Stochastic renormalization-group approach to space-dependent supercritical branched chain processes

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A model for physicochemical and biological space-dependent branched chain processes is suggested based on the assumption that the number and the positions of the direct offspring of a particle are random variables characterized by a given point process. The generating functional of the random point process characterizing the number and the position of the total offspring from a given generation may be computed by repeated iteration of the generating functional attached to the offspring of a particle. A stochastic renormalization transformation is introduced by assuming that for each generation there is a constant probability that the growth of the population stops and that the actual state of the population is an average of the contributions of different generations. A detailed analysis is performed for supercritical self-similar branching processes for which the distance of a particle from its direct ancestor is a random variable selected from a symmetric jump probability density. In this case all Janossy and product densities of the population can be computed exactly. The fluctuations of particle concentrations for a given generation have an intermittent behavior. The asymptotic behavior of the total number N of particles is independent of the spatial distribution: the probabilty of N has a long tail of the inverse power law type modulated by a periodic function of $\ln N$. In contrast, the spatial distribution of particles depends both on the jump probability density and on the total number of particles. If the moments of the jump probability density are finite then the positions $\mathbf{r}_1, \ldots, \mathbf{r}_N$ of N particles are Gaussian random variables with a correlation matrix obeying a logarithmic scaling law $\langle \mathbf{r}_i \cdot \mathbf{r}_j \rangle \sim \delta_{ij} \ln N$, as $N \to \infty$. If the jump probability density has a long tail then the probability density of $\mathbf{r}_1, \ldots, \mathbf{r}_N$ for $N \to \infty$ is given by a separable Lévy law with a dimensional parameter given by a second scaling law $b \sim (\ln N)^{1/\varepsilon}$ as $N \to \infty$, where ε is the fractal exponent of the Lévy law.

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I. INTRODUCTION

The branched chain processes have been used for modeling a broad class of growth phenomena from physics, chemistry, and biology [1]. Examples include nuclear or chemical chain reactions [1,2], high energy hadron collisions [3], antiproton-nucleus annihilation [4], stimulated emission of photons [5], polymer or crack growth [6,7], population growth [1,8], etc. Recently two approaches for the study of branched chain processes have been suggested in both physical and biological contexts: the first approach consists of incorporating memory into the evolution equations [9–11] and the second one makes a connection between the branched chain dynamics and the Shlesinger-Hughes stochastic renormalization method [12,13]. The interference of the memory effects with stochastic renormalization has also been investigated [14].

The purpose of this article is to introduce a model for space-dependent branched chain processes. The spatial distribution of a branched chain process has been analyzed merely in connection with the applications in evolutionary biology and mathematical theory of epidemics [1,15,16]; the corresponding models are very complicated and cannot be studied analytically. In contrast, our model is sufficiently simple that a detailed analytical study is possible, yet complicated enough to be physically interesting. The motivation of our research is related to several apparently uncorrelated physical and biological problems: the growth of cellular aggregates [17], chemical chain reactions [18], random walk theory [19], and the study of multifragmentation processes [20]. The mathematical techniques used are the theory of random point processes [21], in particular a method developed by the present authors for the study of fractal random processes [22] and space- and time-dependent colored noise [23]; the Shlesinger-Hughes stochastic renormalization method [24,25], and the theory of Lévy flights [26].

The structure of the paper is as follows. In Sec. II we formulate the problem of population growth in terms of the theory of random point processes. In Sec. III we derive a general equation for the generating functional of the population at the qth generation; in Sec. IV this equation is used for computing the fluctuations of the population density. In Sec. V a renormalization transformation is introduced by assuming that the growth process can stop after a random number of generations. In Secs. VI–VIII we deal with the application of the theory to a self-similar process for which the distance of a newborn

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individual from its ancestor is a random variable selected from a constant probability law. Finally, in Sec. IX the possibilities of application of our approach in physics and biology are discussed.

II. FORMULATION OF THE PROBLEM

We study the replication of a population made up of a large number of components (molecules, particles, cracks, cells, etc.). We follow the common assumption [1] that the growth of the population is a succession of independent events; for each event one component (the ancestor) is replaced by a random number ν of components (the offspring) placed at random distances $\Delta\mathbf{r}_1, \ldots, \Delta\mathbf{r}_\nu$ from the ancestor. By using the formalism of random point processes [21–23] the stochastic properties of the number ν and of the positions $\Delta\mathbf{r}_1, \ldots, \Delta\mathbf{r}_\nu$ of the offspring of a component can be described by a set of Janossy densities

$$R_0 \quad \text{and} \quad R_\nu(\Delta\mathbf{r}_1, \ldots, \Delta\mathbf{r}_\nu)d\Delta\mathbf{r}_1 \cdots d\Delta\mathbf{r}_\nu \; . \tag{1}$$

