Unified nonlinear orbit dynamics of an equilibrium electron in a helical wiggler with a positive or reversed axial-guide magnetic field at magnetoresonance

Junsheng Yu
*China Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China; Southwestern Institute of Physics, P.O. Box 432, Chengdu 610041, People's Republic of China; and Plasma Physics Group, Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom**

(Received 13 July 1994)

Nonlinear orbit dynamics are investigated analytically for a single relativistic electron in an ideal helical wiggler with a positive or reversed axial-guide magnetic field at magnetoresonance. An algebraic equation is derived for determining the maximum perpendicular velocity. The upper limit for a transverse electron orbit excursion is found to be proportional to $\varepsilon^{1/3}$ and ε for the positive and reversed guide field, respectively, where ε is the ratio of the wiggler field to the guide field. The analytical results are in agreement with numerical calculations.

PACS number(s): 41.60.Cr

I. INTRODUCTION

The free electron laser (FEL) operating in a combined helical wiggler with either positive or reversed axial-guide magnetic field has been studied experimentally [1–3] and theoretically [4–6]. It has been recognized that three regimes exist for FEL operation: group I regime at the relatively weak guide field, group II regime at the stronger guide field, and "resonance regime" where the wiggler-induced frequency approaches the cyclotron frequency. Many theoretical works show that there should be an advantageous amplification of FEL instability near magnetoresonance because of the increase of the transverse electron velocity [4,7–10]. However, there is little experimental support for a substantial efficiency enhancement. Instead, a total breakdown of the system performance has been reported for the conventional orientation of the guide field [3,11–13], and a large dip in output power observed recently by Conde and Bekefi for the reversed orientation [3]. The most important reason suggested for these experimental observations is the electron orbit loss, i.e, when the electron transverse excursions become too large, the electrons strike the drift tube wall and are lost [14–18]. So, the following questions arise. Does the upper limit of the transverse electron excursion exist at magnetoresonance for a single electron in a helical wiggler with either positive or reversed guide field? If so, how large is it and what is the dependence on parameters? What is the difference between the orbit dynamics of a positive and a reversed guide field? Although an explanation for the Conde-Bekefi experiment has been given by a nonlinear simulation based on the three-dimensional (3D) wiggler [15], analytical answers to the above questions are still awaited, even in the frame of the 1D wiggler. The present paper attempts to provide these answers. As is well known, an analytical calculation based on the 3D wiggler is very complex. An analytically tractable problem may be achieved, and some physical outlines obtained based on the 1D wiggler. On the other hand, the limit of the 1D wiggler approximation is satisfied by most of the experimental parameters. Here, we employ the 1D wiggler $\mathbf{B}_w = B_w[\cos(k_w z)\mathbf{e}_x + \sin(k_w z)\mathbf{e}_y]$ with a uniform guide magnetic field $\mathbf{B}_0 = hB_0\mathbf{e}_z$, where $h = 1, -1$ represents the positive and the reversed guide field, respectively. In addition, the beam self-fields are neglected and the parameter $\varepsilon = B_w/B_0$ is assumed to be small enough for the electron motion not to become stochastic. In Sec. II, the transverse electron excursion is obtained in terms of the Lorentz force equation. In Sec. III, the maximum transverse velocity is determined by an algebraic equation, which is simplified from the potential equation, and then the maximum perpendicular orbit excursion is obtained. The characteristic of electron motion for the positive and the reversed guide field is discussed. In Sec. IV, numerical calculations of the transverse velocity and orbit excursion are shown. Finally, conclusions and discussions are summarized in Sec. V.

II. TRANSVERSE ORBIT EXCURSION

The Lorentz force equation can be written as $d\boldsymbol{\beta}/dt = -e\boldsymbol{\beta}\times(\mathbf{B}_w + \mathbf{B}_0)/(mc)$, where $\boldsymbol{\beta} = \mathbf{v}/c$, $\gamma = \sqrt{1-\beta^2}$, $m = \gamma m_0$, c is the light speed, and e and m_0 are the electron charge and mass, respectively. The component equations are

$$\frac{d\beta_x}{d\tau} = -h\beta_y + \varepsilon\beta_z \sin k_w z \ , \tag{1a}$$

$$\frac{d\beta_y}{d\tau} = h\beta_x - \varepsilon\beta_z \cos k_w z \ , \tag{1b}$$

$$\frac{d\beta_z}{d\tau} = -\varepsilon(\beta_x \sin k_w z - \beta_y \cos k_w z) \ , \tag{1c}$$

where $\tau = \Omega_0 t$, $\Omega_0 = eB_0/(\gamma m_0 c)$ is the cyclotron

---

*Mailing address.

