Improved fluid-dynamic model for vehicular traffic

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The fluid-dynamic traffic model of Kerner and Konhäuser [Phys. Rev. E **48**, 2335 (1993); **50**, 54 (1994)] is extended by an equation for the vehicles' velocity variance. It is able to describe the observed increase of velocity variance immediately before a traffic jam develops. Another modification takes into account the finite amount of space that each vehicle needs. As a consequence, the improved traffic model does not produce densities that exceed the maximum vehicle density or negative velocities, like former models did.

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I. INTRODUCTION

During the past decades, physicists and engineers have tried to develop traffic models with the aim of optimizing traffic flow. Because of the drastical deterioration of the traffic situation, this field has recently become very urgent. However, there is still lively debate about the most suitable approach for an efficient real-time traffic control. Apart from some *follow-the-leader models* [1–4] and some *microscopic models* of driver behavior [4–9], many of the proposed models utilize the analogy of vehicular traffic with gases and fluids. In particular *gas kinetic* (Boltzmann-like) *models* [10–13], *fluid-dynamic models* [14–23], and *cellular automaton models* [24,25] have been suggested. However, since cellular automaton models contain an extreme number of variables, their numerical simulation is rather time consuming and needs the application of supercomputers. Otherwise simplifications have to be made that lead to a coarse-grained description (so-called low-fidelity models). The problem with gas kinetic models is that the introduced average *phase space density* $\hat{\rho}(r,v,t)$ of vehicles per lane with speed v at place r and time t is a very small quantity so that the corresponding empirical data are subject to considerable fluctuations. Moreover, Boltzmann-like models include too many variables to be solved with computers in real time and to be applicable to on-line traffic control. Therefore, some authors have preferred to apply fluid-dynamic models that can be derived from gas kinetic traffic equations as mean value equations for collective variables such as the average spatial *density* $\rho(r,t) = \int dv\, \hat{\rho}(r,v,t)$ per lane or the *average velocity* $V(r,t) = \int dv\, v\hat{\rho}(r,v,t)/\rho(r,t)$ [11–13,26].

This paper is organized as follows. In Sec. II the Lighthill-Whitham model [27] and the Kerner-Konhäuser model [20,21] will be briefly discussed since they are the basis of the "improved model" presented in Sec. III. This introduces a modification that takes into account the safe distance each driver keeps. Moreover, it extends the Navier-Stokes-like equations of Kerner and Konhäuser by a third differential equation for the velocity variance of vehicles. In Sec. IV a linear stability analysis about the stationary and spatially homogeneous solution shows the instability of traffic flow above a critical density and reveals significant differences with respect to the Kerner-Konhäuser model, including the onset of instability at higher densities and shorter wavelengths. Computer simulations for a small initial perturbation presented in Sec. V confirm the instability, but also demonstrate substantially different predictions for the detailed traffic flow that develops from this perturbation.

II. FLUID-DYNAMIC TRAFFIC MODELS

The simplest fluid-dynamic model is based on the *continuity equation*

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial r}(\rho V)\,. \tag{1}$$

Equation (1) describes the conservation of the number of vehicles on a road without entrances and exits. Since $V(r,t)$ is an unknown quantity, Lighthill and Whitham [27] suggested to assume

$$V(r,t) = V_e[\rho(r,t)] \tag{2}$$

with a static velocity-density relation $V_e(\rho)$ that could be determined from traffic data or derived from the equilibrium solution of Boltzmann-like traffic models [10]. As several authors showed [14,27–29], Eqs. (1) and (2) can be applied to the description of *nonlinear density waves* and the formation of *shock waves* (i.e., discontinuous density changes) in traffic flow.