Here $R_\nu(\Delta\mathbf{r}_1, \ldots, \Delta\mathbf{r}_\nu)d\Delta\mathbf{r}_1 \cdots d\Delta\mathbf{r}_\nu$ is the probability that the direct offspring number of a component is ν and that the new components are placed at distances between $\Delta\mathbf{r}_l$ and $\Delta\mathbf{r}_l + d\Delta\mathbf{r}_l, l = 1, \ldots, \nu$, from their direct ancestor. We do not impose any restrictions on the relative positions of the new components and thus a $1/\nu!$ Gibbs factor should be introduced in the normalization condition for R_ν:

$$R_0 + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \int \cdots \int R_\nu(\Delta\mathbf{r}_1, \ldots, \Delta\mathbf{r}_\nu) d\Delta\mathbf{r}_1 \cdots d\Delta\mathbf{r}_\nu = 1 \; . \tag{2}$$

All the information concerning the random behavior of the variables $\nu, \Delta\mathbf{r}_1, \ldots, \Delta\mathbf{r}_\nu$ is also contained in the generating functional

$$L[W(\Delta\mathbf{r})] = R_0 + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \int \cdots \times \int R(\Delta\mathbf{r}_1, \ldots, \Delta\mathbf{r}_\nu) \times W(\Delta\mathbf{r}_1) \cdots W(\Delta\mathbf{r}_\nu) \times d\Delta\mathbf{r}_1 \cdots d\Delta\mathbf{r}_\nu \; . \tag{3}$$

where $\mathbf{W}(\Delta\mathbf{r})$ is a suitable test function.

The state of the population at the qth generation can also be described by a set of Janossy densities

$$Q_0^{(q)} \quad \text{and} \quad Q_N^{(q)}(\mathbf{r}_1, \ldots, \mathbf{r}_N)d\mathbf{r}_1 \cdots d\mathbf{r}_N \; , \tag{4}$$

with the normalization condition

$$Q_0^{(q)} + \sum_{N=1}^{\infty} \frac{1}{N!} \int \cdots \int Q_N^{(q)}(\mathbf{r}_1, \ldots, \mathbf{r}_N) d\mathbf{r}_1 \cdots d\mathbf{r}_N = 1 \; , \tag{5}$$

or by the generating functional

$$\mathcal{G}^{(q)}[W(\mathbf{r})] = Q_N^{(0)} + \sum_{N=1}^{\infty} \frac{1}{N!} \int \cdots \int Q_N^{(q)}(\mathbf{r}_1, \ldots, \mathbf{r}_N) W(\mathbf{r}_1) \cdots W(\mathbf{r}_N) d\mathbf{r}_1 \cdots d\mathbf{r}_N \; , \tag{6}$$

where $Q_N^{(q)}(\mathbf{r}_1, \ldots, \mathbf{r}_N)d\mathbf{r}_1 \cdots d\mathbf{r}_N$ is the probability that at the qth generation there are N components with positions between $\mathbf{r}_l$ and $\mathbf{r}_l + d\mathbf{r}_l, l = 1, \ldots, N$ and $W(\mathbf{r})$ is another test function. If the growth process stops after a random number of generations then the final state of the population can be described by a set of renormalized Janossy densities

$$\tilde{Q}_0 = \sum_{q=0}^{\infty} \chi_q Q_0^{(q)} \; ,$$
$$\tilde{Q}_N(\mathbf{r}_1, \ldots, \mathbf{r}_N)d\mathbf{r}_1 \cdots d\mathbf{r}_N = \sum_{q=0}^{\infty} \chi_q Q_N^{(q)}(\mathbf{r}_1, \ldots, \mathbf{r}_N) d\mathbf{r}_1 \cdots d\mathbf{r}_N \tag{7}$$

with the normalization condition

$$\tilde{Q}_0 + \sum_{N=1}^{\infty} \frac{1}{N!} \int \cdots \int \tilde{Q}_N(\mathbf{r}_1, \ldots, \mathbf{r}_N) d\mathbf{r}_1 \cdots d\mathbf{r}_N = 1 \; , \tag{8}$$

or by the generating functional

$$\tilde{\mathcal{G}}[W(\mathbf{r})] = \tilde{Q}_0 + \sum_{N=1}^{\infty} \frac{1}{N!} \int \cdots \int \tilde{Q}_N(\mathbf{r}_1, \ldots, \mathbf{r}_N) \times W(\mathbf{r}_1) \cdots W(\mathbf{r}_N) \times d\mathbf{r}_1 \cdots d\mathbf{r}_N \; , \tag{9}$$

where χ_q is the probability that the growth process stops at the qth generation.

Our aim is to answer the following two questions.

