PHYSICAL REVIEW E VOLUME 51, NUMBER 5 MAY 1995

Estimating perturbative coefficients in quantum field theory and statistical physics

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(Received 4 March 1994)

We present a method to estimate perturbative coefficients in quantum field theory and statistical physics. We show that our method works in a large number of cases and in a wide variety of areas. The method is so reliable that it has enabled us to find several errors in various publications.

PACS number(s): 05.50.+q, 11.10.Jj, 11.15.Bt, 75.10.−b

I. INTRODUCTION

It has long been a hope in perturbative quantum field theory (PQFT) to be able to estimate, in a given order, the result for the coefficient, without the brute force evaluation of all the Feynman diagrams contributing in this order. As one goes to higher and higher order, the number of diagrams, and the complexity of each, increases very rapidly. Feynman suggested that even a way of determining the sign of the contribution would be useful.

The standard model (SM) of particle physics seems to work extremely well. This includes quantum chromodynamics (QCD), the electroweak theory as manifested in the Weinberg-Glashow-Salam model, and quantum electrodynamics (QED). In each case, however, we must use perturbation theory and compute large numbers of Feynman diagrams. In most of these calculations, however, we have no idea of the size or sign of the result until the computation is completed.

Recently we proposed [1–5] a method to estimate coefficients in a given order of PQFT, without actually evaluating all of the Feynman diagrams in this order. In this paper we would like to describe the method in detail and present some results.

II. METHOD

Our method makes use of Padé approximants (PA). There are many good references for PA (see, for example Refs. [6–10]). We begin by defining PA (type I) as

$$[N/M]=\frac{a_0+a_1x+\cdots+a_Nx^N}{1+b_1x+\cdots+b_Mx^M} \tag{2.1}$$

to the series

$$S=S_0+S_1x+\cdots+S_{N+M}x^{N+M} \tag{2.2}$$

where we set

$$[N/M]=S+O(x^{N+M+1}) \tag{2.3}$$

and write an equation for the coefficients of each power of x.

We have written a computer program which solves Eq. (2.3) and then predicts the coefficient of the next term, S_{N+M+1}. It works for arbitrary N and M. Furthermore, we have derived algebraic formulas for the $[N/1],[N/2],[N/3]$, and $[N/4]$ PA's where N is arbitrary. To illustrate the method, consider the simple example

$$\frac{\ln(1+x)}{x}=1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{c}\ . \tag{2.3a}$$

We write the $[1/1]$ Padé as follows:

$$[1/1]=\frac{a_0+a_1x}{1+b_1x}\ . \tag{2.3b}$$

It is easy to show that

$$a_0=1,\quad b_1=\tfrac{2}{3},\quad a_1=\tfrac{1}{6},\quad c=\tfrac{9}{2}\ .$$

We can see that the prediction for c is close to the correct value of $c=4$. For $x=1$, we get $[1/1]=\frac{7}{10}$, close to the correct result $\ln 2=0.6931$. This is much better than the partial sum

$$1-\tfrac{1}{2}+\tfrac{1}{3}=\tfrac{5}{6}=0.8333 \tag{2.3c}$$

If we now take the series

$$\frac{\ln(1+x)}{x}=1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}\ ,$$

we have $S_0=1$, $S_1=-\frac{1}{2}$, $S_2=\frac{1}{3}$, $S_3=-\frac{1}{4}$. Then

$$[1/2]=\frac{a_0+a_1x}{1+b_1x+b_2x^2}=\frac{1+x/2}{1+x+x^2/6}$$

and for $x=1$ we obtain

$$[1/2]=\tfrac{9}{13}=0.6923$$

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very close to the correct value of $\ln 2 = 0.6931$ (the partial sum is 0.58). The Padé approximant prediction (PAP) for the [1/2] is

$$S_4 = 7/36 = 0.1944 ,$$

very close to the correct value of $\frac{1}{5}$.

This result can be obtained in a way similar to before, either from first principles or by using Eq. (2.5). These results and the previous results, along with those for the [0/3] and the [2/1] Padé's are shown in Table I.

We have applied this method to many different examples in the perturbative quantum field theory (PQFT), including quantum electrodynamics (QED) and quantum chromodynamics (QCD) as well as statistical physics, atomic physics, and condensed matter theory. We will present many successful examples in this paper. However, we will not be able to present them all here due to space limitations. Some have already been presented in previous papers [1–5]. It will be seen that the PAP's are very accurate in a large number of cases in a wide class of expansions.

Consider now the $[N/M]$ PA, given by Eqs. (2.1), (2.2), and (2.3). By cross multiplying Eq. (2.1) and combining like powers of x, we obtain the following equations:

$$\begin{aligned}
a_{N+1} &= 0 = S_{N+1} + S_N b_1 + S_{N-1} b_2 + \cdots + S_{N-M+1} b_M \\
a_{N+2} &= 0 = S_{N+2} + S_{N+1} b_1 + S_N b_2 + \cdots + S_{N-M+2} b_M \\
&\vdots \\
a_{N+M} &= 0 = S_{N+M} + S_{N+M-1} b_1 + S_{N+M-2} b_2 \\
&\quad + \cdots + S_N b_M \\
a_{N+M+1} &= 0 = S_{N+M+1} + S_{N+M} b_1 + S_{N+M-1} b_2 \\
&\quad + \cdots + S_{N+1} b_M
\end{aligned} \tag{2.4}$$

The first M equations in Eq. (2.4) represent M linear equations for the M unknown b_i's and the last equation in (2.4) gives the prediction for the next term S_{N+M+1} in terms of the now known b_i's. Note that we do not need to solve for the a_i's, $i = 1, \ldots, N$. Note also that our prediction for the next term S_{N+M+1} is independent of the value of the expansion parameter x.

In general, the $[N/M]$ PAP is given by [8]

$$S_{N+M+1} = \frac{\begin{vmatrix} S_{N-M+1} & S_{N-M+2} \cdots S_{N+1} \\ \vdots & \vdots \quad \vdots \\ S_N & S_{N+1} \cdots S_{N+M} \\ S_{N+1} & S_{N+2} S_{N+M+1} \end{vmatrix}}{\begin{vmatrix} S_{N-M+1} & S_{N-M+2} \cdots S_{N+1} \\ \vdots & \vdots \quad \vdots \\ S_N & S_{N+1} \cdots S_{N+M-1} \end{vmatrix}} , \tag{2.5}$$

where $S_j = 0$ for $j < 0$.

In this way it is easy to obtain the following results:

$$\begin{aligned}
S_2 &= S_1^2 / S_0 & [0/1] & \\
S_3 &= S_2^2 / S_1 & [1/1] & \quad \mathrm{I} \\
S_4 &= S_3^2 / S_2 & [2/1] & \\
S_3 &= 2S_1S_2/S_0 - S_1^3/S_0^2 & [0/2] & \quad \mathrm{II} \\
S_4 &= \frac{2S_1S_2S_3 - S_0S_3^2 - S_2^3}{S_1^2 - S_0S_2} & [1/2] & \quad \mathrm{III} .
\end{aligned} \tag{2.6}$$

We now define A_n and ϵ_n given by [19]

$$A_n = 1 + \epsilon_n = \frac{S_n S_{n+2}}{(S_{n+1})^2} . \tag{2.7}$$

The condition

$$\begin{aligned} A_n + A_n^{-1} &= 2 , \\ \epsilon_n &= 0 , \end{aligned} \tag{2.8}$$

ensures that the prediction for [1/1] in Eq. (2.5) agrees with the prediction for [0/2]. It also ensures that the [2/1] prediction agrees with [1/2]. We now show that Eq. (2.8) also ensures that the $[N/M]$ PAP for arbitrary N and M is exact.

Consider now the geometric series given by

$$S_n = (\pm 1)^n a r^n, \quad n = 0, 1, \ldots, N+M . \tag{2.9}$$

From Eqs. (2.4) it can easily be shown that

$$S_{N+M+1} = (\pm 1)^{N+M+1} a r^{N+M+1} \tag{2.10}$$

and the PAP is exact. It should be emphasized that this is a sufficient condition for the PA to be exact but not a necessary condition. As we shall see later, there are cases where $\epsilon_n \neq 0$ and yet the PAP is exact. Also there are many examples where $\epsilon_n \to 0$ and yet the series is *not* geometric. In general we shall see the condition $\epsilon_n \ll 1$ is a sufficient but not a necessary condition for the PAP to be accurate, if not exact.

TABLE I. Padé approximants to the series $\ln(1+x)/x$ with $x = 1$. PS is partial sum.

	Series ($x=1$) Value	$\mathcal{P}_{\text{error}}$	Next term Value	$\mathcal{P}_{\text{error}}$
S_3				
[1/1]	0.700	1.0	−0.222	11.1
P/S	0.833	20.2		
Exact result	0.6931		−0.250	
S_4				
[1/2]	0.6923	0.12	0.1944	2.8
[0/3]	0.6857	1.07	0.1736	13.2
[2/1]	0.6905	0.38	0.1875	6.3
PS	0.5833	15.8		
Exact result	0.6931		0.200	

III. RESULTS FOR PA PREDICTIONS

We now present our results for the [2/3] and [3/4] PA predictions. As we shall see, the reason we have chosen these values of N and M is that from the $[M-1/M]$ PA, where $N=M-1$, we can step up or step down in N and obtain any $[N/M]$ PAP. Thus from the [2/3] result, we can obtain any $[N/3]$ result and from the [3/4] PAP we can obtain any $[N/4]$ result, where $N=0,1,2,\ldots,M-2$ or $N=M,M+1,\ldots,$ by stepping down in N or up in N, respectively. The result for the [2/3] PAP is given by

$$S_6=A/B\ , \tag{3.1}$$

where

$$\begin{aligned}A=&2S_2^2S_3S_5-2S_1S_3^2S_5+2S_0S_3S_4S_5\\&-2S_1S_2S_4S_5+S_1^2S_5^2-S_0S_2S_5^2+S_2^2S_4^2\\&-3S_2S_3^2S_4+2S_1S_3S_4^2-S_0S_4^3+S_3^4\end{aligned} \tag{3.2}$$

and

$$B=S_2^3-2S_1S_2S_3+S_0S_3^2-S_0S_2S_4+S_1^2S_4\ . \tag{3.3}$$

From this result we can obtain the [1/3] PAP by stepping down $S_j\to S_{j-1}$ and putting $S_{-1}=0$. This procedure can be repeated to obtain the [0/3] PAP. By stepping up $S_j\to S_{j+1}$ we obtain the [3/3] PAP. This procedure can be repeated to obtain the [4/3] PAP, etc. This is why the procedure we used in the first paper [1] of applying the last four terms in the [1/2] PAP formula is correct. It is equivalent to using the [2/2] PAP, the [3/2] PAP, etc.

We now present our result for the [3/4] PAP. This result is much more complicated and the numerator C contains 50 terms. Our result is

$$S_8=C/D\ , \tag{3.4}$$

where

$$\begin{aligned}C=&2(2S_2S_3S_4^2S_7-S_3^3S_4S_7-S_1S_4^3S_7-S_2^2S_4S_5S_7+S_1S_3S_4S_5S_7+S_3^2S_4^2S_6+2S_1S_4^2S_5S_6\\&+S_2^2S_4S_6^2-S_1S_3S_4S_6^2-S_2S_3S_4S_5S_6-S_2S_4^3S_6+S_3^2S_4S_5^2+S_2S_4^2S_5^2-2S_3S_4^3S_5-S_1S_4S_5^3+S_2S_3^2S_5S_7\\&+S_0S_4^2S_5S_7+S_1S_2S_5^2S_7-S_0S_3S_5^2S_7-S_2^2S_4S_5S_7-S_1S_3S_4S_5S_7-S_2^2S_3S_6S_7-S_0S_3S_4S_6S_7-S_1^2S_5S_6S_7\\&+S_0S_2S_5S_6S_7+S_1S_2S_4S_6S_7+S_1S_3^2S_6S_7-S_1S_2S_3S_7^2-S_3^3S_5S_6-S_1S_3S_4S_6^2-S_1S_2S_5S_6^2+S_1S_3S_5^2S_6\\&+S_0S_3S_5S_6^2-S_2S_3S_5^3)-3S_0S_4S_5^2S_6+S_2^3S_7^2+S_0S_3^2S_7^2+S_1^2S_4S_7^2-S_0S_2S_4S_7^2+S_2S_3^2S_6^2+S_1^2S_6^3-S_0S_2S_6^3\\&+S_3^2S_4S_5^2+S_2^2S_5^2S_6+S_4^5+S_2S_4^2S_5^2+S_3^2S_4^2S_6-S_2S_4^3S_6+S_0S_4^2S_6^2+S_0S_5^4\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}D=&2(S_1S_3^2S_5+S_1S_2S_4S_5-S_1S_3S_4^2-S_2^2S_3S_5-S_1S_2S_3S_6-S_0S_3S_4S_5)+3S_2S_3^2S_4\\&-S_3^4-S_1^2S_5^2+S_1^2S_4S_6+S_0S_4^3+S_0S_2S_5^2+S_0S_3^2S_6-S_0S_2S_4S_6+S_2^3S_6-S_2^2S_4^2\ .\end{aligned} \tag{3.6}$$

These results can be checked by observing that the number of factors in each term must be the same and the sum of the indices in each term must also be the same. For example, for the [2/3] PAP, each term in the numerator of the PAP given in Eq. (3.2) has four factors and the sum of indices is equal to 12. The term in the denominator B given in Eq. (3.3) has three factors and the sum of indices is equal to 6. Thus S_6 given by Eq. (3.1) has one factor and an index number equal to 6 as expected. A similar analysis can be made for the [3/4] PAP. Each term in the numerator C given in Eq. (3.5) has five factors and the sum of the indices is equal to 20. The terms in the denominator D given in Eq. (3.6) have four factors and the sum of indices is equal to 12. Thus S_8 given by Eq. (3.4) has one factor and index number equal to 8 as expected. Using MAPLE we have derived the formulas for the [4/5] PAP and the [5/6] PAP. However, these formulas are too long to be given here.