However, by (2), Lighthill and Whitham's model assumes that the velocity V is always in equilibrium. Therefore, it does not supply an exact description of ramps or bottlenecks (where traffic flow is not in equilibrium). For the same reason, it does not describe the well-known self-organization of *stop-and-go traffic* with a typical oscillation frequency and wavelength [30–32]. In order to remove this shortcoming it was suggested to handle $V(r,t)$ as an independent dynamic quantity that is to be determined by an additional differential equation. Similarly to Kühne [18,19], Kerner and Konhäuser [20] gave reasons for applying the Navier-Stokes equation

$$\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial r}=-\frac{1}{\rho}\frac{\partial P}{\partial r}+\frac{1}{\tau}[V_e(\rho)-V] \tag{3}$$

for an unidirectional n-lane road. Analogous to gas kinetic models,

$$P(r,t):=\rho(r,t)\Theta(r,t)-\eta_0\frac{\partial V(r,t)}{\partial r} \tag{4}$$

is called the *pressure* of the "vehicular gas." $\Theta(r,t):=\int dv\,(v-V)^2\hat{\rho}(r,v,t)/\rho(r,t)$ is the vehicular *velocity variance* and η_0 denotes a *viscosity coefficient.* The viscosity term $-\eta_0\partial V/\partial r$ causes sudden velocity changes to be smoothed [18,19]. The term $[V_e(\rho)-V]/\tau$, where τ denotes a *relaxation time* that corresponds to the average acceleration time, describes the drivers' tendency to drive with the density-dependent velocity $V_e(\rho)$. This velocity $V_e(\rho)$ is given by the *average desired velocity* W of the drivers diminished by an amount that is determined by necessary deceleration processes when slower cars cannot be overtaken. $V_e(\rho)$ depends on the physical situation, which means the number n of lanes, the gradient of the road, the weather conditions, etc.

Kerner and Konhäuser [20,21] simulated Eqs. (1), (3), and (4) on the assumptions [33]

$$\Theta(r,t):=\Theta_0 \tag{5}$$

and

$$V_e(\rho):=V_0(\{1+\exp[(\rho/\rho_{\max}-0.25)/0.06]\}^{-1} -3.72\times10^{-6}) \tag{6}$$

(see Fig. 1), where $\rho_{\max}=1/l$ is the *maximum density* (l is the *average vehicle length*). Their numerical results are

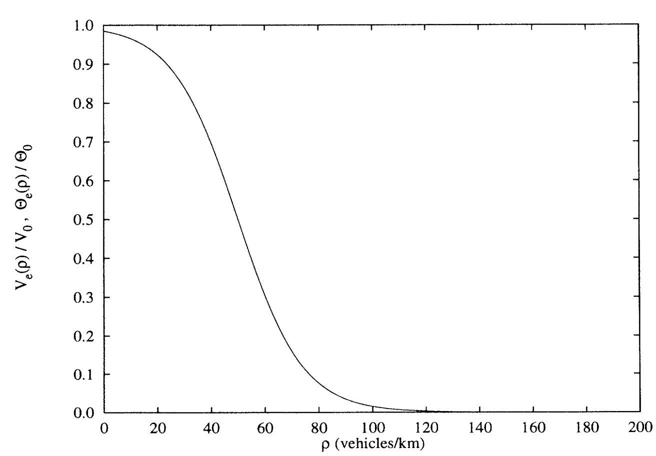

FIG. 1. Relation between the average velocity V and the density ρ [cf. Eq. (6)] and the velocity variance Θ and the density ρ [cf. Eq. (9)], respectively, in the stationary and spatially homogeneous case (indicated by a subscript e standing for equilibrium).

very encouraging. However, assumption (5) is obviously not justified since the velocity variance $\Theta(r,t)$ should decrease with decreasing velocity $V(r,t)$ and vanish if $V(r,t)$ vanishes. Moreover, for certain parameter values, e.g., $V_0=140$ km/h, $\Theta_0=(45\text{ km/h})^2$, $\tau=0.5$ min, and $\eta_0=600$ km/h, Eqs. (1), (3), and (4) produce densities $\rho(r,t)$ that exceed the maximum density per lane $\rho_{\max}=200$ vehicles/km. In the following an improved model shall be introduced that does not have these shortcomings.

