Mass multifractality in fragmentation

Marcelo A. F. Gomes
Departamento de Fisica, Universidade Federal de Pernambuco, 50732-910 Recife, PE Brazil

Tasso R. M. Sales
The Institute of Optics, University of Rochester, Rochester, New York 14623
(Received 24 October 1994)

It is shown that a diffusive fragmentation process leads to distributions that exhibit mass multifractality. The model is based on a power-law distribution for propagating cracks in a solid host that are modeled as random walks of finite length. This dynamic generates fragment distributions characterized by exponents that are consistent with several experiments and results in a geometric configuration of cracks that shows a multifractal character at the time of maximal fragment diversity.

PACS number(s): 05.40.+j, 05.70.Ln

Multifractal features have been found in a number of physical problems including turbulence [1], chaos [2], diffusion-limited aggregation [3], and conduction in random resistors [4]. In general, one associates the term multifractal with an infinite set of exponents that characterize the scaling properties of the moments of some probability measure relevant to a particular process [5]. If the measure is the fraction of the total mass in a box centered at each point of the system, we obtain the special case of mass (or geometrical) multifractality [6]. Mass multifractality was observed in 1987 by Tél and Vicsek [6] for analytically tractable growth models and more recently by Vicsek, Family, and Meakin [7] for diffusion-limited aggregates. Although fragmentation dynamics usually generate a complex geometric distribution of cracks and clusters, studies available deal at most with fractal dimensions. A more complete analysis would be important for a better understanding of the geometric aspects involved in such processes. In this work, computer calculations are used to show the existence of mass multifractality in fragmentation. In our model, dynamics fragmentation is induced by cracks that are simulated as noninteracting random walks with the number of steps controlled by a power-law distribution function.

Many phenomena in nature exhibit fragmentation; that is, a system is partitioned into discrete domains as a consequence of impulsive (sudden) or dynamic processes. Irreversible fragmentation dynamics are relevant to a variety of phenomena in physics, chemistry, and biology. Recently it has been observed that many fragmentation dynamics display a singular time t_D where the number of different sizes of fragments or diversity is maximal [8]. If $n(s,t)$ denotes the number of fragments of size s at time t, diversity is then defined as $\sum_s \theta[n(s,t)]$, where θ is zero if its argument is zero and one otherwise. A maximum in diversity means a maximum in the number of size scales and so the relationship with fractals [9] (i.e., systems without characteristic size scales) needs to be examined. This work deals with the internal geometry of this maximal diversity state. We are particularly interested in a diffusive fragmentation model (DFM) recently proposed [10], which seems to be consistent with experimental work on chemical and nuclear explosions and with the size distribution in sudden breakage obtained by the use of entropy maximization. Furthermore, the DFM provides new insights in cluster dynamics since certain critical exponents obtained with this model are also in agreement with those found in other nonequilibrium processes (see Refs. in [10]).

The rules of our diffusive fragmentation model are implemented on discrete square lattices with L^2 sites all initially occupied. The rules are (I) a site on the lattice is chosen at random; (II) from this site propagates a random walk (RW) of n steps obeying the power law

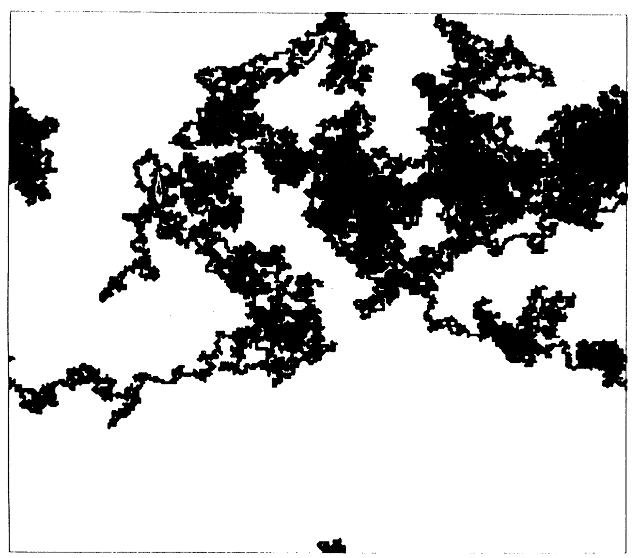

FIG. 1. Distribution of clusters at the time of maximal diversity for the model described in the text on a square lattice with 300^2 sites. The occupied sites belonging to the clusters are in white. The black regions denote the subset of the lattice formed by all points visited by some random walk (cracks) from the time $t=0$.