frequency, $\beta_x=(\Omega_0/c)dx/dt$, $\beta_y=(\Omega_0/c)dy/dt$, and $\beta_z=(\Omega_0/c)dz/dt$. The transverse electron excursion $\Delta x=x(\tau)-x(\tau_0)$ and $\Delta y=y(\tau)-y(\tau_0)$ can be obtained from Eqs. (1a) and (1b), respectively,

$$\Delta x=\frac{c}{h\Omega_0}[\beta_y(\tau)-\beta_y(\tau_0)]+\frac{\varepsilon}{hk_w}[\sin\{k_wz(\tau)\}-\sin\{k_wz(\tau_0)\}]\ , \tag{2a}$$

$$\Delta y=-\frac{c}{h\Omega_0}[\beta_x(\tau)-\beta_x(\tau_0)]-\frac{\varepsilon}{hk_w}[\cos\{k_wz(\tau)\}-\cos\{k_wz(\tau_0)\}]\ , \tag{2b}$$

where τ_0 is the initial normalized time. Setting $\tau_0=0$, $z(0)=0$, and $\beta_x(0)=\beta_y(0)=0$, $(\Delta r)^2$ can be expressed as

$$(\Delta r)^2=\frac{c^2\beta_\perp^2}{\Omega_0^2}+2\frac{c}{\Omega_0}\frac{\varepsilon}{k_w}(\beta_y\sin k_wz+\beta_x\cos k_wz)+\frac{\varepsilon^2}{k_w^2}\ , \tag{3}$$

where $\Delta r=\sqrt{(\Delta x)^2+(\Delta y)^2}$. Equation (3) implies that the transverse orbit excursion is contributed by the Larmor motion in the guide field, the wiggling motion in the wiggler field, and their coupling. It is obvious that the transverse electron excursion has an upper limit because $|\sin k_wz|\le 1$, $|\cos k_wz|\le 1$, and $|\beta_{x,y}|<1$.

III. MAXIMUM TRANSVERSE VELOCITY AND ORBIT EXCURSION

We next analyze the dependence of the upper limit on parameters. Along the line of Freund and Drobot [14], Eqs. (1a)–(1c) can be written as follows:

$$\frac{d\beta_1}{d\tau}=\beta_2\left[\frac{ck_w}{\Omega_0}\beta_3-h\right]\ , \tag{4a}$$

$$\frac{d\beta_2}{d\tau}=-\varepsilon\beta_3-\beta_1\left[\frac{ck_w}{\Omega_0}\beta_3-h\right]\ , \tag{4b}$$

$$\frac{d\beta_3}{d\tau}=\varepsilon\beta_2\ , \tag{4c}$$

in the frame of the wiggler, where $\mathbf{e}_1=\mathbf{e}_x\cos k_wz+\mathbf{e}_y\sin k_wz$, $\mathbf{e}_2=-\mathbf{e}_x\sin k_wz+\mathbf{e}_y\cos k_wz$, and $\mathbf{e}_3=\mathbf{e}_z$; and $\beta_1=\beta_x\cos k_wz+\beta_y\sin k_wz$, $\beta_2=-\beta_x\sin k_wz+\beta_y\cos k_wz$, and $\beta_3=\beta_z$. Multiplying Eq. (4a) by ε and substituting Eq. (4c) into it yields

$$C_h=\varepsilon\beta_1-\frac{ck_w}{2\Omega_0}\beta_3^2+h\beta_3\ , \tag{5}$$

which is a constant of the motion, i.e., $C_h(\tau)=C_h(\tau_0)$. Using the "energy" conservation condition $\beta_1^2+\beta_2^2+\beta_3^2=\beta_0^2$, and Eq. (4c), gives the potential equation

$$\frac{1}{4}\left[\frac{d\beta_\perp^2}{d\tau}\right]^2+V(\beta_\perp^2)=0\ , \tag{6}$$

where $V(\beta_\perp^2)=(\beta_0^2-\beta_\perp^2)\{[C_h-h\sqrt{\beta_0^2-\beta_\perp^2}+(ck_w/2\Omega_0)(\beta_0^2-\beta_\perp^2)]^2-\varepsilon^2\beta_\perp^2\}$ and $\beta_\perp^2=\beta_1^2+\beta_2^2$. It is noted that the electron motion is allowed in the region where $V(\beta_\perp^2)\le 0$. Therefore, the maximum value $\beta_{\perp\max}^2$ can be determined by