III. THE IMPROVED MODEL

Similar to Eq. (3) it is reasonable to introduce an additional differential equation for the velocity variance $\Theta(r,t)$. This equation reads

$$\frac{\partial\Theta}{\partial t}+V\frac{\partial\Theta}{\partial r}=-\frac{2P}{\rho}\frac{\partial V}{\partial r}-\frac{1}{\rho}\frac{\partial J}{\partial r}+\frac{2}{\tau}[\Theta_e(\rho)-\Theta], \tag{7}$$

analogous to the equation for thermal conduction [35,36]. (Of course, here Θ does not have the interpretation of "heat" but only of velocity variance.) Equation (7) can be derived from a Boltzmann-like traffic model [13,26]. The quantity

$$J(r,t)=-\kappa_0\frac{\partial\Theta(r,t)}{\partial r}, \tag{8}$$

where κ_0 is a *kinetic coefficient*, describes a *flux of velocity variance* leading to a spatial smoothing of $\Theta(r,t)$. Similar to the viscosity term in Eq. (4), it results from the finite reaction and braking time, which causes a delayed adaption of velocity to the respective traffic situation [13,26,35,36]. The term $2[\Theta_e(\rho)-\Theta]/\tau$ originates in the same way as the term $[V_e(\rho)-V]/\tau$ in Eq. (3). On the one hand, it results from the drivers' attempt to drive with their desired velocities and, on the other hand, from the drivers' interactions [37] (i.e., from deceleration maneuvers in situations where a fast vehicle cannot overtake a slow one when approaching it). The *desired velocity* w varies from one driver to another. Therefore, even for $\rho\approx0$ a finite velocity variance $\Theta_e(\rho)$ of the vehicles is expected. Due to a lack of an empirical relation for $\Theta_e(\rho)$ we shall assume

$$\Theta_e(\rho):=\Theta_0(\{1+\exp[(\rho/\rho_{\max}-0.25)/0.06]\}^{-1} -3.72\times10^{-6}), \tag{9}$$

which is at least phenomenologically justified [compare to (6) and Fig. 1].

It is not necessary to derive equations for higher moments $\overline{(v-V)^k}:=\int dv\,(v-V)^k\hat{\rho}(r,v,t)/\rho(r,t)$ of the vehicles' velocities v. For example,

$$\overline{(v-V)^3}=\gamma\Theta^{3/2} \tag{10}$$

is a negligible quantity since the *skewness* γ is small for low densities ρ and the velocity variance Θ is small for high densities ρ. This is confirmed by traffic data [12].

One additional advantage of approach (7)–(9) in comparison with (5) is described in the following. Assume that we have a sudden increase $\partial\rho/\partial r > \rho[V_e(\rho) - V]/(\tau\Theta_0)$ of density ρ at the end of a traffic jam that is not moving ($V = 0$). In this case Eqs. (3)–(5) predict that the vehicles will begin to drive with negative velocity due to

$$-\frac{1}{\rho}\frac{\partial P}{\partial r} = -\frac{\Theta_0}{\rho}\frac{\partial \rho}{\partial r} < 0, \tag{11}$$

which leads to $\partial V/\partial t < 0$. In contrast to this, (7)–(9) give rise to non-negative velocities because of $\Theta \approx 0$, which implies $(1/\rho)\partial P/\partial r \approx 0$.