(i) If we know the Janossy densities of the population at zeroth generation and of the direct offspring of an individual, which are stochastic properties of the population at the qth generation?

(ii) If the growth process stops after a random number of generations and the stopping rule is known, which are the stochastic properties of the population that eventually emerges?

III. EVOLUTION EQUATIONS FOR GENERATING FUNCTIONALS

The Janossy densities for the different generations can be computed step by step by means of a chain of probability balance equations

$$Q_N^{(q)}(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \sum_{N'} \frac{1}{N'!} \underset{\sum \nu_\alpha = N}{\sum_{\nu_1} \cdots \sum_{\nu_{N'}}} \frac{1}{\nu_1! \cdots \nu_{N'}!} \int \cdots \int d\mathbf{r}'_1 \cdots d\mathbf{r}'_{N'} Q_{N'}^{(q-1)}(\mathbf{r}'_1, \ldots, \mathbf{r}'_{N'}) \times R_{\nu_1}(\mathbf{r}_1 - \mathbf{r}'_1, \ldots, \mathbf{r}_{\nu_1} - \mathbf{r}'_1) \cdots \times R_{\nu_{N'}}(\mathbf{r}_{N-\nu_{N'}+1} - \mathbf{r}'_{N'}, \ldots, \mathbf{r}_N - \mathbf{r}'_{N'}), \quad q = 1, 2, \ldots \; . \tag{10}$$

By multiplying Eq. (10) by the test functions $W_q(\mathbf{r}_1) \cdots W_q(\mathbf{r}_N)$, integrating over $\mathbf{r}_1, \ldots, \mathbf{r}_N$, and summing over N we can establish a relationship between the generating functionals of two successive generations; after some standard algebraic manipulations we arrive at

$$\mathcal{G}_q[W_q(\mathbf{r})] = \mathcal{G}_{q-1}[W_{q-1}(\mathbf{r}') = L[W(\Delta\mathbf{r}) = W_q(\mathbf{r}'+\Delta\mathbf{r})]] ; \quad (11)$$

Eq. (11) can be rewritten symbolically as

$$\mathcal{G}_q[W_q] = \mathcal{G}_{q-1}[L[W_q]], \quad q=1,2,\ldots . \quad (12)$$

By applying Eq. (12) recursively from $q=0$ to q we get

$$\mathcal{G}_q[W_q] = \mathcal{G}_0[L[L \cdots L[W_q] \cdots]] , \quad (13)$$

where there are q iterations on the right-hand side; that is, the evaluation of the generating functionals corresponding to different generations can be reduced to a repeated functional iteration of the generating functionals $\mathcal{G}_0[W(\mathbf{r})]$ and $L[W(\Delta\mathbf{r})]$. Equation (13) can be rewritten in a simpler form if we introduce the generating functional $G_q[W(\mathbf{r})]$ of the offspring at the qth generation of a component from zeroth generation placed at $\mathbf{r}=\mathbf{0}$. For one component at the zeroth generation placed at $\mathbf{r}=\mathbf{0}$ we have

$$Q_N^{(0)} = \delta_{N1}\delta(\mathbf{r}), \quad \text{i.e.,} \quad \mathcal{G}_0[W(\mathbf{r})] = W(\mathbf{r}) , \quad (14)$$

and thus

$$G_q[W(\mathbf{r})] = \mathcal{G}_q[\mathcal{G}_0[W(\mathbf{r})] = W(\mathbf{r})] = L[L[\cdots L[W(\mathbf{r})] \cdots]] = L^{*(q)}[W(\mathbf{r})] , \quad (15)$$

that is, $G_q[W(\mathbf{r})]$ is the qth functional iterate of the generating functional $L[W(\Delta\mathbf{r})]$. By applying Eq. (15) from $q=0$ to q and from $q=q$ to 0, respectively, we get two chains of functional relations

$$G_q[W_q(\mathbf{r})] = G_{q-1}[L[W_q(\mathbf{r})]], \quad q=1,2,\ldots , \quad (16)$$

with $G_0[W_0(\mathbf{r})] = W_0(\mathbf{r})$ and

$$G_q[W_q(\mathbf{r})] = L[G_{q-1}[W_q(\mathbf{r})]], \quad q=1,2,\ldots \quad (17)$$

with $G_0[W_0(\mathbf{r})] = W_0(\mathbf{r})$. Together with the relationship

$$\mathcal{G}_q[W(\mathbf{r})] = \mathcal{G}_0[G_q[W(\mathbf{r})]] , \quad (18)$$

which can be derived from Eqs. (13) and (15), Eq. (16) or (17) determines completely the stochastic properties of the number and the positions of the components from the qth generation.