IV. THEOREM FOR STEPPING UP OR DOWN IN N

We now show how one can step up or down in N from the known result for the $[M-1/M]$ PAP. In this section we state and prove the theorem.

Theorem. (a) To step down from PAP $[M-1/M]$ to $[M-2/M]$, we step $S_j\to S_{j-1}$ and put $S_{-1}=0$. To step down further, we repeat this procedure as many times (once per step) as desired until we reach $[0/M]$. (b) To step up from $[M-1/M]$ to $[M/M]$, we step $S_j\to S_{j+1}$. To step up further we repeat this procedure as many times (once per step) as desired for any $[N/M]$, $N=M,M+1,\ldots,$.

Proof. We write $[N/M]$ as given in Eq. (2.1) and obtain the linear equations given in Eqs. (2.4). For $N=M-1$, these equations become

$$
\begin{aligned}
a_M &= 0 = S_M + S_{M-1}b_1 + S_{M-2}b_2 + \cdots + S_0 b_M \\
a_{M+1} &= 0 = S_{M+1} + S_M b_1 + S_{M-1}b_2 + \cdots + S_1 b_M \\
&\vdots \\
a_{2M-1} &= 0 = S_{2M-1} + S_{2M-2}b_1 + S_{2M-3}b_2 \\
&\qquad + \cdots + S_{M-1}b_M \\
a_{2M} &= 0 = S_{2M} + S_{2M-1}b_1 + S_{2M-2}b_2 + \cdots + S_M b_M \ .
\end{aligned} \tag{4.1}
$$

The first M equations in Eq. (4.1) enable one to solve for the M unknowns, $b_1, b_2, \ldots, b_M$ and the last equation in Eqs. (4.1) allow us to solve for the unknown PAP S_{2M} in terms of the now known b_i, $i=1,2,\ldots,M$.

(i) The corresponding equations for $N=M-2$ are

$$
\begin{aligned}
a_{M-1} &= 0 = S_{M-1} + S_{M-2}b_1 + S_{M-3}b_2 + \cdots + S_{-1}b_M \\
a_M &= 0 = r_M + S_{M-1}b_1 + S_{M-2}b_2 + \cdots + S_0 b_M \\
&\vdots \\
a_{2M-2} &= 0 = S_{2M-2} + S_{2M-3}b_1 + S_{2M-4}b_2 \\
&\qquad + \cdots + S_{M-2}b_M \\
a_{2M-1} &= 0 = S_{2M-1} + S_{2M-2}b_1 + S_{2M-3}b_2 \\
&\qquad + \cdots + S_{M-1}b_M
\end{aligned} \tag{4.2}
$$

where

$$S_{-1} = 0 \ .$$

It can be easily seen that Eq. (4.2) can be obtained from Eq. (4.1) by merely using the replacement $S_j \to S_{j-1}$ and setting $S_{-1}=0$. The first M equations in Eq. (4.2) allow one to solve for the M b_i's, $i=1,2,\ldots,M$ and the last equation gives us the PAP for S_{2M-1}. It is easily verified that one can repeat this process to obtain all $[N/M]$ PAP for $N=0,1,\ldots,M-2$.

(ii) The equations for $N=M$ are

$$
\begin{aligned}
a_{M+1} &= 0 = S_{M+1} + S_M b_1 + S_{M-1}b_2 + \cdots + S_1 b_M \ , \\
a_{M+2} &= 0 = S_{M+2} + S_{M+1}b_1 + S_M b_2 + \cdots + S_2 b_M \ , \\
&\vdots \\
a_{2M} &= 0 = S_{2M} + S_{2M-1}b_1 + S_{2M-2}b_2 + \cdots + S_M b_M \ , \\
a_{2M+1} &= 0 = S_{2M+1} + S_{2M}b_1 + S_{2M-1}b_2 + \cdots + S_{M+1}b_M \ .
\end{aligned} \tag{4.3}
$$

Again, the first M equations in Eq. (4.3) give us the M b_i's, $i=1,2,\ldots,M$ and the last equation in Eq. (4.3) gives us the PAP for S_{2M+1}. It can also be seen that one can go from Eq. (4.1) to Eq. (4.3) by merely replacing $S_j \to S_{j+1}$. It can also be seen that one can repeat this process to obtain all $[N/M]$ PAP for all $N=M$, $M+1,\ldots$. Thus starting from the $[M-1/M]$ PAP one can obtain the results for all $[N/M]$ PAP's $N=0,1,2,\ldots$. The information for the general $[N/M]$ PAP is contained in the $[M-1/M]$ PAP. It should be stressed that one cannot go from the $[N/M]$ PAP where $N<M-1$ to PAP's for larger N. One must start with the $[N/M]$ PAP with $N \geq M-1$.

V. PADÉ APPROXIMANTS AND THE R AND R_τ RATIOS IN PERTURBATIVE QCD

In this section we will consider the R ratio and the R_τ ratio in perturbative QCD [11–14]. They are defined as follows:

$$R_\tau = \frac{\Gamma(\tau \to \nu + \text{hadrons})}{\Gamma(\tau \to e\nu\bar{\nu})} \tag{5.1}$$

and R is given by

$$R = \frac{\sigma_{\text{tot}}(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \ . \tag{5.2}$$

We first consider R in the general MS-type scheme given by the parameter t,

$$\Lambda_t = e^{-t/2}\Lambda_{\overline{MS}} \ . \tag{5.3}$$

Obviously $t=0$ corresponds to the $\overline{MS}$ scheme, $t=\ln 4\pi - \gamma = 1.95$ represents the MS scheme, $t=-2.0$ for the G scheme, and $t=4\zeta(3)-\frac{11}{2}=-0.692$ yields our MS scheme [11]. The scale-dependent R (in the general MS-type scheme) is given by

$$R = 3\sum Q_f^2 R(t) - 1.24\left[\sum Q_f\right]^2 x^3 \ , \tag{5.4}$$

where $x=\alpha_s/\pi$ and N_f is the number of fermions (quarks). We neglect the second term in Eq. (5.4) as it is small in all cases of interest. $R(t)$ is given by

$$
\begin{aligned}
R(t) = 1 + x &+ x^2[(1.9857+2.75t) - N_f(0.1153+0.1667t)] \\
&+ x^3[(-6.6369+17.2964t+7.5625t^2) \\
&\qquad - N_f(1.2001+2.0877t+0.9167t^2) + N_f^2(-0.0052+0.0384t+0.0278t^2)] \ .
\end{aligned} \tag{5.5}
$$

Our results for $t=2$, 4, and 10 are shown in Tables II, III, and IV, respectively [30]. It can be seen that the method works very well and we can predict the unknown next term (NT) and next-next (NNT) term. The NNT terms from the [0/3] and the [1/2] of Eq. (2.5) agree very well with those from the [2/1] and so are not listed in our tables. In Figs. 1 and 2, we plot the estimated and exact terms as a function of t for two representative values of

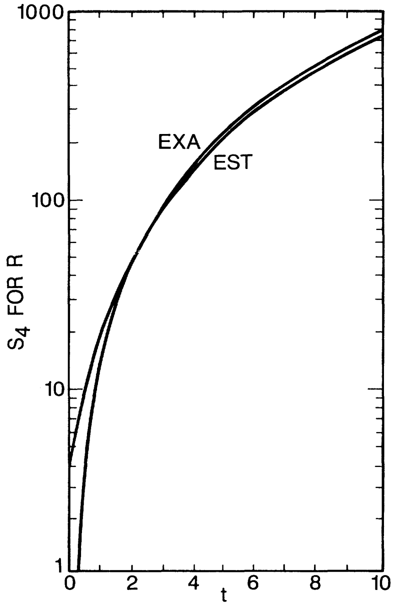

FIG. 1. The exact (EXA) and the estimated (EST) coefficients vs t for the x^3 coefficient of $R(t)$ for $N_f=1$.

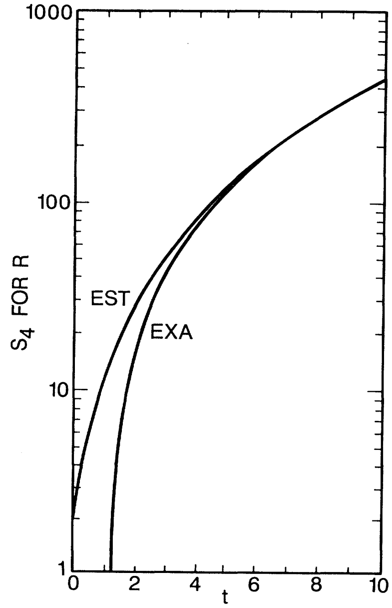

FIG. 2. The exact (EXA) and the estimated (EST) coefficients vs t for the x^3 coefficient of $R(t)$ for $N_f=5$.

N_f ($N_f=1$ and $N_f=5$, respectively). It can be seen that the agreement is excellent for $t>1$ and improves as t increases. The reason for this behavior can be seen as follows.

From the [1/1] of Eqs. (2.6) and (5.5) we obtain

$$S_3=S_2^2/S_1=3.943+10.92t+7.5625t^2 -N_f(0.458+1.2962t+0.9167t^2) +N_f^2(0.0133+0.0384t+0.0278t^2)\ . \quad (5.6)$$

The exact result is given by the x^3 term in Eq. (5.5). It can be seen by comparing this term with Eq. (5.6) that the t^2, t^2N_f, $t^2N_f^2$, and tN_f^2 coefficients agree. In fact, this agreement is exact. Now we understand why the estimate and the exact result agree so well for large t.

The reason for the agreement of the coefficients can be seen as follows [11]. Consider the function given by

$$D(q^2/\mu^2,\alpha_s)=\sum_{i=0}^{\infty} R_i\left[\frac{\alpha_s}{\pi}\right]^i =R(t)\ . \quad (5.7)$$

The perturbation series for D is

$$D=1+D_0\left[\frac{\alpha_s}{4\pi}\right]+(D_1-\beta_0D_0\ln q^2/\mu^2)\left[\frac{\alpha_s}{4\pi}\right]^2 +[D_2-(\beta_1D_0+2\beta_0D_1)\ln q^2/\mu^2 +(\beta_0^2D_0\ln^2 q^2/\mu^2]\left[\frac{\alpha_s}{4\pi}\right]^3 , \quad (5.8)$$

where β_0 and β_1 are the first two coefficients of the β function. Since $t=\ln q^2/\mu^2$, one can see that

$$r_2^2/r_1=(D_1-\beta_0D_0t)^2/D_0\ . \quad (5.9)$$

The t^2 coefficient is $\beta_0^2D_0$ which agrees with the coefficient of the $t^2\alpha_s^3$ term even if $D_0\neq 1$. The cross term in Eq. (13) is $-2\beta_0D_1t$. This agrees with one of the $t\alpha_s^3$ terms. However, the other one does not have an N_f^2 contribution. Thus the tN_f^2 coefficients also agree.