We will now introduce some corrections which are due to the fact that vehicles are not pointlike objects, as is implicitly assumed with Eqs. (4) and (8), but objects that occupy a space of length

$$s(V) = l + V\Delta T\,. \tag{12}$$

Here l is the *average vehicle length*, ΔT is about the *reaction time*, and $V\Delta T$ the velocity-dependent *safe distance*. As a consequence of (12) one would have to calculate a *virial expansion* in $\rho(r,t)s(V)$ for P and J. Since this results in *approximate* relations for *moderate* densities only [35,36], we will, as a first approach, assume as corrected (interpolated) relations [26]

$$P(r,t) := \frac{\rho(r,t)\Theta(r,t)}{1 - \rho(r,t)s[V(r,t)]} - \eta\frac{\partial V(r,t)}{\partial r}\,, \tag{13a}$$

$$\eta(\rho, V) := \frac{\eta_0}{1 - \rho s(V)}, \tag{13b}$$

and

$$J(r,t) := -\kappa\frac{\partial \Theta(r,t)}{\partial r}\,, \qquad \kappa(\rho, V) := \frac{\kappa_0}{1 - \rho s(V)}\,. \tag{14}$$

These relations are at least plausible from a phenomenological point of view [26]. The first term of (13a) was chosen to be completely analogous to the pressure formula of van der Waals for a *real gas*. Equations (13) and (14) take into account the "repulsive" interactions between vehicles [38], which become more and more important for high densities because the vehicles are queuing. According to (13) and (14), the repulsive effect of the pressure and the smoothing effect of the kinetic coefficients η, κ grow with increasing density ρ (compare Fig. 5 to Fig. 4). For $\rho \to \rho_{\max}$ we have $\rho s(V) \to \rho_{\max}(l + 0\Delta T) = 1$ so that the diverging pressure P suppresses an increase of velocity and the term $-(1/\rho)\partial P/\partial r$ causes a homogenization of traffic flow. This prevents $\rho(r,t)$ from exceeding the maximum density $\rho_{\max}$, which is confirmed by numerical results. In order to determine the exact functional dependence of P, η, and κ on $\rho s(V)$, future investigations are necessary which may lead to slight modifications of (13) and (14).

IV. LINEAR STABILITY ANALYSIS

The stationary and spatially homogeneous solution of Eqs. (1), (3), and (7) is

$$\rho(r,t) = \rho_e\,, \quad V(r,t) = V_e(\rho_e)\,, \quad \Theta(r,t) = \Theta_e(\rho_e)\,, \tag{15}$$

where ρ_e denotes the mean vehicle density on the entire road. The unpleasant thing about Eqs. (1), (3), and (7) is that the homogeneous solution (15) becomes unstable for a large traffic volume $\rho_e V_e$. As a consequence, there develops stop-and-go traffic that is less efficient than a homogeneous flow according to (15). The region of instability is found via a linear stability analysis with

$$\begin{aligned} \rho(r,t) &= \rho_e + \tilde{\rho}e^{ikr-\lambda t}\,, \\ V(r,t) &= V_e + \tilde{V}e^{ikr-\lambda t}\,, \\ \Theta(r,t) &= \Theta_e + \tilde{\Theta}e^{ikr-\lambda t}\,. \end{aligned} \tag{16}$$

This leads to the following condition for $\omega := \lambda - ikV_e$:

$$\begin{aligned} &\left(\frac{\kappa_e}{\rho_e}k^2 + \frac{2}{\tau} - \omega\right)\left[\omega^2 - \omega\left(\frac{ik}{\rho_e}\frac{\partial P_e}{\partial V_e} + \frac{\eta_e k^2}{\rho_e} + \frac{1}{\tau}\right)\right. \\ &\qquad \left. + \left(k^2\frac{\partial P_e}{\partial \rho_e} + \frac{ik\rho_e}{\tau}\frac{\partial V_e}{\partial \rho_e}\right)\right] \\ &\qquad -\omega\frac{2k^2 P_e}{\rho_e^2}\frac{\partial P_e}{\partial \Theta_e} + \frac{2k^2}{\tau}\frac{\partial P_e}{\partial \Theta_e}\frac{\partial \Theta_e}{\partial \rho_e} \stackrel{!}{=} 0\,, \end{aligned} \tag{17}$$

where

$$P_e := \frac{\rho_e\Theta_e}{1 - \rho_e s(V_e)}\,, \quad \eta_e := \eta(\rho_e, V_e)\,, \quad \kappa_e := \kappa(\rho_e, V_e)\,. \tag{18}$$