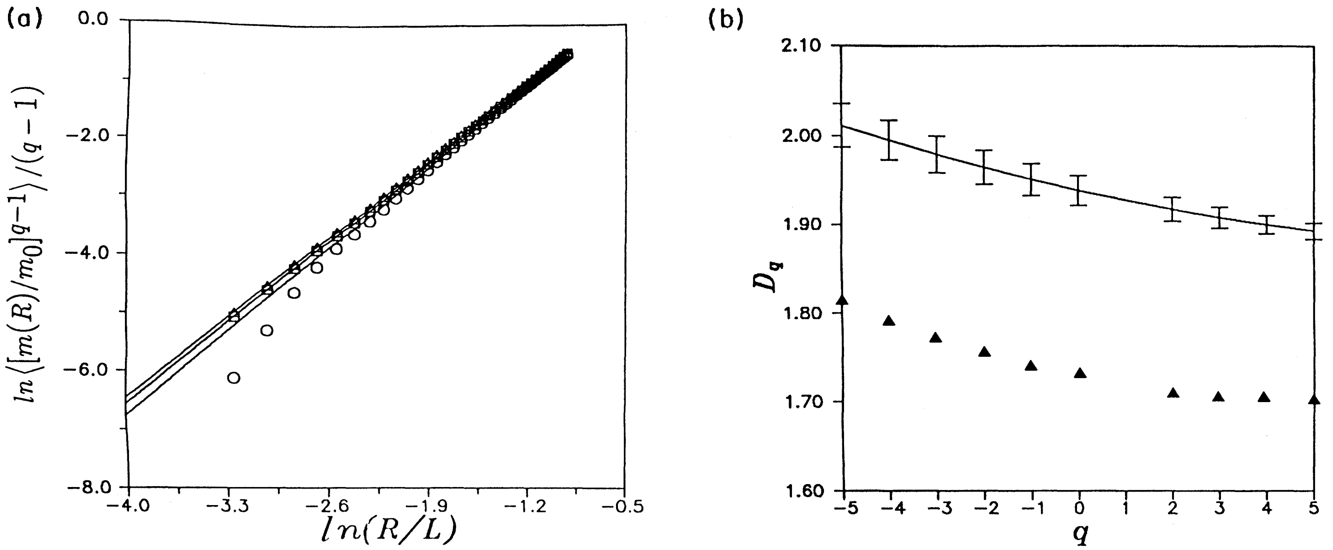

FIG. 2. (a) $\ln\langle[m(R)/m_0]^{q-1}\rangle/(q-1)$ vs $\ln(R/L)$ for $q=-5$ (○) ($D_{-5}=2.03$); 0 (□) ($D_0=1.96$); and 5 (△) ($D_5=1.91$) for the subset of fragments at the time of maximum diversity. These data were obtained for a typical experiment on a lattice with $L=1000$ over near 56 000 randomly selected sandbox centers. (b) Curves for the generalized dimensions in our model (upper curve, averaged over ten experiments) and for the DLA (based on the work of Ref. [7]).

$P(n)\sim n^{-B}$, $B>0$; and (III) all the m occupied sites visited by a RW of $m\leq n$ steps become unoccupied and the mass of the system is diminished of m units while the time increases of n units. Thus, $P(n)$ is the probability for the destructive random walk to be of length n. These rules are repeated up to the extinction of the entire initial mass of the system. Here, the exponent B is set equal to 1 and $L=1000$. These simple rules offer a plausible picture to simulate fragmentation in solids. For example, in brittle material, cracks appear, grow, and propagate as a result of dynamical processes such as rock blasting [11]. Fragments are then formed when the crack density is sufficiently high to fully surround small pieces of matter. In our model, the RW's simulate the cracks, and the sites chosen at random by rule (I) simulate the points or defects in the material where the RW's (cracks) defined by (II) begin to grow.

To study the multifractal features in the DFM at the time t_D of maximal diversity, we have used the generalized sandbox method [6,7]. First, we implement the dynamics of the DFM and perform the statistics of clusters as explained in Ref. [10]. The distributions of clusters for different times are stored and then the time of maximal diversity is identified. In order to allow the reader an opportunity to develop some intuition regarding a typical geometric distribution of the model, we display in Fig. 1 the distribution of clusters (occupied sites belonging to the region in white) for the DFM at t_D for a lattice of 300^2 available sites. As the following step, we measure the masses $m_i(R)$ within square boxes of size R, where the center of the ith box is randomly chosen among a fraction of 10% of the sites. Using the set of masses $\{m_i(R)\}$ we calculate the moments $M_q=\langle[m(R)/m_0]^{q-1}\rangle$, where $\langle\cdots\rangle$ denotes the average over the centers and m_0 is the total mass. For mass multifractals, M_q scales as $(R/L)_q^{(q-1)D}$. In this case, the generalized exponents D_q can be easily obtained for q not unity.