We now turn to R_τ. In the general *MS*-type scheme, R_τ^{pert} is given by [12,13,14]

$$R_\tau^{\text{pert}}=3R_\tau(t)\ ,$$

where [11]

$$R_\tau(t)=1+x+x^2[(6.3399+2.75t)-N_f(0.3792+0.1667t)] +x^3[(48.5832+41.2443t+7.5625t^2)-N_f(7.8795+4.9905t+0.9167t^2)+N_f^2(0.1579+0.1264t+0.0278t^2)]\ . \quad (5.10)$$

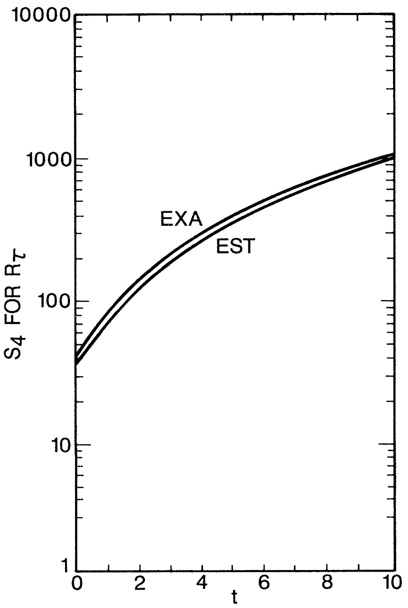

FIG. 3. The exact (EXA) and the estimated (EST) coefficients vs t for the x^3 coefficient of $R_\tau(t)$ for $N_f=1$.

The results for $t=0$, 4, and 10 are shown in Tables V, VI, and VII, respectively [30]. It can be seen that the method works very well and we can predict the NT and the NNT terms. The NNT terms from II and III of Eq. (4) agree very well with those from I and so are not listed in our tables. In Figs. 3 and 4 we plot the estimated and exact terms as a function of t for two representative values of N_f ($N_f=1$ and $N_f=5$, respectively). It can be seen that

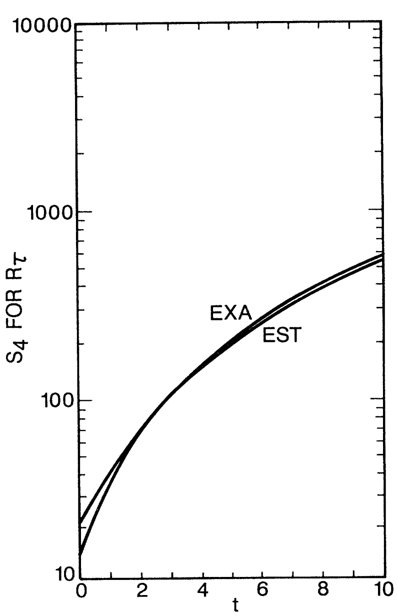

FIG. 4. The exact (EXA) and the estimated (EST) coefficients vs t for the x^3 coefficient of $R_\tau(t)$ for $N_f=5$.

in this case, the agreement is excellent, even for $t=0$, and, again, improves as t increases. Again, we can see why we get this behavior.

From I of Eqs. (2.6) and (5.10) we obtain

$$\begin{aligned} S_3=S_2^2/S_1=&40.1943+34.8695t+7.5625t^2\\ &-N_f(4.8082+4.1989t+0.9167t^2)\\ &+N_f^2(0.1438+0.1264t+0.0278t^2)] \;. \end{aligned} \tag{5.11}$$

The exact results is given by the x^3 term of Eq. (5.10). It can be seen again that the t^2, t^2N_f, $t^2N_f^2$, and tN_f^2 coefficients agree. Again this agreement is exact. Moreover, the t^2, t^2N_f, and $t^2N_f^2$ of Eqs. (5.5) and (5.10) also agree exactly. These equalities can be seen from an analysis, similar to the one presented for R.

VI. ERRORS IN THE PAP

In this section we study the % error in the PAP. The % error $\mathcal{P}_{\text{error}}$ can be expressed completely in terms of the ϵ_n where the ϵ_n are defined as before in Eq. (2.6). Recall

$$A_n=1+\epsilon_n=\frac{S_nS_{n+2}}{(S_{n+1})^2} \tag{6.1}$$

and we write

$$\mathcal{P}_{\text{error}}=100p \;. \tag{6.2}$$

It can easily be shown that, for the [0/1] PAP,

$$p=-\epsilon_0/(1+\epsilon_0) \;. \tag{6.3}$$

By our step-up theorem, it follows that, for the $[N/1]$ PAP,

$$p=-\epsilon_N/(1+\epsilon_N) \;. \tag{6.4}$$

For the [1/2] PAP, we get

$$p=\frac{\epsilon_1^2/\epsilon_0-\epsilon_2(1+\epsilon_1)^2}{(1+\epsilon_1)^2(1+\epsilon_2)} \;. \tag{6.5}$$

The % difference between the [1/2] and the [2/1] PAP's is obtained by setting $\epsilon_2=0$ in Eq. (6.5). Again from the step-up theorem we use the same equation for the $[N/2]$ PAP,

$$p=\frac{\epsilon_N^2/\epsilon_{N-1}-\epsilon_{N+1}(1+\epsilon_N)^2}{(1+\epsilon_N)^2(1+\epsilon_N)} , \quad N\ge 1 \;. \tag{6.6}$$

To get the [0/2] PAP we step down and set $\epsilon_{-1}=-1$. From Eqs. (3.1), (3.2), and (3.3) we can get the [2/3] PAP % error. We write

$$r_6=C/D \;, \tag{6.7}$$

where

$$\begin{aligned} C=A_3^4[&2A_2^2A_3-2A_1A_2^2A_3+2A_0A_1^2A_2^3A_3-2A_1A_2^3A_3\\ &+A_1^2A_2^4A_3^2-A_0A_1^2A_2^4A_3^2+A_2^2\\ &-3A_2+2A_1A_2^2-A_0A_1^2A_2^3+1] \end{aligned} \tag{6.8}$$

and

$$D=A_2^3A_3^6[1-2A_1+A_0A_1^2-A_0A_1^2A_2+A_1^2A_2] \tag{6.9}$$

and

$$p=\frac{C/D-A_4}{A_4} \ . \tag{6.10}$$

Again we can step up or down to obtain the % error for all $(N,3)$ PAP's.

The formula for the [3/4] PAP % error is much more complicated but can be derived in a straightforward manner from Eqs. (3.4), (3.5), and (3.6). It is given by

$$r_8=E/F \ , \tag{6.11}$$

where

$$\begin{aligned}E=A_4^5A_5^{10}\{&2[2A_2A_3^3A_4^2A_5-A_3^3A_4^2A_5-A_1A_2^2A_3^3A_4^2A_5-A_2^2A_3^4A_4^2A_5+A_1A_2^2A_3^4A_4^2A_5+A_3^2A_4+2A_1A_2^2A_3^3A_4\\&+A_2^2A_3^4A_4^2-A_1A_2^2A_3^4A_4^2-A_2A_3^3A_4-A_2A_3^2A_4+A_3^2+A_2A_3^2-2A_3-A_1A_2^2A_3^3+A_2A_3^4A_4^2A_5\\&+A_0A_1^2A_2^3A_3^4A_4^2A_5+A_1A_2^3A_3^5A_4^2A_5-A_0A_1^2A_2^3A_3^5A_4^2A_5-A_2^2A_3^4A_4^2A_5-A_1A_2^2A_3^4A_4^2A_5\\&-A_2^2A_3^5A_4^3A_5-A_0A_1^2A_2^3A_3^5A_4^3A_5-A_1^2A_2^4A_3^6A_4^3A_5+A_0A_1^2A_2^4A_3^6A_4^3A_5+A_1A_2^3A_3^5A_4^3A_5\\&+A_1A_2^2A_3^5A_4^3A_5-A_1A_2^3A_3^6A_4^4A_5^2-A_3^3A_4-A_1A_2^2A_3^4A_4^2-A_1A_2^3A_3^5A_4^2+A_1A_2^2A_3^4A_4\\&+A_0A_1^2A_2^3A_3^5A_4^2-A_2A_3^3]-3A_0A_1^2A_2^3A_3^4A_4+A_2^3A_3^6A_4^4A_5^2+A_1^2A_2^4A_3^6A_4^4A_5^2+A_0A_1^2A_2^3A_3^6A_4^4A_5^2\\&-A_0A_1^2A_2^4A_3^6A_4^4A_5^2+A_2A_3^4A_4^2+A_1^2A_2^4A_3^6A_4^3-A_0A_1^2A_2^4A_3^6A_4^3+A_3^2+A_2^2A_3^4A_4+1\\&+A_2A_3^2+A_3^2A_4-A_2A_3^2A_4+A_0A_1^2A_2^3A_3^4A_4^2+A_0A_1^2A_2^3A_3^4\} \ ,\end{aligned} \tag{6.12}$$

$$\begin{aligned}F=A_3^4A_4^8A_5^{12}\{&2[A_1A_2^2A_3+A_1A_2^3A_3-A_1A_2^2-A_2^2A_3-A_1A_2^3A_3^2A_4-A_0A_1^2A_2^3A_3]+3A_2-1-A_1^2A_2^4A_3^2\\&+A_1^2A_2^4A_3^2A_4+A_0A_1^2A_2^3+A_0A_1^2A_2^4A_3^2+A_0A_1^2A_2^3A_3^2A_4-A_0A_1^2A_2^4A_3^2A_4+A_2^3A_3^2A_4-A_2^2\} \ ,\end{aligned} \tag{6.13}$$

and

$$p=\frac{E/F-A_6}{A_6} \ . \tag{6.14}$$

Once again we can step up or down to obtain the % error for all $[N/4]$ PAP's.

Since the % error can be expressed completely in terms of the $A_n=1+\epsilon_n$ it can be seen that for $\epsilon_n \ll 1$ the % error is very small. However, this is a sufficient condition and not a necessary one. That is, even if the ϵ_n are not small, the % error may still be small due to cancellations.

As will be seen in Sec. XIII, a sufficient but not necessary condition for the PAP to yield accurate results is that

$$\lim_{n\to\infty} A_n=\pm 1 \ , \tag{6.15}$$

where

$$A_n=1+\epsilon_n=\frac{S_nS_{n+2}}{(S_{n+1})^2} \tag{6.16}$$

When this condition is satisfied, the PAP will be very good. However, even if this condition is not satisfied, it may still be very good due to cancellations. This condition is satisfied in QED for the n-bubble diagram [15] contribution to $g-2$ of the muon and electron.

For the electron,

$$a_n=\frac{n!/2}{6^n}e^{-10/3} \tag{6.17}$$

and, hence,

$$A_n=\frac{n+2}{n+1}\to 1 \ . \tag{6.18}$$

For the muon,

$$a_n=\frac{n!/2}{6^n}e^{-10/3}\left[\frac{m_u}{m_e}\right]^4 \tag{6.19}$$

and again

$$A_n=\frac{n+2}{n+1}\to 1 \ . \tag{6.20}$$

Thus in both cases the PAP is very good.

It can be seen that for

$$S_n=(-1)^na_n \tag{6.21}$$

the series is asymptotic to

$$E_1(x)=E(x/6) \ , \tag{6.22}$$

where $E(x)$ is given in Eq. (12.7). For

$$S_n=a_n \ , \tag{6.23}$$

the series is asymptotic to

$$E_1'(x)=E'(x/6)\ , \tag{6.24}$$

where $E'(x)$ is given in Eq. (12.8).

VII. EXAMPLES FROM STATISTICAL PHYSICS

We now present in Tables IX–XX some examples of the PAP from statistical physics. Most of our examples can be found in Refs. [16] and [17]. Note that in order to keep this paper from growing much too long we present only some of our results. In the tables and the text (N,M) is the same as $[N/M]$. Table IX gives our results for the high-temperature susceptibility series of the square lattice Ising model [18].

One can see that the method works and the PAP's are very accurate, with the % error decreasing to approximately 10^{-10}% for the last known term. The prediction for the NT should be very accurate. We use $\epsilon_{n+2}'=\epsilon_n$.

Table X gives the number of closed polygons on a square lattice [19] and the corresponding PAP's. It can be seen that the % error decreases steadily as one goes to higher order. The number in parentheses is the power of 10, thus $0.27(-7)=0.27\times10^{-7}$. It can be seen that the PAP's are very accurate and the next term (NT) is predicted. This prediction should be very accurate. It can also be seen that $\epsilon' \ll 1$ and decreases steadily as one goes to higher order. This is a sufficient condition, but not a necessary condition, for the PAP's to be accurate estimates.