Equation (17) is satisfied for three complex roots $\omega(\rho_e, k)$ that have to be determined numerically. The homogeneous solution (15) is unstable if the real part Re $\omega(\rho_e, k)$ of at least one of the roots $\omega(\rho_e, k)$ is negative, i.e., if

$$\min\ \mathrm{Re}\ \omega(\rho_e, k) < 0\,. \tag{19}$$

Figure 2 illustrates for which values of ρ_e and k this is the case given that the road had infinite length. (For a circular road of length L only the values $k = 2\pi m/L$ with $m = \pm 1, \pm 2, \ldots$ are admissible due to the periodic boundary conditions [20]. The conservation of the number of vehicles excludes $m = 0$.) Figure 3 depicts the region of instability for the traffic model of Kerner and Konhäuser [20]. The corresponding instability condition for their equations (1), (3), and (4) reads

$$\frac{\rho_e}{\sqrt{\Theta_0}}\left|\frac{\partial V_e}{\partial \rho_e}\right| > 1 + \tau k^2\frac{\eta_0}{\rho_e} \tag{20}$$

and can be explicitly derived via an analogous linear stability analysis [19]. All parameters that have been chosen for Figs. 2 and 3 are specified in Eq. (21) of Sec. V.

A comparison of Figs. 2 and 3 shows that the improved traffic model (1), (3), (7), (13), and (14) predicts unstable traffic at higher densities ρ_e and larger *wave numbers* k (i.e., shorter wavelengths $2\pi/k$) than the Kerner-Konhäuser model. A qualitative interpreta-

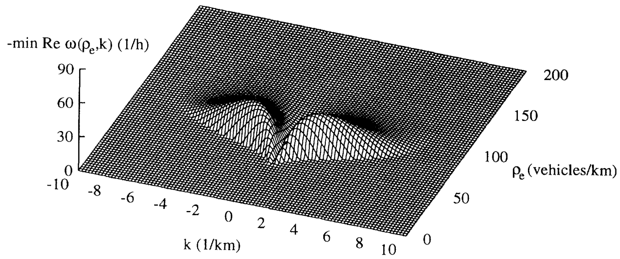

FIG. 2. The instability region of the improved model (indicating the development of stop-and-go traffic) is the region of the k-ρ_e plane where the quantity $-\min\ \mathrm{Re}\ \omega(\rho_e, k)$ is positive (i.e., where the real part of the smallest eigenvalue of the linearized traffic equations is negative). For illustrative reasons the vertical coordinate was set to zero at values ρ_e, k for which the fluid-dynamic traffic equations are stable.

tion can be obtained from Eq. (20) by replacing Θ_0 with $\Theta_e(\rho_e)$ and η_0 with $\eta(\rho_e, V_e)$. On the one hand, the circumstance $\Theta_e(\rho_e) \leq \Theta_0$ causes a larger instability region. On the other hand, the increase of viscosity η with density ρ_e reduces the instability, especially for $k \neq 0$. This also causes, on average, a less rapid growth of unstable perturbations which is reflected by the smaller values of $-\min\ \mathrm{Re}\ \omega(\rho_e, k)$ in Fig. 2 compared to Fig. 3. Nevertheless, one should keep in mind that the instability condition for the improved model includes additional terms which stem from the dynamic Eq. (7) for the variance $\Theta(r, t)$. In order to obtain the exact instability condition, however, Eq. (17) would have to be solved analytically.