$$[C_h-h\sqrt{\beta_0^2-\beta_{\perp\max}^2}+(ck_w/2\Omega_0)(\beta_0^2-\beta_{\perp\max}^2)]^2-\varepsilon^2\beta_{\perp\max}^2=0\ , \tag{7}$$

which is related to the initial condition C_h, β_0, ck_w/Ω_0, and the parameters ε and h. Equation (7) can also be derived as follows: summing Eq. (4a) multiplied by β_1 and Eq. (4b) multiplied by β_2 yields $d\beta_\perp^2/dt=-\varepsilon\beta_2\beta_3$. The necessary condition when the transverse velocity $\beta_\perp$ approaches maximum magnitude is $\beta_2=0$ when $\beta_3\ne 0$ is assumed. Together with Eq. (5) and the condition of energy conservation, Eq. (7) is obtained. Taking the initial velocities as $\beta_1=\beta_2=0$ yields $\beta_0=\beta_{\parallel 0}$, $C_h=\beta_{\parallel 0}(h-ck_w\beta_{\parallel 0}/2\Omega_0)$. Taking the first three terms in the expansion $(\beta_{\parallel 0}^2-\beta_{\perp\max}^2)^{1/2}\cong\beta_{\parallel 0}[1-(\beta_{\perp\max}/\beta_{\parallel 0})^2/2-(\beta_{\perp\max}/\beta_{\parallel 0})^4/8]$, and applying the magnetoresonance condition $ck_w\beta_{\parallel 0}/\Omega_0=1$, Eq. (7) can be simplified to an algebraic equation

$$\frac{h}{8}\left[\frac{\beta_{\perp\max}}{\beta_{\parallel 0}}\right]^3+\tfrac{1}{2}(h-1)\left[\frac{\beta_{\perp\max}}{\beta_{\parallel 0}}\right]\mp\varepsilon=0\ , \tag{8}$$

which determines the maximum transverse velocity as follows:

$$\beta_{\perp\max}\cong\begin{cases}2\beta_{\parallel 0}\varepsilon^{1/3}\ , & h=1 \quad (9a)\\ \beta_{\parallel 0}\varepsilon\ , & h=-1\ . \quad (9b)\end{cases}$$

The expression of $\beta_{\perp\max}$ for $h=1$ is in agreement with the scaling laws [10]. Substituting (9) into (3), and taking $\beta_1=\beta_{\perp\max}$, we obtain the upper limit of the transverse orbit excursion,

$$(\Delta r)_{\max}\cong\begin{cases}\dfrac{2}{k_w}\varepsilon^{1/3}+\dfrac{\varepsilon}{k_w}\ , & h=1 \quad (10a)\\[2ex] \dfrac{\varepsilon}{k_w}+\dfrac{\varepsilon}{k_w}\ , & h=-1\ . \quad (10b)\end{cases}$$

The first term on the right side of Eq. (10) refers to the Larmor motion, the second term to the wiggling motion. Equation (10) shows that the maximum perpendicular orbit excursion is dominated by the Larmor motion in the case of the positive guide field, and by both the Larmor motion and the wiggling motion in the case of the reversed guide field. Equations (9) and (10) also show that the maximum transverse velocity and excursion for the positive guide field are much greater than those for the reversed guide field.

IV. NUMERICAL ANALYSIS

The transverse velocity β_x ($\beta_{\perp\max}$) and the transverse orbit excursion Δr ($\Delta r_{\max}$) are calculated from Eq. (1) [Eqs. (9) and (10)] at magnetoresonance $ck_w\beta_{\parallel 0}=\Omega_0$, which are shown in Fig. 1 for the positive guide field