Table XI describes the chain-generating function for self-avoiding walks on the triangular lattice [19]. Again the results are very interesting with the % error decreasing steadily as one goes to higher order. Also, as before, $\epsilon' \ll 1$ and decreases. The PAP for the (4,4) is infinity since the (3,3) is exact. This is an example of our theorem, proved in Sec. XII. The prediction for NT should be very accurate.

Table XII gives the zero-field coefficients U_r for the honey-combed lattice in the two-dimensional (2D) Ising model [18]. Again, the results are very good. The two predictions for the NT agree and should be very precise.

When we first ran this series, the predictions for -24 and -1368 were much too large in magnitude. This occurred because we omitted the first two terms of the series, 1.5 and 0. In Table XII we started from the beginning of the series, and, as one can see, the predictions for -24 and -1368 are now very good.

Table XIII gives the zero-field coefficients, U_r, for the plane-triangular lattice in the 2D Ising model [18]. The results are very good and the two predictions for the NT agree. Infinity for the [2,2] and the [4,5] are the result of our theorem since the (1,1) PAP and the (3,4) PAP are exact. The (2,3) infinity arises because the (1,2) PAP is 0/0.

Table XIV shows the zero-field coefficients U_r for the square lattice in the 2D Ising model [18]. Again the results are very good and the two predictions for the NT agree.

Table XV gives the spontaneous magnetization coefficients for the honey-combed lattice [18]. The accu-

TABLE IX. High-temperature susceptibility series of the square-lattice Ising model. NT refers to the next (unknown) term.

(N,M)	Padé	Exact	% error	$-\epsilon'$
(0,1)	16	12	33	0.25
(1,2)	108	100	8	0.074
(3,3)	1972	1972	0	0.61(−2)
(3,4)	5188	5172	0.31	0.016
(4,3)	5188	5172	0.31	0.016
(5,4)	34856	34876	0.057	0.91(−2)
(5,5)	89764	89764	0	0.43(−2)
(5,6)	229 704	229 628	0.033	0.61(−2)
(7,6)	1486 858	1486 308	0.037	0.44(−2)
(7,7)	3764 311	3763 460	0.023	0.25(−2)
(8,7)	9496 081	9497 380	0.014	0.34(−2)
(12,12)	36 212 337 725	36 212 402 548	0.18(−3)	0.10(−2)
(12,13)	89 896 881 041	89 896 870 204	0.12(−4)	0.12(−2)
(20,20)	68 849 212 197 681(3)	68 849 212 197 172(3)	0.74(−9)	0.4(−3)
(20,21)	169 150 097 346(6)	169 150 097 365(6)	0.11(−7)	0.46(−3)
(21,21)	41 541 963 877(7)	41 541 963 949(7)	0.17(−6)	0.36(−3)
(21,22)	101 981 266 329(6)	1019 816 266 253(6)	0.75(−8)	0.41(−3)
(24,23)	36 912 183 773 288(6)	36 912 183 772 985(6)	0.82(−9)	0.35(−3)
(24,24)	90 466 431 959 184(6)	90 466 431 959 612(6)	0.47(−9)	0.28(−3)
(24,25)	22 164 947 092 629(7)	22 164 947 092 555(7)	0.33(−9)	0.32(−3)
(25,24)	221 649 470 925 546(6)	221 649 470 925 555(6)	0.38(−11)	0.32(−3)
(26,25)	13 294 400 774 266(8)	13 294 400 774 247(8)	14(−9)	0.29(−3)
(26,26)	32 546 159 798 889(8)	32 546 159 798 489(8)	0.12(−8)	0.24(−3)
(27,26)	79 654 880 661 744(8)	79 654 880 659 405(8)	0.29(−8)	0.27(−3)
(26,27)	79 654 880 659 339(8)	79 654 880 659 405(8)	0.83(−10)	0.27(−3)
(27,27)	194 906 447 358 589(8)	NT		

TABLE X. Number of closed polygons on a square lattice. NT refers to the next (unknown) term.

(N,M)	Padé	Exact	% error	ϵ'
(0,1)	4	7	43	0.75
(1,1)	24.5	28	13	0.14
(1,2)	114.3	124	7.8	0.11
(3,3)	15641.4	15 268	2.4	0.04
(3,4)	81603.2	81 826	0.27	0.031
(5,4)	2520776.4	2521 270	0.02	0.021
(5,5)	14382759.9	14 385 376	0.018	0.017
(5,6)	83301403.3	83 290 424	0.013	0.015
(7,7)	17332403704.6	17 332 874 364	0.27(−2)	0.97(−2)
(7,8)	104653043328.9	104 653 427 012	0.37(−3)	0.86(−2)
(8,8)	636737111378.9	636 737 003 384	0.17(−4)	0.77(−2)
(8,9)	3900768657365.8	3900 770 002 646	0.34(−4)	0.69(−2)
(9,8)	3900768645591.6	3900 770 002 646	0.35(−4)	0.69(−2)
(9,9)	24045477087166.3	24 045 500 114 388	0.96(−4)	0.62(−2)
(9,10)	149059818372329	149 059 814 328 236	0.27(−5)	0.56(−2)
(10,9)	149059814952508	149 059 814 328 236	0.42(−6)	0.56(−2)
(10,10)	928782402852355	928 782 423 033 008	0.22(−5)	0.51(−2)
(10,11)	5814401458255866	5814 401 613 289 290	0.27(−5)	0.47(−2)
(11,10)	5814400906586723	5814 406 163 289 290	0.12(−4)	0.47(−2)
(11,11)	36556766563181916	36 556 766 640 745 936	0.21(−6)	0.43(−2)
(11,12)	230757492329413778	230 757 492 737 449 632	0.18(−6)	0.40(−2)
(12,11)	230757492299121126	230 757 492 737 449 632	0.19(−6)	0.40(−2)
(12,12)	1461972664107671386	1461 972 662 850 874 880	0.86(−7)	0.37(−2)
(12,13)	9293993426515280515	9293 993 428 791 900 928	0.24(−7)	0.34(−2)
(13,12)	9293993426247752549	9293 993 428 791 900 928	0.27(−7)	0.34(−2)
(13,13)	592709055867(8)	NT		

TABLE XI. Chain-generating function for self-avoiding walks on the triangular lattice. NT refers to the next (unknown) term.

(N,M)	Padé	Exact	% error	$-\epsilon'$
(0,1)	36	30	2.0	167
(1,1)	150	138	8.7	0.08
(1,2)	606	618	1.9	0.026
(2,1)	634.8	618	2.7	0.026
(2,2)	2742	2730	0.44	0.01
(2,3)	11 994	11 946	0.4	0.0094
(3,2)	11 973	11 946	0.23	0.0094
(3,3)	51 882	51 882	0	0.0075
(3,4)	223 914	224 130	0.096	0.005
(4,3)	223 914	224 130	0.096	0.005
(4,4)	∞	964 134	∞	0.0042
(4,5)	4135 167	4133 166	0.048	0.003
(5,4)	4134 438	4133 166	0.031	0.003
(5,5)	17 673 567	17 668 938	0.026	0.0028
(5,6)	75 361 184	75 355 206	0.0079	0.002
(6,5)	75 355 045	75 355 206	0.0002	0.002
(6,6)	320 734 924	320 734 686	0.00007	0.002
(6,7)	1 322 798 287	1362 791 250	0.00052	0.0017
(7,6)	1 362 798 648	1362 791 250	0.00054	0.0017
(7,7)	5 781 565 021	5781 765 582	0.0035	0.0015
(7,8)	24 497 359 134	24 497 330 332	1.2(−4)	0.0013
(8,7)	24 497 638 541	24 497 330 332	1.3(−3)	0.0013
(8,8)	103 673 993 600	103 673 967 882	2.5(−5)	1.2(−3)
(8,9)	438 296 768 643	438 296 739 594	6.6(−6)	1.0(−3)
(9,8)	438 296 789 461	438 296 739 594	1.1(−5)	1.0(−3)
(9,9)	1 851 231 418 710	1851 231 376 374	2.3(−6)	9.4(−4)
(9,10)	7 812 439 589 810	7812 439 620 678	4(−7)	8.4(−4)
(10,9)	7 812 439 615 570	7812 439 620 678	1(−7)	8.4(−4)
(10,10)	32 944 292 663 100	NT		

TABLE XII. Zero-field coefficients U_r for the honey-combed lattice in the 2D Ising model.

(N,M)	Padé	Exact	% error	ϵ'
(2,2)	−27	−24	12.5	−0.68
(2,3)	82.5	93	11.3	1.4
(3,2)	81.75	93	12.1	1.4
(3,3)	−138	−180	23.3	0.50
(3,4)	606	639	5.2	0.83
(4,3)	594.8	639	6.9	0.83
(4,4)	−1402.9	−1368	2.5	−0.40
(4,5)	4618.2	4653	0.75	0.59
(5,4)	4608.8	4653	0.95	0.59
(5,5)	−10 572.9	−10605	0.30	−0.33
(5,6)	35 011.5	35169	0.45	0.46
(6,5)	35 005.3	35169	0.47	0.46
(6,6)	−84 365.2	−83664	0.84	−0.28
(6,7)	272 803.1	272 835	0.012	0.37
(7,6)	272 683.4	272 835	0.056	0.37
(7,7)	−669 849.3	−669 627	0.033	−0.25
(7,8)	2158 678.9	2157 759	0.043	0.31
(8,7)	2157 454.1	2157 759	0.014	0.31
(8,8)	−5421 719.1	−5423 280	0.029	−0.22
(8,9)	17 310 559.5	17 319 837	0.054	0.27
(9,8)	17 321 197.7	17 319 837	0.0079	0.27
(9,9)	−44 357 521.9	−44 354 277	0.0073	−0.20
(9,10)	140 662 210.4			
(10,9)	140 653 337.5			

racy improves steadily as one goes to higher order. $\mathcal{P}_{\text{error}}=1.0\times10^{-6}$ % for the [8/7] PAP.

Table XVI shows the low-temperature ferromagnetic susceptibility coefficients for the simple-cubic lattice in the Ising model [18]. Again the results are very interesting and the PAP's for the NT agree. Table XVII shows the low-temperature ferromagnetic susceptibility coefficients for the diamond lattice in the Ising model [18]. One can see that the results are very good and the predictions for the NT agree. Table XVIII gives the low-temperature ferromagnetic susceptibility coefficients for the honey-combed lattice in the Ising model [18]. Again the results are very good. Table XIX shows the spontaneous magnetization coefficients for the plane-triangular lattice in the Ising model [18]. The results are reasonable.

Table XX shows the spontaneous magnetization coefficients for the simple-cubic lattice in the Ising model [18]. The results are very good.

In Tables XXI–XXV we use the first few coefficients and then predict *all* the rest of them [30]. For example, in Table XXI we give $d\ln\chi/dw$ where χ is the magnetic susceptibility for the 2D square lattice Ising model of ferromagnetism [20] (high-temperature expansion). We use, as an input, S_0, S_1, S_2, S_3, S_4, and S_5 and then predict all the rest of the terms as shown in Table XXI. As expected, the % error now increases as one goes to higher order as the error accumulates. The results, however, are surprisingly good. By comparing these results to those of Table XXVI, where we use all of the previous known terms and predict only one coefficient at a time, one can see that the % error for the [7/6] PAP has a 10.2% error in Table XXI while the % error in Table XXVI is 0.009%. Table XXII shows the same series where we now use as input S_0, S_1, S_2, S_3, and S_4 and predict all of the rest of the terms. The results are similar to those of Table XXI. Note that the [7/7] PAP's for the NT in Tables XXVI, XXI, and XXII are very close to each other.

Table XXIII shows the chain-generating function for self-avoiding walks on the triangular lattice [20] and should be compared with Table XI. We use S_0, S_1, S_2, S_3, and S_4 as input and predict all the rest of the terms. The results in Table XXIII show, as expected, that the

TABLE XIII. Zero-field coefficients U_r for the plane-triangular lattice in the 2D Ising model. NT refers to the next (unknown) term.

(N,M)	Padé	Exact	% error	ϵ'
(0,1)	12	12	0	0
(1,1)	24	24	0	0
(1,2)	48	54	11.1	0.125
(2,1)	48	54	11.1	0.125
(2,2)	∞	138	∞	0.14
(2,3)	∞	378	∞	0.072
(3,3)	1056	1080	2.2	0.04
(3,4)	3186	3186	0	0.0325
(4,3)	3148	3186	1.2	0.0325
(4,4)	9588	9642	0.56	0.026
(4,5)	∞	29784	∞	0.021
(5,5)	93508	93552	0.047	0.0168
(5,6)	297 897	297 966	0.023	0.014
(6,5)	297 993	297 966	0.0091	0.014
(6,6)	960 229	960 294	0.0067	0.012
(6,7)	3126 390	3126 408	0.00056	0.0102
(7,6)	3126 175	3126 408	0.0074	0.0102
(7,7)	10 268 597	10 268 688	0.00088	0.0089
(7,8)	33 989 699	NT		
(8,7)	33 989 266	NT		

TABLE XIV. Zero-field coefficients U_r for the square lattice in the 2D Ising model. NT refers to the next (unknown) term.