V. COMPUTER SIMULATIONS

In this section the results of the different proposed fluid-dynamic traffic models will be compared for the instability region ($\rho_e = 60$ vehicles/km). In the computer simulations the following values typical for Germany were used for the model parameters (cf. [18,39]):

$$\begin{aligned} \tau &= 0.5\ \mathrm{min}, & V_0 &= 120\ \mathrm{km/h}, \\ \Theta_0 &= (45\ \mathrm{km/h})^2, & \eta_0 &= 600\ \mathrm{km/h}, \\ \kappa_0 &= 600\ \mathrm{km/h}, & l &= 5\ \mathrm{m}, \\ \Delta T &= 0.75\ \mathrm{s}. \end{aligned} \tag{21}$$

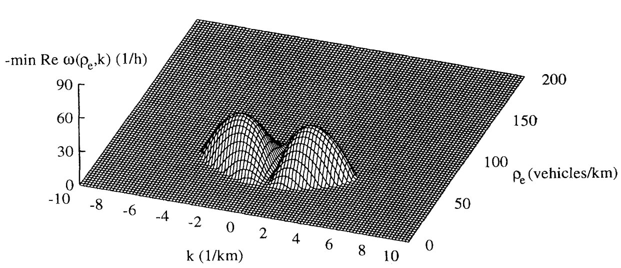

FIG. 3. Instability region of the Kerner-Konhäuser model (compare to Fig. 2).

For the numerical solution a two-step Lax-Wendroff method [40] was applied. According to several tests, this method gives very reliable results. Concerning the discretization, spatial intervals of $\Delta r \leq 100$ m length and time intervals of $\Delta t \leq 0.1$ s duration showed to be suitable. In order to obtain plausible results, it is very important that Δt and ΔT are sufficiently small.

For reasons of simplicity periodic boundary conditions were used: $\rho(L, t) = \rho(0, t)$, $V(L, t) = V(0, t)$, and $\Theta(L, t) = \Theta(0, t)$. The length L of the simulated circular road was set to 10 km. Moreover, the initial condition $\rho(r, 0) := \rho_e$ and $V(r, 0) := V_e(\rho_e)[1 + 0.01 \sin(2\pi r/L)]$ was taken since this corresponds to the simplest perturbation and it seems quite natural that some vehicles drive a little faster while others move a little slower than the average. In addition, $\Theta(r, 0) := \Theta_e(\rho_e)$ was chosen for the improved traffic model of Sec. III. The results of the numerical computations are depicted in Figs. 4 and 5. These show significant differences, which should be empirically observable. According to Fig. 4 the model of Kerner and Konhäuser shows the formation of a *cluster* on a finite circular road. The back of the cluster has a *shock* structure since its slope is almost perpendicular.

Cluster formation can also be observed for the improved model illustrated in Fig. 5. However, the propa-

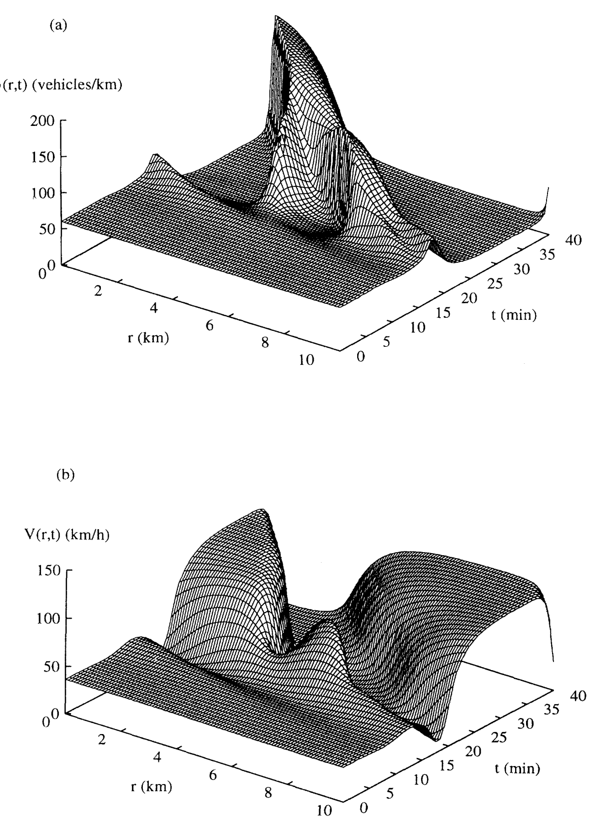

FIG. 4. Temporal development of (a) density $\rho(r, t)$ and (b) velocity $V(r, t)$ according to the model of Kerner and Konhäuser.

