions of those pseudodifferential-Schrödinger equations form Lorentz invariant subspaces in the linear space of all solutions to the free Klein-Gordon equation.

However, we cannot directly infer from the above facts any information about how a given pseudodifferential-Schrödinger stochastic jump process is perceived by different relativistic observers. For example the *normalization* of Schrödinger wave functions is not a relativistically covariant notion.

At this point we adopt the standard definition of the Klein-Gordon scalar product [34]:

$$(\phi_1,\phi_2) : = \int_{R^3} d^3x\, [\overline{\phi_1}\sqrt{-\triangle + m^2}\phi_2 + (\sqrt{-\triangle + m^2}\overline{\phi_1})\phi_2] \tag{108}$$

which is independent of the specific spacelike surface of integration. Both positive and negative energy solutions are covered in this definition, albeit separately, with no superposition. The integrand in (108) should be compared with the timelike component $j^0(x)$ of the conserved four-current $j^\mu(x)$, Eq. (106).

Since the Newton-Wigner position operator

$$[\hat{\vec{x}}_{NW}\phi](x) \equiv i\left[\nabla_{\vec{k}} - \frac{\vec{k}}{2(\vec{k}^2+m^2)}\Phi\right](k_0,\vec{k}) \qquad (109)$$

$k_0 = \sqrt{\vec{k}^2+m^2}$, $j = 1,2,3$, see (107), is Hermitian with respect to the scalar product (108), one can introduce a covariant [41] localization notion for all positive energy solutions of the free Klein-Gordon (and hence pseudodifferential-Schrödinger) equation [19,34]. Indeed, given a positive energy solution of the free Klein-Gordon equation $\phi(x)$, which we know to solve the pseudodifferential equation as well, then, we can introduce a *new* solution for both of those equations as follows [19]:

$$\phi(x) \to \phi_{NW}(x) := [(-\triangle + m^2)^{1/4}\phi](x). \qquad (110)$$

If we take for granted the Klein-Gordon scalar product normalization, Eq. (108), of $\phi(x)$, we realize that the common solution of the Klein-Gordon and pseudodifferential-Schrödinger equations: $\psi(\vec{x},t) := \phi_{NW}(\vec{x},t)$, may be consistently normalized according to the Schrödinger equation rule: $\rho(x) := |\psi(x)|^2 \Rightarrow \int_{R^3} d^3x\rho(\vec{x},t) = 1$. As a consequence, and in part because of this normalization, we have succeeded to associate the previously investigated stochastic jump process with the Newton-Wigner localization. Clearly, we deal with the probability measure *identifying* the probability that the Newton-Wigner "particle" can be found at time t in the spatial volume A, with the probability of spatial jumps bounded by this volume, at time t,

$$\begin{aligned}\text{Prob}[\vec{X}(t) \in A] &= \int_A \rho(\vec{x},t)d^3x \\ &= \int_A |[(-\triangle + m^2)^{1/4}\phi](\vec{x},t)|^2 d^3x. \qquad (111)\end{aligned}$$

The inhomogeneous orthochronous Lorentz mapping $x' = \Lambda x + a$ ($x'^\mu = \Lambda^\mu_\nu x^\nu + a^\mu, \Lambda^0_0 > 1$) can be associated with the scalar transformation rule for Klein-Gordon wave functions $\phi'(x'_0,\vec{x}') = \phi(x_0,\vec{x}) \Rightarrow \phi'(x) = \phi(\Lambda^{-1}(x-a))$. The transformation acts invariantly in positive and negative energy subspaces of solutions, respectively. This fact implies an extension to pseudodifferential-Schrödinger equations of motion.

The Klein-Gordon equation is form invariant: $(\Box' + m^2)\phi'(x') = (\Box + m^2)\phi(x) = 0$, which allows us to associate with $\phi'(x')$, while normalized according to (108), a pseudodifferential-Schrödinger stochastic process according to (110)

$$\begin{gathered}\tilde{\psi}'(x') := [(-\triangle' + m^2)\phi']^{1/4}(x'),\\ i\partial_{t'}\tilde{\psi}'(\vec{x}',t') = \sqrt{-\triangle' + m^2}\tilde{\psi}'(\vec{x}',t'),\\ \Downarrow \\ i\partial_{t'}\psi'(\vec{x}',t') = \left[\sqrt{-\triangle' + m^2} - m\right]\psi'(\vec{x}',t'),\\ \psi'(\vec{x}',t') := \exp(imt')\tilde{\psi}'(\vec{x}',t').\end{gathered} \qquad (112)$$

The transfer of data about the pseudodifferential-Schrödinger process from one inertial observer to another is completely determined by Eq. (112).