(N,M)	Padé	Exact	% error	ϵ'
(0,1)	8	8	0	0
(1,1)	16	24	33.3	0.5
(1,2)	∞	84	∞	0.17
(2,1)	72	84	14.3	0.17
(2,2)	306	328	6.7	0.116
(2,3)	1323	1372	3.6	0.071
(3,2)	1363	1372	0.63	0.071
(3,3)	5960	6024	1.07	0.05
(3,4)	27336	27 412	0.28	0.036
(4,3)	27730	27 412	1.16	0.036
(4,4)	128 066	128 228	0.13	0.028
(4,5)	613 094	613 160	0.011	0.022
(5,4)	612 665	613 160	0.081	0.022
(5,5)	2984 645	2985 116	0.016	0.018
(5,6)	14 753 869	14 751 592	0.015	0.015
(6,5)	14 750 984	14 751 592	0.004	0.015
(6,6)	73 823 947	73 825 416	0.002	0.013
(6,7)	373 484 271	373 488 764	0.0012	0.011
(7,6)	373 488 763	373 488 764	2.7(−7)	0.011
(7,7)	1907 330 705	1907 334 616	2.1(−4)	0.0094
(7,8)	9820 750 775	NT		
(8,7)	9837 387 041	NT		

error accumulates as one goes to higher order.

In Table XXIV we present the number of closed polygons on a square lattice [20]. Table XXIV should be compared with Table X. Again the error in Table XXIV increases.

Finally, Table XXV shows the spontaneous magnetization coefficients for the honey-combed lattice [18] and should be compared with Table XV. Again, as expected, the error in Table XXV increases.

In Table XXVI we present the results for the $d\ln\chi/dw$ where χ is the magnetic susceptibility [8] for the 2D square lattice Ising model of Ferromagnetism (high-temperature expansion). One sees that the results are very good with the % error decreasing to 0.009% for the [7/6].

TABLE XV. Spontaneous magnetization coefficients for the honey-combed lattice. NT refers to the next (unknown) term.

(N,M)	Padé	Exact	% error	ϵ'
(2,1)	522.7	534	2.1	0.022
(2,2)	1718	1732	0.81	0.020
(2,3)	5704	5706	0.035	0.016
(3,2)	5721	5706	0.27	0.016
(3,3)	19029	19038	0.048	0.013
(3,4)	64 179.6	64176	0.0055	0.010
(4,3)	64 133.3	64176	0.067	0.010
(4,4)	218 221.7	218 190	0.015	0.0086
(4,5)	746 944.5	747 180	0.032	0.0072
(5,4)	747 250.6	747 180	0.0094	0.0072
(5,5)	2574 272.6	2574 488	0.0084	0.0062
(5,6)	8918 791.1	8918 070	0.0081	0.0053
(6,5)	8917 936.8	8918 070	0.0015	0.0053
(6,6)	31 036 314	31 036 560	7.9(−4)	0.0047
(6,7)	108 456 649	108 457 488	7.7(−4)	0.0041
(7,6)	108 457 188	108 457 488	2.8(−4)	0.0041
(7,7)	380 390 643	380 390 574	1.8(−5)	0.0037
(7,8)	1338 495 468	133 8495 492	1.8(−6)	0.0033
(8,7)	1338 495 478	133 8495 492	1.0(−6)	0.0033
(8,8)	4723 664 030	NT		

TABLE XVI. Low-temperature ferromagnetic susceptibility coefficients for the simple-cubic lattice in the Ising model. NT refers to the next (unknown) term.

(N,M)	Padé	Exact	% error	ϵ'
(0,1)	16.3	135	88	7.3
(1,1)	−1301.8	−276	370	−0.79
(1,2)	1637.1	1520	7.7	1.7
(2,1)	564.3	1520	63	1.7
(2,2)	−4171	−4056	2.8	−0.52
(22,3)	17 928	17 778	0.85	0.64
(3,2)	18 055	17 778	1.6	0.64
(3,3)	−54 137	−54392	0.47	−0.30
(3,4)	214 098	213 522	0.27	0.28
(4,3)	214 295	213 522	0.36	0.28
(4,4)	−702 351	−700 362	0.28	−0.16
(4,5)	2602 225	2601 674	0.021	0.13
(5,4)	2603 977	2601 674	0.089	0.13
(5,5)	−8835 471	−8836 812	0.015	−0.086
(5,6)	31 921 447	31 925 046	0.011	0.064
(6,5)	31 923 924	31 925 046	0.0035	0.064
(6,6)	−110 325 319	−110 323 056	0.0021	−0.043
(6,7)	392 994 671	393 008 712	0.0036	0.031
(7,6)	392 997 811	393 008 712	0.0028	0.031
(7,7)	−1369 504 263	−1369 533 048	0.0021	−0.022
(7,8)	4844 433 325	NT		
(8,7)	4844 450 320	NT		

VIII. ERRORS FOUND BY PAP's

This method is so reliable that we have found several errors in publications based on the PAP's as follows.

(1) In volume 3 of Ref. [18] and Table XVI, we get the results

$[n/m]$	PAP	Exact	% error
[6/6]	838 513 161	828 623 100	1.20
[6/7]	4 292 338 178	NT	
[7/6]	2 637 328 855	NT	

where NT is the next-unknown term. If however, we use the correct value from Sykes *et al.* [21] the results are

$[n/m]$	PAP	Exact	% error
[6/6]	838 513 161	838 623 100	0.013
[6/7]	3 761 459 955	NT	
[7/6]	3 761 255 448	NT	

and the PAP's for the NT agree with each other very well.

(2) In Ref. [22], Eq. (8) for $\beta(g)$ has a negative $\zeta(7)$

TABLE XVII. Low-temperature ferromagnetic susceptibility coefficients for the diamond lattice in the Ising model. NT refers to the next (unknown) term.

(N,M)	Padé	Exact	% error	$-\epsilon'$
(0,1)	64	44	45	0.3125
(1,1)	242	208	16.3	0.14
(1,2)	899	984	8.6	−0.0074
(2,1)	983	984	0.074	−0.0074
(2,2)	4655	4584	1.6	0.015
(2,3)	21 412	21 314	0.46	0.0019
(3,3)	98 904	98 292	0.62	0.0082
(3,4)	448 491	448 850	0.08	0.0098
(4,3)	452 236	448 850	0.75	0.0098
(4,4)	2015 940	2038 968	1.13	0.005
(5,4)	9159 502	9220 346	0.66	0.0045
(5,5)	41 426 168	41 545 564	0.29	0.0036
(5,6)	186 654 128	186 796 388	0.076	0.0022
(6,5)	186 897 493	186 796 388	0.054	0.0022
(6,6)	838 513 161	838 623 100	0.013	0.0015
(6,7)	3761 459 955	NT		
(7,6)	3761 255 448	NT		

TABLE XVIII. Low-temperature ferromagnetic susceptibility coefficients for the honey-combed lattice in the Ising model. NT refers to the next (unknown) term.

(N,M)	Padé	Exact	% error	$-\epsilon'$
(0,1)	36	27	33	0.25
(1,1)	121.5	122	0.41	−0.0041
(1,2)	551	516	6.8	0.064
(2,1)	551	516	6.8	0.064
(2,2)	4659	2148	117	0.016
(2,3)	8718	8792	0.84	0.017
(3,2)	8906	8792	1.3	0.017
(3,3)	35 696	35 622	0.21	0.010
(3,4)	143 361	143 079	0.20	0.0087
(4,3)	143 239	143 079	0.11	0.009
(4,4)	570 671	570 830	0.028	0.007
(4,5)	2264 738	2264 649	0.004	0.0056
(5,4)	2264 668	2264 649	9.(−4)	0.006
(5,5)	8942 977	8942 436	6.1(−3)	0.0047
(5,6)	35 167 051	35 169 616	7.3(−3)	0.004
(6,5)	35 155 272	35 169 616	0.041	0.004
(6,6)	137 841 646	137 839 308	1.7(−3)	0.0035
(6,7)	538 597 298	NT		
(7,6)	538 595 547	NT		

term. This term should be positive as later confirmed by the authors. The results with the minus sign are the following:

n	[1/2]	[2/1]	Exact	% error
0	828.7	792.9	39.4	2003
1	1187.1	1133.2	90.2	1216
2	1615.2	1537.8	165.5	876
3	2116.2	2009.7	268.6	688
4	2692.8	2551.9	402.9	568
5	3347.3	3166.7	571.8	485
6	4081.7	3856.6	778.6	424

where $\mathcal{P}_{\rm error}=100([1/2]-(\text{Exact}))/(\text{Exact})$. It can be seen that the PAP's do not agree with the exact results. If, however, the correct positive sign is used the results are improved tremendously.

n	[1/2]	[2/1]	Exact	% error
0	828.7	792.9	1002	17.3
1	1187.1	1133.2	1424.3	16.7
2	1615.2	1537.8	1922.3	16.0
3	2116.2	2009.7	2499.3	15.3
4	2692.8	2551.9	3158.8	14.8
5	3347.3	3166.7	3904.2	14.3
6	4081.7	3856.6	4738.7	13.9

(3) Consider the Euler Numbers 1, 5, 61, 1385, 50521.

TABLE XIX. Spontaneous magnetization coefficients for the plane-triangular lattice in the Ising model. NT refers to the next (unknown) term.

(N,M)	Padé	Exact	% error	ϵ'
(2,2)	−238.1	−228	4.4	−0.98
(2,3)	3859	4050	4.7	42.
(3,4)	30 841	30 960	0.38	29
(4,3)	29 965	30 960	3.2	29
(4,4)	−17 720	−17 670	0.28	−0.96
(4,5)	241 752	242 402	0.27	23.
(5,4)	241 734	242 402	0.28	23
(5,5)	−143 733	−152 520	5.8	−0.95
(5,6)	1932 202	1932 000	0.01	19
(6,5)	1928 973	1932 000	0.16	19
(6,6)	−1312 749	−1312 844	0.0072	−0.95
(6,7)	15 609 972	15 612 150	0.014	17
(7,6)	15 610 019	15 612 150	0.014	17
(7,7)	−11 345 873	−11 297 052	0.43	−0.94
(7,8)	127 516 921	NT		
(8,7)	127 541 268	NT		

TABLE XX. Spontaneous magnetization coefficients for the simple-cubic lattice in the Ising model. NT refers to the next (unknown) term.

(N,M)	Padé	Exact	% error	ϵ'
(1,2)	777.7	792	1.8	0.93
(2,2)	−2127	−2148	0.97	−0.34
(2,3)	7548.6	7716	2.2	0.32
(3,2)	7519.1	7716	2.6	0.32
(3,3)	−24 751	−23 262	6.4	−0.16
(3,4)	80 773.6	79 512	1.6	0.13
(4,3)	78 792	79 512	0.91	0.13
(4,4)	−252 340	−252 054	0.11	−0.073
(4,5)	846 473	846 628	0.018	0.060
(5,4)	846 652	846 629	0.0028	0.060
(5,5)	−2752 719	−2753 520	0.029	−0.032
(5,6)	9205 508	9205 800	0.0032	0.028
(6,5)	9174 623	9205 800	0.34	0.028
(6,6)	−30 381 945	−30 371 124	0.036	−0.013
(6,7)	101 941 199	101 585 544	0.35	0.014
(7,6)	101 543 745	101 585 544	0.041	0.014
(7,7)	−338 086 890	−338 095 596	0.0026	−0.005
(7,8)	1133 485 933	NT		
(8,7)	1133 487 959	NT		

The results are shown in Table XXVIII. Here again the results are very good. The results are much improved when the correct series is used. Somehow the PAP works for what we may call the natural series (NS) and not for the unnatural series (UNS) which is what the series with an error apparently were. NS are derived from functions with sensible analytic continuations outside their radius of convergence and UNS are not, as the PA is a form of approximant analytic continuation.