gation velocity of the cluster and its shape are different from those depicted in Fig. 4. Obviously, the spatiotemporal density and velocity changes are not as extreme for the improved model as for the model of Kerner and Konhäuser. This is due to the additional smoothing effect of (13b) for high densities. Note that the velocity variance $\Theta(r,t)$ in Fig. 5 increases immediately before the traffic jam develops [i.e., directly before the density $\rho(r,t)$ increases]. This empirically observed fact was reported by Kühne [18,30].

VI. CONCLUSIONS AND OUTLOOK

During the past years, fluid-dynamic models have become promising candidates for the description of vehicular traffic. On the one hand, they are complex enough for a detailed calulation of traffic flows and, on the other hand, they are sufficiently simple for real-time traffic simulations.

However, even the most advanced fluid-dynamic models showed some shortcomings that were overcome in this paper by introducing two improvements. First, the model of Kerner and Konhäuser was extended by an equation for the vehicles' velocity variance. This allowed for the description of the observed increase of velocity variance immediately before a traffic jam develops. Second, the finite amount of space that each vehicle needs was taken into account. This prevented the traffic model from producing densities that exceed the maximum vehicle density. On a finite circular road the improved model showed the formation of a cluster, like the model of Kerner and Konhäuser did. However, the density and velocity changes were not as extreme.

Further work will focus on the investigation of road networks and flow optimization with the aim of developing efficient methods of traffic control. In addition, a gas kinetic derivation of the improved fluid-dynamic model is planned to be presented in a forthcoming paper [26]. The underlying gas kinetic model is based on reasonable assumptions about individual driver behavior concerning acceleration, overtaking, deceleration, and lane-changing maneuvers. It takes into account multiple lanes, the variation of desired and actual speeds, distance keeping, as well as reaction and braking times.

The same behavioral assumptions are applied in recent "microscopic" traffic simulations of a large number of interacting vehicles for which the model parameters can be determined relatively easily. From these simulations, the model parameters η_0 and κ_0 of the fluid-dynamic traffic model can be obtained. Something similar holds for the equilibrium velocity-density relation $V_e(\rho)$, the equilibrium variance-density relation $\Theta_e(\rho)$, and other collective quantities. One advantage of evaluating microscopic simulation results is that spatial variations of the density $\rho(r,t)$, the average velocity $V(r,t)$, and the velocity variance $\Theta(r,t)$ can be easily determined, whereas traffic data are normally collected at single intersections of a road only. Another advantage is that the initial traffic conditions can be prepared arbitrarily. By this one can achieve that specific terms of the fluid-dynamic traffic equations are negligible for a certain time interval. In contrast, for real traffic data the separation of single terms is difficult since the data would have to be collected at the right place at the right time. For example, $V_e(\rho)$ and $\Theta_e(\rho)$ can best be evaluated in quasistationary and quasihomogeneous traffic conditions. With a bit of effort this is still feasible, but some traffic situations that are essential for special measurements may be hard to find in reality.