At a given time in the evolution of the system, there are two major subsets that can be analyzed: fragments and cracks. We have performed calculations of D_q for both cases. For fragments, the results are not convincing

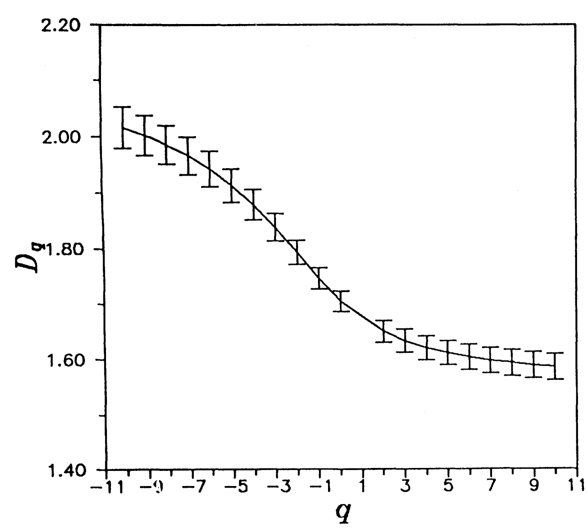

FIG. 3. The generalized dimensions D_q as a function of q for the set of cracks at t_D. These data refer to simulations on square lattices with size $L=1000$. Error bars denote extrema in D_q for ten similar experiments.

in the sense that they do not allow one to decide if multifractal features are indeed present or not. Although in some experiments the curve of D_q versus q shows the typical form, in general the scalings in single experiments do not show significant differences. As a result, error bars are considerably large when compared with the interval of variation for the dimensions; consequently, we refrain from asserting anything in this case. Further studies on larger lattice sizes would be necessary to decide this question. It is interesting to compare with the diffusion-limited aggregation (DLA) case [7] where multifractal behavior shows up for large values of L (around 3000). This may be one of the reasons why we could not detect it for the fragmentation process studied here since, within our computational power, we could go only up to $L=1000$. In Fig. 2 these results are illustrated.

The subset of cracks, however, revealed multifractal behavior clearly with a wide range of variation for the D_q (approximately from 1.6 to 2) and in particular with $D_0=1.70\pm0.02$ for the value of the fractal dimension. The dimensions are shown in Fig. 3 for q between -10 and 10. As noted, we found no problems in computing generalized dimensions even for large negative values of q as it is common in certain cases. For Fig. 3, m_0 was approximately 4×10^5.

Two basic facts serve as motivations to our study. First, is the analogy between the concept of fractals and the presence of the largest number of size scales at the time of maximal diversity t_D. Second (perhaps more important), is a better understanding of the geometric structure of clusters and cracks generated in fragmentation dynamics. In our model, a low-dimensional isotropic system is submitted to a noninteracting crack dynamics ruled by a power law and a single exponent. We have not been able to decide about the geometry of the subset of fragments mainly due to a computational limitation. On the other hand, the subset of cracks exhibits a clear multifractal nature. The fact that we have considered t_D in our study implies stringent constraints concerning the relation fragments cracks. Requiring the largest number of size scales in a given time means that the crack distribution must organize in a complex pattern to satisfy such a condition. In such a circumstance, the pattern of cracks is a mass multifractal. Our work is, in a sense, complementary to the discovery of mass multifractality in the DLA [7] holding also some analogies with a class of automata processes akin to growth phenomena [12]. In all cases, multifractality is associated with the set generated by the "active" elements of the dynamics, in this case random walks. One advantage here, however, is that we needed smaller lattices. In this context, the model presented in this work offers an excellent framework to study mass multifractality.

This work was partially supported by the Brazilian Agencies CAPES, CNPq, and FINEP.

[1] B. B. Mandelbrot, J. Fluid Mech. **62**, 331 (1974).

[2] P. Grassberger, Phys. Lett. A **97**, 227 (1983); H. G. E. Hentschel and I. Procaccia, Physica D **8**, 435 (1983); T. C. Halsey, M. H. Hensen, L. P. Kadanoff, I. Procaccia, and B. I. Shairman, Phys. Rev. A **33**, 1141 (1986).

[3] T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).

[4] L. de Arcangelis, S. Redner, and A. Coniglio, Phys. Rev. B **34**, 4656 (1986).

[5] For a review, see, G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987).

[6] T. Tél and T. Vicsek, J. Phys. A **20**, L835 (1987); T. Vicsek, Physica A **168**, 490 (1990).

[7] T. Vicsek, F. Family, and P. Meakin, Europhys. Lett. **12**, 217 (1990).

[8] K. R. Coutinho, M. A. F. Gomes, and S. K. Adhikari, Europhys. Lett. **18**, 119 (1992).

[9] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1983).

[10] M. A. F. Gomes and T. R. M. Sales, Phys. Lett. A **177**, 215 (1993).

[11] Z. Jaeger and R. Englman, J. Appl. Phys. **59**, 4048 (1986).

[12] T. R. M. Sales, Phys. Rev. E **48**, 2418 (1993).