Examples of NS are given in this paper, for instance,

NS $\tanh X$

$1,2,3,4$

UNS $\tanh X+10^7X^4$

$1,2,3,10^7$

(4) Consider the spontaneous magnetization

TABLE XXVI. $d\ln\chi/dw$ where χ is the magnetic susceptibility for the 2D square lattice Ising model of ferromagnetism (high-temperature expansion). See Tables XXI and XXII. NT refers to the next (unknown) term.

(N,M)	No. of input coefficients	Padé	Exact	% error
(1,1)	3	98	48	104
(1,2)	4	201	164	22.8
(2,1)	4	82	164	49.8
(2,2)	5	288	296	2.8
(2,3)	6	961	956	0.48
(3,2)	6	963	956	0.76
(3,3)	7	1820	1760	3.4
(3,4)	8	4876	5428	10.2
(4,3)	8	5172	5428	4.7
(4,4)	9	10 160	10 568	3.9
(4,5)	10	33 584	31 068	8.1
(5,4)	10	33 932	31 068	9.2
(5,5)	11	67 746	62 640	8.2
(5,6)	12	177 201	179 092	1.1
(6,5)	12	178 461	179 092	0.35
(6,6)	13	370 472	369 160	0.36
(6,7)	14	1033 105	1034 828	0.17
(7,6)	14	1034 923	1034 828	0.009
(7,7)	15	2172 702	NT	

TABLE XXVII. Bernoulli numbers: 1/6, 1/30, 1/42, 1/30, 5/66, ...

(N/M)	Padé	Exact	% Error	$-\epsilon'$
(2,1)	0.0467	0.7576	38.4	4.62
(2,2)	0.1986226	0.25311	21.5	1.47
(2,3)	1.00236	1.1667	14.1	1.38
(3,3)	6.4426	7.09216	9.2	1.32
(3,4)	51.7136	54.9712	5.9	1.28
(4,4)	508.917	529.124	3.8	1.24
(4,5)	6040.26	6192.12	2.5	1.22
(5,5)	85 220.6	86 580.3	1.6	1.20
(5,6)	1411 219.5	1425 517.2	1.0	1.18
(6,6)	27 123 755.9	27 298 231.1	0.64	1.16
(6,7)	599 135 504.1	601 580 873.9	0.41	1.15
(7,7)	15 077 304 588.9	15 116 315 767.1	0.26	1.14
(7,8)	4289 118(5)	4296 146(5)	0.16	1.13
(8,8)	13 697 456(6)			

TABLE XXVIII. Euler numbers: 1,5,61,1385,50521.

(N,M)	Padé	Exact	% Error	$-\epsilon'$
(2,1)	31 446	505 21	37.8	1.61
(2,2)	2147 961	2702 765	20.5	1.47
(2,3)	173 959 381	199 360 981	12.7	1.38
(3,3)	17 684 137 933	19 391 512 145	8.8	1.32
(3,4)	2273 197 781 041	2404 879 675 441	5.5	1.28
(4,4)	356 692 782 234 136	370 371 188 237 525	3.7	1.24
(4,5)	67 755 523 470 897 901			

TABLE XXIX. Spontaneous magnetization coefficients for the body-centered-cubic lattice (PAD 6).

(N,M)	Padé	Exact	% Error	$-\epsilon'$
(3,3)	−6088.2	−6264	2.8	1.74
(3,4)	9972.7	9744	2.3	0.59
(4,4)	9206.9	10 014	8.1	1.66
(4,5)	−90 185.7	−86 976	3.7	9.75
(5,5)	202 969.7	205 344	1.2	0.73
(5,6)	−80 292.9	−80 176	0.15	0.83
(6,6)	−1003 165.4	−1009 338	0.61	33.2
(7,6)	3573 063.9	3579 568	0.18	1.3
(7,7)	−4587 323.8	−4575 296	0.26	0.64
(7,8)	−8306 222.5	−8301 024	0.06	2.4
(8,8)	53 984 689.5	54 012 882	0.05	4.6
(9,8)	−112 671 954.3	−112 640 896	0.03	0.68
(9,9)	5194 161.5	5164 464	0.58	0.98
(10,9)	694 850 424.4	694 845 120	0.0008	2935
(10,10)	−2160 968 860.3	−2160 781 086	0.009	1.0
(10,11)	2231 643 228.6			

TABLE XXX. Spontaneous magnetization coefficients for the face-centered-cubic lattice (PAD 7).

(N,M)	Padé	Exact	% Error	$-\epsilon'$
(5,5)	−12 489.6	−12 924	1.3	13.5
(5,6)	20 502.2	19 536	4.9	0.88
(6,5)	19 254.3	19 536	1.4	0.88
(6,6)	−3 493.3	−3062	14.1	0.90
(6,7)	7 086.9	8280	14.4	−16.3
(7,6)	7 939.9	8280	4.1	−16.3
(7,7)	−26 890.7	−26 694	0.74	−0.19
(7,8)	−144 697.1	−153 536	5.8	2.8
(8,8)	873 274.9	507 948	71.9	1.58
(8,9)	−429 070.1	−406 056	5.7	0.76
(9,9)	45 663.5	79 532	42.6	0.76
(9,10)	−712 454.8	−729 912	2.4	−45.9
(10,10)	−619 406.2	−631 608	1.9	1.09
(11,10)	9 251 173.9	9279 376	0.30	18.0
(11,11)	−1 5856 119.2	−15 771 600	0.54	0.88
(11,12)	7 462 809.8	7467 336	0.061	0.72
(12,12)	−10 983 055.1	−10 935 114	0.44	−2.09
(13,12)	21 878 888.0	21 835 524	0.20	−0.36
(13,13)	112 690 152.0	112 752 684	0.055	3.59
(13,14)	−400 706 509.2	−405 576 168	0.033	−1.69
(14,14)	410 377 748.2	410 287 368	0.022	0.71
(14,15)	−23 3554 029.9			

coefficients for the face-centered-cubic lattice [18] (PAD 7). The results with $S_{24}=-467336$ are given in Table XXXI. However, this term was incorrectly copied from the paper Ref. [21] to the volume Ref. [18]. The correct value is $S_{24}=7467336$. When this is used one obtains (see Table XXX above)

$[N/M]$	Padé	Exact	% error	ϵ'
[11/12]	7462 809.8	7467 336	0.061	0.72
[12/12]	−10983 055.1	−10935 114	0.44	−2.09
[13/12]	21878 888.0	21835 524	0.20	−0.36
[13/13]	112690 152.0	112 752 684	0.055	3.59
[13/14]	−400 706 509.2	−405 576 168	0.033	1.69
[14/14]	410 377 748.2	410 28 768	0.022	0.71
[14/15]	−233 554 029.9			

TABLE XXXI. Spontaneous magnetization coefficients for the face-centered-cubic lattice (PAD 7) (with error).

(N,M)	Padé	Exact	% Error	ϵ'
(11,12)	7462 809.8	−467 336	1697	1.02
(12,12)	774 506 321.2	−10 935 114	7183	−789
(13,12)	2263 518 931	21 835 524	10 270	1.09
(13,13)	−4720 901 059	112 752 684	4287	3.59
(13,14)	−4081 493 897	−400 576 168	919	1.69
(14,14)	2218 178 476	41 287 368	441	.71
(14,15)	4179 549 493			

Thus again the PAP works for the correct series which is apparently a NS while it does not for the incorrect series, which is an UNS.

IX. THE R_τ RATIO IN τ LEPTON DECAYS

We consider the QCD result for [13,23]

$$R_\tau=\frac{\Gamma(\tau\to\text{hadrons})}{\Gamma(\tau\to e\nu\bar{\nu})}\ . \tag{9.1}$$

We considered this quantity in our first paper [1] but only for $N_c=3$ and $N_f=3$, where N_c is the number of colors [Gauge group in SU(N_c)] and N_f is the number of fermions (quarks)]. R_τ is known up to four-loop order for arbitrary N_c and N_f and we can predict the next term. Here we consider R_τ for a range of values of N_c and N_f. We compare the PAP with the known result at four-loop order. Our results are shown [30] in Table VIII. In all cases $S_0=1$. The [1/1] PAP gives S_3 and our estimate for S_4 is given by the [1/2] and [2/1] PAP's.

It can be seen that the agreement between the predicted S_3 is remarkably good. Moreover, the [1/2] and [2/1] PAP's for S_4 agree very well also.

X. BERNOULLI NUMBERS, EULER NUMBERS, AND FIBBINACCI NUMBERS

The Bernoulli numbers are 1/6, 1/30, 1/42, 1/30, 5/66, . . . , and, as we shall see, form a NS. The results are shown in Table XXVII. It can be seen that the results are interesting and the % error decreases monotonically, reaching 0.16% for the [7/8] PAP.

The Euler Numbers are 1, 5, 61, 1385, 50521, . . . , and also form a NS. The results are shown in Table XXVIII. Again the results are very good.

The Fibbinacci Numbers are 1,1,2,3,5,8,13, . . . , . It can be shown that in this case the $[N/M]$ PAP is exact where $M \geq 2$. Moreover if one generalizes to a, b, $a+b$, $a+2b$, $2a+3b$, . . . , the PAP is still exact.

XI. SERIES WITH UNCLEAR SIGNS

Although we will discuss series with unusual sign patterns (Sec. XIII), there are some series for which *no* sign pattern is apparent. We will consider here four such series [18] which we denote as PAD 6, PAD 7, PAD 9, and PAD 10. PAD stands for Padé approximant data. We will see that, in spite of the fact that no sign pattern can be discerned, the PAP's are still accurate and the estimation method works. In fact, one obtains not only the correct sign, but also an accurate value.

In Table XXIX we present the results for the spontaneous magnetization coefficients (SMC's) for the body-centered-cubic lattice (PAD 6) [18]. It can be seen that the results are very good.

Table XXX shows the results for the SMC for the face-centered-cubic lattice (PAD 7) [18]. Again the results are excellent. Table XXXII gives the results for the low-temperature ferromagnetic susceptibility coefficients (LTFSC's) for the body-centered-cubic lattice (PAD 9) [18]. Again the PAP's are very good. Table XXXIII shows the results for the LTFSC for the face-centered-cubic lattice (PAD 10) [18]. Again the results are interesting.

To emphasize how remarkable it is that the PAP method works for PAD 6, 7, 9, and 10, we present the signs of these series in Table XXXIV. For PAD 6, once $S_0, \ldots, S_6$ are given (note the slash) then $S_7, S_8, \ldots, S_{21}$ give the correct signs in each case. The probability for that, if the signs were random, would be $p=1/2^{15}=3\times10^{-5}$. Similarly, for PAD 7 all the signs from $S_{11}, \ldots, S_{29}$ are correct. In this case $p=2\times10^{-6}$. Similarly, for PAD 9, $p=4\times10^{-6}$ and for PAD 10, $p=6\times10^{-8}$. It should be emphasized that the numbers, as well as the signs, are necessary for this to work. Although we do not understand these remarkable results, it is apparent that these series are what we have called NS.

XII. THEOREMS

In this section we present several new theorems which apply to the PAP's. First we present two known theorems [9].

Theorem I. If S is in a Stieljes series for $f(x)$ then the diagonal Padé (N,N) provides an upper bound and the $(N-1,N)$ and $(N,N-1)$ provide lower bounds to $f(x)$ where

$$S=\sum_{n=0}^{\infty} a_n(-x)^n\ , \qquad S_n=(-1)^n a_n\ , \tag{12.1}$$

and

$$a_n=\int_0^{\infty} t^n p(t)dt, \quad n=0,1,2,\ldots\ . \qquad p(t)\geq 0, \quad 0\leq t<\infty\ . \tag{12.2}$$

Theorem II. For every Stieljes series there is a Stieljes

TABLE XXXII. Low-temperature ferromagnetic susceptibility coefficients for the body-centered-cubic lattice (PAD 9).

(N,M)	Padé	Exact	% Error	$-\epsilon'$
(3,3)	−541.4	−576	6.0	1.0
(4,3)	373.3	519	28.1	0.61
(4,4)	2927.2	3264	10.3	8.0
(4,5)	−12 251.9	−12 468	1.7	1.6
(5,5)	21 263.6	20 568	3.4	0.57
(5,6)	26 571.5	26 662	3.7	1.8
(6,6)	−217 251.0	−215 568	0.78	7.2
(7,6)	527 274.7	528 576	0.25	0.70
(7,7)	−159 529.3	−164 616	3.1	0.87
(7,8)	−3017 585.1	−3014 889	0.089	59.8
(8,8)	10 902 952.4	10 894 920	0.074	1.2
(9,8)	−13 792 082.3	−13 796 840	0.034	0.65
(9,9)	−29 926 176.9	−29 909 616	0.055	2.7
(9,10)	190 397 178.7	190 423 962	0.014	3.9
(10,10)	−399 555 494.7	−399 739 840	0.046	0.67
(10,11)	−22 638 007.7	−22 768 752	0.57	1.0
(11,11)	2802 235 075.0	280 3402 560	0.042	2163
(12,11)	−874 325 9456	−874 3064 909	0.0022	0.97
(11,12)	−8755 658 406	−874 3064 909	0.14	0.97
(12,12)	8732 589 890			

TABLE XXXIII. Low-temperature ferromagnetic susceptibility coefficients for the face-centered-cubic lattice (PAD 10).