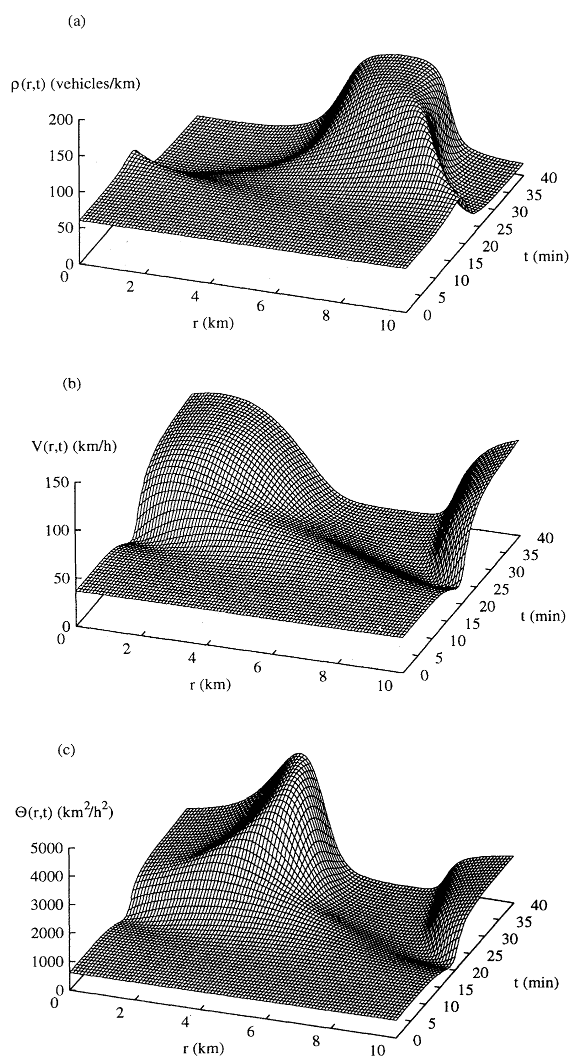

FIG. 5. Temporal development of (a) density $\rho(r,t)$, (b) velocity $V(r,t)$, and (c) velocity variance $\Theta(r,t)$ on a circular road for the improved traffic model.

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[32] A linear stability analysis about the stationary and homogeneous solution analogous to Sec. IV shows that density variations are marginally stable [i.e., Re $\omega(\rho_e, k) = 0$] so that perturbations are neither damped nor amplified.
[33] Kerner and Konhäuser do not state in their papers [20,21] why they use this special kind of velocity-density relation and whether it is fitted to empirical data. Due to the considerable scatter of corresponding traffic data, the discussion about the correct form of the velocity-density relation is still continuing. Consequently, many different proposals exist [34,2,23]. However, Eq. (6) has qualitatively the right form since it shows a small slope at low and high densities (cf. [12,31]). This relation was applied here for reasons of comparability with Kerner and Konhäuser's results. Assumption (5) of constant Θ probably comes from the circumstance that Θ was originally interpreted as square $(c_0)^2$ of the relative propagation velocity c_0 of perturbations [30,18,19]. However, the analogy with the velocity of sound in ordinary fluids, which is assumed here, is only valid in the limit $\tau \to \infty$ and $\eta_0 \to 0$. Moreover, a comparison with Boltzmann-like traffic models [12,13] shows that Θ has the meaning of a velocity variance. Therefore, Θ is expected to vary with density ρ [12], which is the reason for the alternative proposal (9).
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[35] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics* (Pergamon, Oxford, 1980 and 1981), Vols. 5 and 10.
[36] Y. L. Klimontovich, *Statistical Physics* (Harwood Academic, Chur, Switzerland, 1986).
[37] In gas kinetic traffic models these are described by the Boltzmann-like interaction terms ("collision terms") [10,11,13,26].
[38] The "repulsive" interactions between vehicles reflect the tendency of drivers to keep a safe distance from the respective vehicle in front, causing them to decelerate when approaching a slower vehicle.
[39] R. D. Kühne and A. Kroen, in *International Conference on Artificial Intelligence Applications in Transportation Engineering, San Buenaventura, CA, 1992*, edited by S. G. Ritchie and C. T. Hendrickson (Institute of Transportation Studies of the University of California, New York, 1992).
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