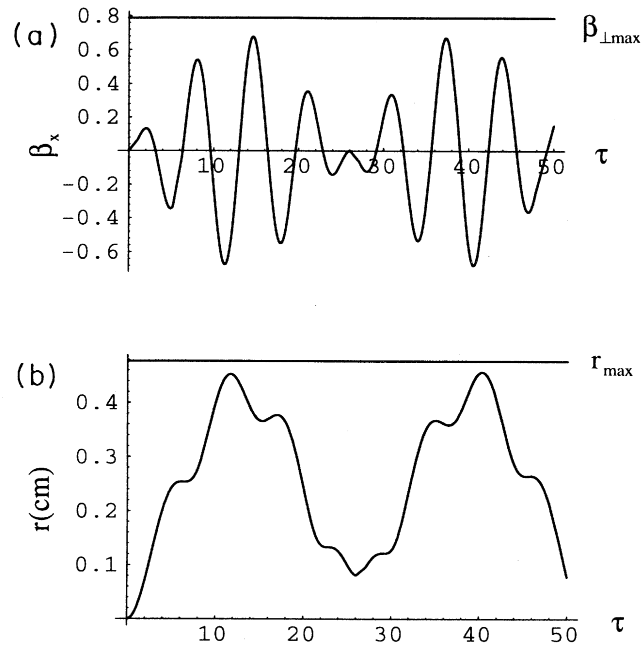

FIG. 1. (a) The transverse velocity β_x vs τ, (b) the transverse orbit excursion Δr or r vs τ, $h=1$, $\varepsilon=0.08$, $x(0)=0$, $y(0)=0$, $z(0)=0$, $\beta_{x0}=0$, $\beta_{y0}=0$, $\beta_{\parallel 0}=0.914$, and $\lambda_w=3.18$ cm.

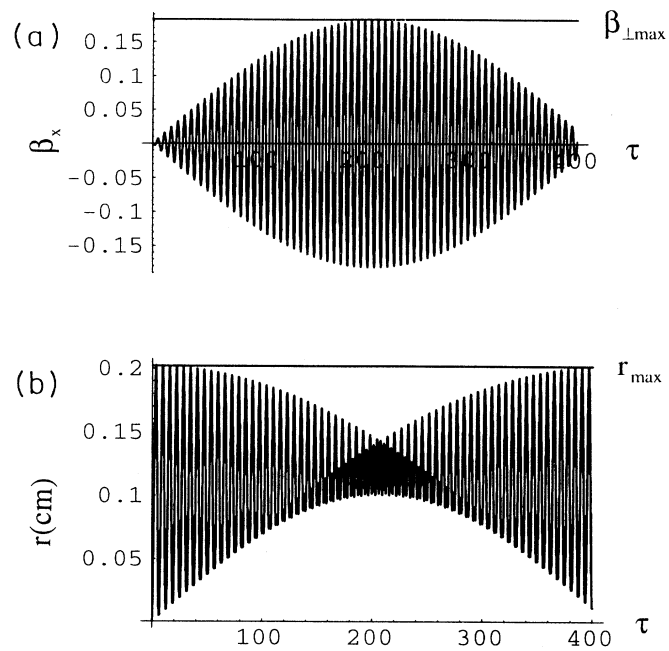

FIG. 2. (a) The transverse velocity β_x vs τ, (b) the transverse orbit excursion Δr or r vs τ, $h=-1$, $\varepsilon=0.2$, $x(0)=0$, $y(0)=0$, $z(0)=0$, $\beta_{x0}=0$, $\beta_{y0}=0$, $\beta_{\parallel 0}=0.914$, and $\lambda_w=3.18$ cm.

$h=1$, and Fig. 2 for the reversed guide field $h=-1$. Taking $\tau_0=0$, $x(0)=0$, $y(0)=0$, $z(0)=0$, $\beta_{x0}=0$, $\beta_{y0}=0$, $\beta_{\parallel 0}=0.914$, and $\lambda_w=3.18$ cm, $\varepsilon=0.08(h=1)$, $\varepsilon=0.2(h=-1)$, and Δr (Δr_{max}) equals r (r_{max}). It can be seen that the numerical results are in agreement with the analytical results. According to both the analytical and numerical studies on the orbit for the positive and reversed guide field, we have found that the dynamical behavior of the orbit is quite different for the two cases. Physically, it may be clarified at the beginning of electron motion in the wiggler. Differentiating Eq. (1a) with respect to τ, substituting Eqs. (1a) and (1b), and neglecting the term of $O(\varepsilon^2)$ yields

$$\frac{d^2\beta_x}{d\tau^2}+\beta_x=\varepsilon\beta_z\left[h+\frac{ck_w\beta_z}{\Omega_0}\right]\cos k_w z \ . \tag{11}$$

During the early stage of the injection of an electron into the wiggler under the condition of magnetoresonance, there are $\beta_z \cong \beta_{z0}$, $k_w z \cong \tau$. Equation (11) can be regarded approximately as a forced oscillation equation with resonance at the fundamental frequency. The amplitude of the force term is nearly $\varepsilon\beta_{z0}(h+1)$, which becomes $2\varepsilon\beta_{z0}$ for $h=1$ and zero for $h=-1$. This means that the electron undergoes a forced resonant oscillation for the positive guide field but an extremely weak forced resonant oscillation for the reversed case.