(N,M)	Padé	Exact	% Error	$-\epsilon'$
(5,5)	−1033.7	−1080	4.3	1.5
(5,6)	657.4	665	1.1	0.78
(6,6)	196.3	384	48.9	1.9
(7,6)	799.1	1968	59.4	−7.9
(7,7)	2180.6	2016	8.2	0.80
(7,8)	−25 139.2	−25 698	2.2	13.4
(8,8)	39 546.0	39 552	0.015	0.88
(9,8)	−9 424.4	−3872	143	0.94
(9,9)	5205 730.1	20 880	2.5(4)	−54.1
(8,10)	20 745.8	20 880	0.64	−54.1
(10,8)	21 746.0	20 880	4.1	−54.1
(7,11)	18 715.6	20 880	10.4	−54.1
(11,7)	22 522.3	20 880	7.9	−54.1
(9,10)	−65 713.9	−65 727	0.020	0.42
(10,10)	−378 156.4	−379 072	0.24	2.8
(11,10)	1279 803.5	1277 646	0.17	1.6
(11,11)	−993 854.7	−986 856	0.71	0.77
(12,11)	173 506.0	176 978	2.0	0.77
(12,12)	−2148 733	−2163 504	0.68	−67.2
(12,13)	−1829 882.1	−1818 996	0.60	1.1
(13,13)	27 914 928.5	2787 108	0.16	19.2
(13,14)	−47 086 318.0	−47 138 844	0.11	0.89
(14,14)	20 503 116.8	20 789 424	1.4	0.74
(15,14)	−36 555 836.9	−36 509 652	0.13	−3.0
(15,15)	76 910 446.3	770 55 330	0.19	−0.20
(15,16)	393 006 968.3	393 046 656	0.010	3.4
(16,16)	−140 3018 924	−140 2934 816	0.0060	1.7
(17,16)	1403 798 649	140 3843 388	0.0032	0.72
(17,17)	−798 532 957.6			

function $F(Z)$ to which it is asymptotic where

$$F(Z)=\int_0^\infty \frac{p(t)}{1+Zt}dt \ . \tag{12.3}$$

Thus $F(Z)$ is the Borel sum of the series S if the Borel sum exists. The Borel sum exists if $|a_n| \le n!C^n$. C^n for some C and all $n > N_0$ for some N_0. In general,

$$G(x)=\sum \frac{a_n(-x)^n}{n!} \tag{12.4}$$

and

$$F(Z)=\int_0^\infty p(t)G(xt)dt \ . \tag{12.5}$$

For example, the series

$$S=\sum_{n=0}^{\infty} S_n x^n \ , \quad S_n=(-n)^n n! \ , \tag{12.6}$$

TABLE XXXIV. Signs for PAD 6, 7, 9, and 10.

PAD 6
7 10 20
$+-0+-++/-++-+--+--+-++-1/2^{15}=3\times10^{-5}$
PAD 7
11 20 25 29
$+-00++-++++/-+-+--+-+--+-+-++-+$
$1/2^{19}=2\times10^{-6}$
PAD 9
7 20 24
$+00+-0+/-++-++-+--+--+--+-$
$1/2^{18}=4\times10^{-6}$
PAD 10
11 2021 30 34
$+0000+-00++/-++++-+-+--+-+--+-+-++-+$
$1/2^{24}=6\times10^{-8}$

is a Stieljes series. This series is Borel summable and is asymptotic to $E(x)$ where

$$E(x)=\int_0^\infty \frac{e^{-t}}{1+xt}dt \ , \tag{12.7}$$

which is finite for any positive value of x.

The results for $x=0.1$, 0.2 have been given in an earlier paper. The results converge to $E(x)$. In fact, we have been able to obtain a 32-figure agreement to $E(0.1)$. For the series $S=\sum a_n x^n$ the integral in this case is

$$E'(x)=\int_0^\infty \frac{e^{-t}}{1-xt}dt \ . \tag{12.8}$$

The integrand has a pole and the integral is not Borel summable. However, if one uses the principal value for $E'(x)$ the results again converge to $E'(x)$ although here we are unable to get the accuracy noted above for $E(x)$.

Theorem III. If the $[N/M]$ PAP is exact then the PAP's $[N+1/M]=[N/M+1]$. The proof of this theorem is straightforward using the linear equations of Sec. II.

Theorem IV. If the $[N/M]$ PAP is given by $[N/M]=C/D$ and the $[N+1/M+1]$ PAP is given by $[N+1/M+1]=N'/D'$, then $D' \propto S_{N+M+1}D-C$. The proof of this theorem follows from the determinental representation [8] for $[N/M]$ and $[N+1/M+1]$. It follows that if $[N/M]$ is exact then $D'=0$. In this case one gets either $[N+1/M+1]=\infty$, or 0/0. In the latter case, the $[N+1/M+1]$ PAP may still be very accurate.

A consequence of theorem III is that if the $[N/M]$ PAP is relatively precise compared with neighboring predictions then the PAP's $[N+1/M]$ and $[N/M+1]$ will be very close to being equal. A consequence of theorem IV is that if the (N,M) PAP is relatively precise compared with neighboring predictions, then the $[N+1/M+1]$ PAP may be very big and much larger than the exact result, and hence, the % error may be large.

Theorem V. If $f(n)=S_{n+1}/S_n$, then

$$\epsilon_n=\frac{f(n+1)-f(n)}{f(n)} \ . \tag{12.9}$$

This is a trivial relation which will be of use later on. If $\epsilon_n=0$ then the PAP is exact. For $\epsilon_n\neq 0$, however, the PAP may still be exact due to cancellations.

Theorem VI. If one makes the scale change $x\to x'=\lambda x$ then the % error for the PAP is unchanged. This can be seen from the % error in terms of the ϵ_n (See Sec. VI).

XIII. SIGNS AND SUMS OF GEOMETRIC SERIES

As we have shown in Sec. II, the PAP $[N/M]$ is exact for a geometric series (GS), $S_n=(\pm 1)^n ar^n$ for all $[N/M]$, $M>0$. Here we generalize this result for sums of GS.

We have proved the following. The [1/2] PAP is exact for the sum of 2 GS. The [2/3] is exact for the sum of 3 GS. The [3/4] is exact for the sum of 4 GS. The generalization of this result is obvious. In each case the PAP's remain exact as one goes up in N and M.

Now we can add different sign sequences as follows (the ${}^nC_n$ are the binomial coefficients):

$$\begin{aligned}
&(-1)^n C_1:\ (\pm 1)^n\\
&(-1)^n C_2:\ (-1)^{n(n-1)/2}=++--\\
&(-1)^n C_3:\ (-1)^{n(n-1)(n-2)/6}=+++-\\
&(-1)^n C_4:\ ++++----\\
&(-1)^n C_5:\ +++++-+-\\
&(-1)^n C_6:\ ++++++--\ .
\end{aligned} \tag{13.1}$$

Now one can calculate the ϵ_n and the $A_n=1+\epsilon_n$. For example, for ${}^nC_1$, $A_n=+1$, and $\epsilon_n=0$. It is obvious from Eq. (6.4) that the % error $\mathcal{P}_{\text{error}}$ is zero,

$$\mathcal{P}_{\text{error}}[n/1]=0, \quad n=1,2,\ldots \ . \tag{13.2}$$

Now for ${}^nC_2$,

$$A_n=1+\epsilon_n=-1 \tag{13.3}$$

and the PAP is exact. This is also true for $A_n=1+\epsilon_n=1$. From Eq. (6.6) we find that

$$\mathcal{P}_{\text{error}}[n/2]=0, \quad n=1,2,\ldots \ , \tag{13.4}$$

and the PAP is exact. From Eq. (6.10) we find that for ${}^nC_3$

$$\mathcal{P}_{\text{error}}[n/3]=0, \quad n=2,3,\ldots , \tag{13.5}$$

for both $A_n=1$ and $A_n=-1$.

From Eq. (6.14) one can easily show that for ${}^nC_4$

$$\mathcal{P}_{\text{error}}[n/4]=0, \quad n=3,4,\ldots , \tag{13.6}$$

for $A_n=1$, $A_n=-1$, and $A_n(-1)^n$. One can continue this process for ${}^nC_5$, etc. Once the PAP is exact it remains exact as one goes from $[n/1]$, $n=0,1,\ldots$, to $[n/2]$, $n=1,2,\ldots$, to $[n/3]$, $n=2,3,\ldots$, to $[n/4]$, $n=3,4,\ldots$, etc.

In general, we have shown that for $m\geq 3$, and

$$S_n=\{-1\}^{{}^nC_m} ar^n \tag{13.7}$$

that the corresponding A_n's are given by

$$A_n=\{-1\}^{{}^nC_{m-2}} \ . \tag{13.8}$$

Thus $A_n=\pm 1$ with the sign sequence given by ${}^nC_{m-2}$. For high enough Padé's, as discussed above, the PAP is exact.

If

$$A_n=1+a/n \ , \tag{13.9}$$

one can easily show that

$$p[n/2]\sim -2a^2/n^2 \tag{13.10}$$

and the error decreases rapidly as $n\to\infty$. If, however,

$$A_n=-1+a/n \tag{13.11}$$

the error decreases more slowly as $n\to\infty$,

$$p[n/2]\sim 4a/n \ , \tag{13.12}$$

TABLE XXXV. Sums of geometric series and different sign sequences.

	${}^nC_1$	${}^nC_2$	${}^nC_3$	${}^nC_4$
(0,1)	1			
(1,2)	2	1		
(0,2)	1			
(2,3)	3	1		
(1,3)	2	1		
(0,3)	1			
(3,4)	4	2	1	1
(4,5)	5			
(3,5)	4	2	1	
(5,6)	6	3		
(7,8)	8	4	2	2
(15,16)	16	8	4	4

where $\mathcal{P}_{\text{error}}=100p$. It is interesting to note [24] that the double-geometric series r, ar, abr, a^2br, $a^2b^2r, \ldots,$ is a special case of the sum of two geometric series. Similarly, for the triple-geometric series, etc.

In each case one has to go to higher-order PAP's to get exact results. One then obtains a table. (Table XXXV) This table shows the number of GS's (summed for which the PAP is exact). Once the sign pattern is learned from one or two complete cycles, the PAP gets all the rest of the signs (and number) correct. This can be understood by realizing that we have (for 1 GS) a sum of GS determined from the sign patterns. For example, for ${}^nC_1$ we need only two terms, for ${}^nC_2$ we need four terms, for ${}^nC_3$ we need eight terms, for ${}^nC_4$ we need eight terms, for ${}^nC_5$ we need 16 terms, etc.

Now using the results in Sec. VI, if we set all $\epsilon_n=\epsilon$, $n=0,1,2$, then we get cancellations in the formulas for the % error and we obtain Table XXXVI. Thus the % error decreases as we go to higher order for two reasons. The power of ϵ increases and ϵ itself decreases.

XIV. OTHER SERIES

In this section we present our results for various other series. First we consider [25] $\tanh X$ where the series converges only for $X<\pi/2$. We give our results in Tables XXXVII and XXXVIII. It can be seen that the PAP's are excellent and we obtain accurate values for $\tanh X$, even for $X>\pi/2$.

One can see that the Padé result is much more accurate than the partial sums. For $x=10$ the partial sums are not reliable, but the PAP and the Padé are very good. The Padé gives reasonable results even for $X=20$.

We next consider $4\tan^{-1}X$, where for $X=1$ the result should be π. Our results are shown in Table XXXIX. Again the PAP's are very accurate and we obtain π to 14 figures.