Now, let us consider the first of equations (87) for the choice $H = \sqrt{-\triangle + m^2}$ of the Hamiltonian: $[i\partial_t - (V - m)]\psi = \sqrt{-\triangle + m^2}\psi$, where we can regard a general potential $V - m$ as a timelike component of a four vector. In case of electromagnetic interactions, the presence of the $(-m)$ term can even be attributed to a gauge transformation $A_\mu \to A_\mu + \partial_\mu\chi$ with $\chi = -mt$ and $\vec{A} \equiv \vec{0}$. In particular, the relativistic stability of matter studies [29] of the existence of bound states for the pseudodifferential-Schrödinger Hamiltonians, involve the Coulomb static potential $eA_0(x) = -(Ze^2/r)$, $r := \sqrt{\vec{x}^2}$. (We recall that pseudodifferential Hamiltonian spectral problems are not limited to the electrostatic potential only, but their range of applicability is much wider [18].)

Complex conjugation converts the forward equation into its time adjoint and it is clear that $[-i\partial_t - (V - m)]\overline{\psi} = \sqrt{-\triangle + m^2}\,\overline{\psi}$ holds true. However, *no* immediate connection with the general form of the Klein-Gordon equation in the presence of electromagnetic interactions (charge e particles), $(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu)\phi(x) = m^2\phi(x)$, can be established, in general.

On the other hand, a pedestrian intuition behind the associated notion of a *relativistic atom* is quite helpful for a deeper understanding of the particular rôle of the Lorentz covariance of the Klein-Gordon and D'Alembert equations in the context of *free* pseudodifferential-Schrödinger equations. Namely, the atom itself is always considered to be at rest, with the frame of reference attached to the nucleus, which in turn is a source of an electrostatic field. In this particular frame of reference the pseudodifferential-Schrödinger equation $[i\partial_t - (V - m)]\psi = \sqrt{-\triangle + m^2}\psi$ is defined. The same pertains to the general pseudodifferential-Schrödinger problem with a minimal electromagnetic coupling due to external fields

$$\begin{aligned}&[i\partial_t - (eA^0 - m)]\psi(\vec{x},t)\\ &\quad = \sqrt{\sum_{j=1}^{3}[(i\nabla_j - eA_j)(i\nabla^j - eA^j)] + m^2}\,\psi(\vec{x},t)\end{aligned} \qquad (113)$$

which is a frame-of-reference-dependent notion, see e.g., [47].

Remark 7. A Euclidean version of (113) was investigated in [21] and an explicit construction was given of the Feynman-Kac path integral formula for the corresponding semigroup kernel. It involves paths, and conditional measures over paths, of a time homogeneous Lévy process.

Remark 8. The problem of how a stochastic process can be perceived by different relativistic observers has been considered before [48] in connection with certain (Markov) rotational diffusions on an S_3 manifold [$SU(2)\times SU(2)$ case specialized to spin 1/2], with the Euler angles parametrization established relative to a fixed three-dimensional orthonormal basis.

Our discussion was confined to the mass $m > 0$ case, but in view of the general nonexistence of the Newton-Wigner localization in the mass $m = 0$ case, an extension of previous relativistic covariance arguments needs some care. The present localization is known to be admissible for massless spin 0 particles. As well, we do not literally need the Newton-Wigner–like position operator and the associated notion of localization at a spatial point to invoke a substantial part of the previous arguments. (In fact, a maximal localizability of photons on a circle, hence in a subset of R^2, was established in [45].) As long as (108) is extended to mass $m = 0$ particles as the normalization condition for wave functions, and the Borel set A is never pointwise, we can safely go through (110)–(112).

Remark 9. There have been numerous attempts to associate the Klein-Gordon equation with stochastic processes. In addition to [19] let us mention a number of other attempts [49–57]. None of them can be viewed as a "derivation" of the Klein-Gordon field from certain "first" stochastic principles. Except for [19], all these attempts exploited a formal similarity of Eqs. (106) to local conservation laws shared by nonrelativistic Markov diffusions [1] and to analogous laws in relativistic kinetic theory [49,58,59]. The status of the Markov property has been found disputable, since its violation is implicit if the relativistic invariance of diffusion (Kolmogorov) equations in Minkowski space is required [60–63]. In the present paper, we have found the Markov property admissible only in the case of the measure preserving (stationary case) stochastic jump dynamics; see e.g., Sec. II.

ACKNOWLEDGMENTS

We express our gratitude to Professor G. G. Emch for a helpful discussion. R. O. was supported by the KBN through Grant No. 2 P302 057 07. P.G. would like to thank the Kosciuszko Foundation for financial support.

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