V. CONCLUSIONS AND DISCUSSIONS

We have presented a unified analysis of the nonlinear orbital dynamics for a single relativistic electron in a helical wiggler with positive- or reversed-axial-guide magnetic field at magnetoresonance. We have found that (1) the transverse electron excursion is upper limited; (2) the upper limit is dominated by the Larmor motion for the positive guide field, but by both the wiggling motion and the Larmor motion for the reversed guide field; (3) the maximum transverse velocity and the maximum transverse electron excursion for the positive guide field are greater than those for the reversed guide field; and (4) at the beginning of the electron motion in the wiggler, the electron undergoes a forced resonant oscillation for the positive guide field but an extremely weak forced resonant oscillation for the reversed guide field.

It should be pointed out that the results in this paper are restricted by the 1D wiggler approximation. The practical details of the electron motion should be modified by the 3D wiggler, when the transverse gradients of the wiggler cannot be neglected [18].

ACKNOWLEDGMENTS

I am grateful to Dr. S. Zhang, Dr. A. Gillespie, Dr. G. Spindler, and Dr. G. Renz for useful discussions. I am particularly grateful to Professor M. Haines for his arrangement of my visit at Blackett Laboratory and A. Knight for his kind help. This work was partly done at Southwestern Institute of Physics, University of Abertay Dundee, and finished at Imperial College London. It was supported by the China Center of Advanced Science and Technology (World Laboratory), the U.K. Royal Society, and Universities' China Committee In London.

[1] S. H. Gold, D. L. Hardesty, A. K. Kinkead, L. R. Barnett, and V. L. Granaststein, Phys. Rev. Lett. **52**, 1218 (1984).

[2] J. Fajans, J. S. Wurtele, G. Bekefi, D. S. Knowles, and K. Xu, Phys. Fluids **28**, 1995 (1985).

[3] M. E. Conde and G. Bekefi, Phys. Rev. Lett. **67**, 3082 (1991); IEEE Trans. Plasma Sci. **20**, 240 (1992).

[4] A. K. Ganguly and H. P. Freund, IEEE Trans. Plasma Sci. **16**, 167 (1988).

[5] H. P. Freund and A. K. Ganguly, IEEE J. Quantum Electron. **21**, 1073 (1985).

[6] H. P. Freund and A. K. Ganguly, IEEE Trans. Plasma Phys. **20**, 245 (1992).

[7] P. Sprangle and V. L. Granatstein, Phys. Rev. A **17**, 1792 (1978).

[8] L. Friedland and J. L. Hirshfield, Phys. Rev. Lett. **44**, 1456 (1980).

[9] W. A. McMullin and R. C. Davidson, Phys. Rev. A **25**, 3130 (1982).

[10] G. Spindler and G. Renz, Phys. Plasmas **1**, 398 (1994).

[11] R. K. Parker, R. H. Jackson, S. H. Gold, H. P. Freund, V. L. Granatstein, P. C. Efthimion, M. Gerndon, and A. K. Kinkead, Phys. Rev. Lett. **48**, 238 (1982).

[12] A. N. Didenko, A. R. Borisov, G. R. Fromenko, A. V. Kosevikov, G. V. Melnikov, Yu. G. Stein, and A. G. Zerlitsin, IEEE Trans. Nucl. Sci. **28**, 3169 (1981).

[13] K. Sakamoto, T. Kobayashi, Y. Kishimoto, S. Kawasaki, S. Musyoki, A. Watanabe, M. Takahashi, H. Ishizuka, and M. Shiho, Phys. Rev. Lett. **70**, 441 (1993).

[14] H. P. Freund and A. T. Drobot, Phys. Fluids **25**, 736 (1982).

[15] H. P. Freund and A. K. Ganguly, IEEE Trans. Plasma Sci. **20**, 245 (1992).

[16] S. Zhang, *Introduction to Free-Electron Lasers* (SWJU, Chengdu, 1993, in Chinese).

[17] Xiao-Jian Shu, Phys. Lett. A **162**, 274 (1992).

[18] K. R. Chu and A. T. Lin, Phys. Rev. Lett. **67**, 3235 (1991).