Once again the Padé and the PAP are excellent while the partial sums are not, especially for large X. In fact, we have obtained π in 32 figures, but due to space limitations we cannot present that many figures here. From Table XXXIX one can see that theorem I of Sec. XII is satisfied here. The reason is that the expansion for $4\tan^{-1}X$ is a Stieljes series with

$$S_n=(-1)^n a_n=\frac{(-1)^n}{2n+1}=\int_0^\infty t^n p(t)dt\ , \tag{14.1}$$

where

$$\begin{aligned} &p(t)=\frac{1}{2\sqrt{t}},\quad 0\le t\le 1\\ &p(t)=0,\quad t>0 \end{aligned} \tag{14.2}$$

and, hence,

$$p(t)\ge 0\ . \tag{14.3}$$

We next consider the series

$$f(n)=S_{n+1}/S_n=\frac{a+bn}{1+cn} \tag{14.4}$$

for arbitrary a, b, and c.

Our result is ($\mathcal{P}_{\text{error}}=100p$)

$$p[n-1/2]\sim\frac{2(b-ac)(ac-bc-b)}{b^2c^2n^4}\ . \tag{14.5}$$

Thus the error decreases rapidly as n increases.

Next consider the series

$$S_n=\frac{a+bn}{1+cn} \tag{14.6}$$

for arbitrary a, b, and c. Our result is

$$p[n-1/2]=\frac{6(b-ac)}{bcn^5} \tag{14.7}$$

and the error decreases even more rapidly.

Consider the series

$$S_n=a+bn\ . \tag{14.8}$$

It is not hard to show that the $(n-1,2)$ and higher PAP's are exact in this case. Now let us consider the

TABLE XXXVI. Relative error p for all $\epsilon_n=\epsilon$.

	M 1	2	3	4	5	6
N 0	$-\epsilon$	$-\epsilon$	$-\epsilon$	$-\epsilon$	$-\epsilon$	$-\epsilon$
1	$-\epsilon$	$-2\epsilon^2$	$-2\epsilon^2$	$-2\epsilon^2$	$-2\epsilon^2$	$-2\epsilon^2$
2	$-\epsilon$	$-2\epsilon^2$	$-6\epsilon^3$	$-6\epsilon^3$	$-6\epsilon^3$	$-6\epsilon^3$
3	$-\epsilon$	$-2\epsilon^2$	$-6\epsilon^3$	$-24\epsilon^4$	$-24\epsilon^4$	$-24\epsilon^4$
4	$-\epsilon$	$-2\epsilon^2$	$-6\epsilon^3$	$-24\epsilon^4$	$-120\epsilon^{51}$	
5	$-\epsilon$	$-2\epsilon^2$	$-6\epsilon^3$	$-24\epsilon^4$		

TABLE XXXVII. Expansion for tanh x and exact value for $x=1$ (tanh 1=0.76 159 415 595 576 489).

(N,M)	Padé (% error)	Partial sum (% error)	Exact NT	Estimated NT (% error)
(1,1)	0.041	5.0	−0.05 396 825	1.18
(1,2)	0.62(−3)	2.0	0.021 186 949	0.046
(2,2)	0.62(−5)	0.83	−0.00 886 324	0.11(−2)
(2,3)	0.42(−7)	0.34	0.00 359 213	0.20(−4)
(3,3)	0.22(−9)	0.14	−0.00 145 583	0.25(−6)
(3,4)	0.85(−12)	0.055	0.00 059 003	0.24(−8)
(4,4)	0.68(−14)	0.022	−0.00 023 913	0.19(−10)
(4,5)	0.94(−14)	0.0091	0.9692(−4)	0.11(−12)
(5,5)	0.94(−14)	0.0037	0.3928(−4)	0.13(−14)
(5,6)	0.94(−14)	0.0015	0.1592(−4)	0.83(−14)

TABLE XXXVIII. Expansion for tanh x and exact value for $x=10$ (tanh 10=0.999 999 995 878).

(N,M)	Padé (% error)	Partial sum (% error)	Exact NT	Estimated NT (% error)
(1,1)	87.0	0.80(7)	−0.05 396 825	1.18
(1,2)	24.3	0.75(9)	0.02 186 949	0.046
(2,2)	11.2	0.77(11)	−0.00 886 324	0.11(−2)
(2,3)	3.3	0.76(13)	0.00 359 213	0.20(−4)
(3,3)	0.92	0.76(15)	−0.00 145 583	0.25(−6)
(3,4)	0.21	0.76(17)	0.00 059 003	0.24(−8)
(4,4)	0.042	0.76(19)	−0.239 129(−3)	0.19(−10)
(4,5)	0.0071	0.76(21)	0.969 154(−4)	0.11(−12)
(5,5)	0.0010	0.76(23)	−0.392 783(−4)	0.13(−14)

TABLE XXXIX. Expansion for $4\tan^{-1}x$ and exact value for $x=1$ ($4\tan^{-1}1=\pi$ $=3.1415\,926\,535\,898$).

(N,M)	Padé	Partial sum	Exact NT	Estimated NT
(3,3)	3.14 161	3.284	−0.066 666	−0.06 657
(3,4)	3.141 589	3.017	+0.058 824	+0.058 800
(4,3)	3.141 587	3.017	+0.058 824	+0.058 794
(5,5)	3.14 159 267	3.232	−0.0434 782	−0.0434 779
(5,6)	3.141 592 650	3.058	+0.039 999 999	+0.039 999 908
(6,5)	3.141 592 649	3.058	0.0399 99 999	0.039 999 892
(7,7)	3.14 159 265 361	3.208	−0.0322 580 645	−0.0322 580 630
(7,8)	3.141 592 653 586	3.079	0.03 030 303 030	0.03 030 302 994
(8,7)	3.141 592 653 586	3.079	0.030 30 303 030	0.03 030 302 990
(8,8)	3.1415 926 535 903	3.200	−0.02 857 142 857	−0.02 857 142 848
(9,9)	3.1415 926 535 898	3.194	−0.025 641 025 641	−0.025 641 025 635
(9,10)	3.1415 926 535 898	3.092	0.024 390 243 902	0.024 390 243 901
(10,9)	3.1415 926 535 898	3.092	0.024 390 243 902	0.024 390 243 901
(10,10)	3.1415 926 535 898	3.189	−0.0232 558 139 535	−0.0232 558 139 531

series

$$S_n = a + bn + cn^2 \ . \tag{14.9}$$

One can show that the relative error for the $(n-1,2)$ PAP goes like

$$p \sim \frac{-4}{n^4} \tag{14.10}$$

and, hence, decreases rapidly with n.

Finally let us consider two simple examples. First we study the series

$$S_n = n^2 \ . \tag{14.11}$$

The $(n-1,2)$ PAP is given by

$$S_{n+2} = \frac{2n^4+4n^3-7n^2-12n-4}{2n^2-4n+1} \tag{14.12}$$

and, hence,

$$p = \frac{-8}{2n^4+4n^3-7n^2-12n+4} \sim \frac{-4}{n^4} \tag{14.13}$$

and decreases rapidly with n.

Finally we discuss the series

$$r_n = n^3 \ . \tag{14.14}$$

The $[n-1/2]$ PAP is given by

$$S_{n+1} = N/D \ , \tag{14.15}$$

where

$$N = 3n^7+6n^6-21n^5-36n^4+13n^3 + 54n^2+36n+8 \tag{14.16}$$

and

$$D = 3n^4-12n^3+15n^2-6n+1 \tag{14.17}$$

and, hence, the % error is given by

$$p = N'/D' \ , \tag{14.18}$$

where

$$N' = -36n^3+72n \ ,$$

$$D' = 3n^7+6n^6-21n^5-36n^4+49n^3+54n^2-36n+8 \ , \tag{14.19}$$

and, hence, the % error decreases rapidly,

$$p \sim -12/n^4 \ . \tag{14.20}$$

In fact, we have shown that, in general, for

$$r_n = n^k, \quad k = 0,1,2,\ldots, \tag{14.21}$$

the $(n-1,2)$ PAP gives accurate results with the error given by

$$p \sim -\frac{2k(k-1)}{n^4} \ . \tag{14.22}$$

We have also considered the example

$$r_n = 1/n^k, \quad k = 0,1,2,\ldots \ . \tag{14.23}$$

Remarkably the error here also goes like $1/n^4$ with the error given by

$$p \sim -\frac{2k(k+1)}{n^4} \ . \tag{14.24}$$

One can get Eq. (14.24) by merely replacing k by $-k$ in Eq. (14.22). We have proved Eqs. (14.22) and (14.24) using MAPLE.

XV. CONCLUSION

We have presented a method to estimate perturbative coefficients in QFT and statistical physics. We have given results for the PAP's and a theorem for stepping up or down in N for the $[N/M]$ PAP. We studied the R and R_τ ratios in PQCD, as well as the errors in the PAP. Using our computer program to solve the linear equations, we presented a large number of examples from statistical physics. We have found a large number of other cases in which the method works well, but have not presented them here due to space limitations.

The method is so reliable that it has enabled us to find several errors in various publications. The method also works in cases where there is no obvious sign pattern.

We have presented several theorems for Padé approximants and the PAP. We have studied sums of geometric series and various sign patterns and have presented systematic results for them. Other mathematical series were also considered.

In summary, this method of estimation works in a large number of cases in a wide variety of areas. It remains to understand which cases are natural series for which the method works and which are unnatural series for which the method does not work. Equation (6.15), however, gives a sufficient condition, but not necessary condition, for a series to be a natural series.

After this work was done we were made aware of two earlier papers on this subject. Luban and Chew [26] considered a_e, however their result for the [1/2] PAP is incorrect. In their notation, it should be

$$[1/2] = \frac{1}{\Delta_3^2}(C_0\Delta_1^2 + 2C_1\Delta_1\Delta_2 + C_2\Delta_2^2) \ . \tag{15.1}$$

Fleisher, Pindor, Raczka, and Raczka [27–29] discussed the R ratio in QCD, but unfortunately used an incorrect four-loop result. For recent results, see Refs. [28,29].

ACKNOWLEDGMENTS

One of us (M.A.S.) would like to thank the theory group at SLAC for its kind hospitality. He would also like to thank the following people for very helpful discussions: David Atwood, Bill Bardeen, Richard Blankenbecler, Eric Braaten, Stan Brodsky, Dean Chlouber, Lisa Cox, N. Deshpande, H. W. Fearing, Steve Godfrey, Pat Kalyyniak, Mike Lieber, Bill Marciano, Leila Meehan, John Ng, Helen Perk, Jacques Perk, Martin Perl, Dominique Pouliot, Helen Quinn, Tom Rizzo, Ken Samuel, Len Samuel, Davison Soper, George Sudarshan, Levan

Surguladze, N. V. V. J. Swamy, and Richard Woloshyn. This work was supported by the U.S. Department of Energy under Grant Nos. DE-FG05-84ER40215 and DE-AC03-76SF00515.

Note added: Since this paper was initially written we have made significant progress as follows:

(1) We have found a way of obtaining error bars for our estimates. This makes our estimates more useful (see Ref. [28]). These error bars are large when the estimate is not accurate. In Table XXX (PAD 7) the [8/8] has a 71.9% error because the % error for the [7/7] is small. See theorem IV. The [9/9] has a relatively large 42.6% error but the error bars here are large. In Table XXXII (PAD 9) the [4/4] has large error bars. In Table XXXIII (PAD 10) the [6/16], [7/6], [9/8], and the [9/9] all have large error-bars. The [9/9] has a very large error. This is due to the fact that the [8/8] has a very small error. See theorem IV.

(2) We have proven the following theorem which provides a sufficient condition for the PAP's to converge to the exact result:

If $g(n)=d^2\ln S_n/dn^2$ then a sufficient condition for the PAP to be accurate is $\lim_{n\to\infty} g(n)=0$. Furthermore, if $\epsilon_n = e^{g(n)}-1$, and $\epsilon_n \sim A/n$ then the relative error for the $[N/M]$ Padé is

$$p \sim \frac{-M!A^M}{N^M} .$$

If $\epsilon_n \sim B/n^2$ then

$$p \sim -\frac{M!B(B+1)(B+2)\cdots(B+M-1)}{N^{2M}}$$

see Ref. [29].

(3) We have found several more errors in various papers.

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[30] See AIP document no. PAPS PLEEE8-51-xxxx-12 for 12 pages of Tables (Tables II–VIII, and XXI–XXV). Table II–VIII give the results for $R(t)$ and $R_\tau(t)$ for various values of t. Tables XXI–XXV present our results when we use only the first few terms as input and then estimate all the rest of the terms. Order by PAPS number and journal reference from American Institute of Physics, Physics Auxiliary Publication Service, Carolyn Gelbach, 500 Sunnyside Blvd., Woodbury, NY 11797-2999. The price is $1.50 for each microfiche (98 pages) or $5.00 for photocopies up to 30 pages, and $0.15 for each additional page over 30 pages. Airmail additional. Make checks payable to the American Institute of Physics.