Equations (10)–(18) describe the dynamics of population growth in an abstract way. In order to clarify their physical significance Table I displays the relationships between the main properties of the populations from two successive generations. From Table I we note that Eq. (10) is in fact a complex convolution product that expresses the stochastic behavior of the number and the

TABLE I. Relationships between the main properties of populations belonging to two successive generations.

Property	$(q-1)$th generation	qth generation	Relationships
Total number of components	N'	N	
Offspring numbers of the different components		the N' components from the $(q-1)$th generation are replaced by $\nu_1, \ldots, \nu_{N'}$ components, respectively	$N=\nu_1+ \cdots +\nu_{N'}$
Position vector of a component	$\mathbf{r}'_l$, $l=1, \ldots, N'$	$\mathbf{r}_u = \mathbf{r}'_{l_u} + \Delta\mathbf{r}_u$	$\mathbf{r}'_{l_u}$ is the position vector of the direct ancestor of the component u; $\Delta\mathbf{r}_u$ is the displacement vector of the component u from qth generation
state probabilities	$Q_{N'}^{(q-1)}(\mathbf{r}'_1, \ldots, \mathbf{r}'_{N'})$ $d\mathbf{r}'_1 \cdots d\mathbf{r}'_{N'}$	$Q_N^{(q)}(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ $d\mathbf{r}_1 \cdots d\mathbf{r}_N$	$Q_N^{(q)} = Q_{N'} * \{R_{\nu_1} \cdots R_{\nu_{N'}}\}$; $*$ is the particle number and space convolution product defined by Eq. (10) and R_ν is the Janossy density of the direct offspring of a component [Eqs. (1) and (2)]
generating functionals	$\mathcal{G}_{q-1}[W_{q-1}(\mathbf{r})]$	$\mathcal{G}_q[W_q(\mathbf{r})]$	$\mathcal{G}_q[W_q(\mathbf{r})] = \mathcal{G}_{q-1}[L[W_q(\mathbf{r})]]$

positions of individuals from the qth generation in terms of the number and the positions of individuals from the $(q-1)$th generation and of the number and the displacement vectors of the direct offspring of an individual. Equation (10) may be symbolically rewritten as

$$Q_N^{(q)}=Q_{N'}^{(q-1)}*\{R_{\nu_1},\ldots,R_{\nu_{N'}}\}\ , \tag{10'}$$

where the centered asterisk denotes the particle number and position convolution product in the $(N,\mathbf{r}_1,\ldots,\mathbf{r}_N)$ space, which corresponds to the sums and integrals from the right-hand side of Eq. (10). The operation of functional iteration from Eq. (12) is the image of this convolution product in the space of test functions $W(\mathbf{r})$.

The above considerations allows us to give a simple diagrammatic representation of the process of population growth. We represent by an arrow the convolution product in the $(N,\mathbf{r}_1,\ldots,\mathbf{r}_N)$ space or the functional iteration in the space of test functions and by $\mathcal{E}(q')$, $q'=0,1,2,\ldots,$ the state of the population from the q'th generation. The corresponding stochastic diagrams for the growth of the first q generations is displayed in Fig. 1. Figure 1 shows that despite the apparent complexity of Eq. (10), the dynamics of population growth is relatively simple. The simple unidirectional linear structure of the stochastic diagram is due to the irreversible nature of population growth, which prevents the occurrence of feedback interactions among the late and early generations.

For a comparison with the theory of space-independent processes [1], we note that the probability $P^{(q)}(N)$ of the total offspring number N at the qth generation can be computed by integrating the Janossy densities $Q_N^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ over $\mathbf{r}_1,\ldots,\mathbf{r}_N$:

$$P^{(q)}(N)=\frac{1}{N!}\int\cdots\int Q_N^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)d\mathbf{r}_1\cdots d\mathbf{r}_N\ ; \tag{19}$$

similarly, the probability $\varphi(\nu)$ that the direct offspring number of an individual is ν can be computed by integrating the Janossy densities $R_\nu(\Delta\mathbf{r}_1,\ldots,\Delta\mathbf{r}_\nu)$ over $\Delta\mathbf{r}_1,\ldots,\Delta\mathbf{r}_\nu$:

$$\varphi(\nu)=\frac{1}{\nu!}\int\cdots\int R_\nu(\Delta\mathbf{r}_1,\ldots,\Delta\mathbf{r}_\nu)d\Delta\mathbf{r}_1\cdots d\Delta\mathbf{r}_\nu\ . \tag{20}$$

The generating functions of the probabilities $P^{(q)}(N)$ and $\varphi(\nu)$ are

$$\mathcal{G}_q(z)=\sum_{N=0}^{\infty}z^N P^{(q)}(N),\quad |z|\le 1 \tag{21}$$

and

$$L(z)=\sum_{\nu=0}^{\infty}z^\nu\varphi(\nu),\quad |z|\le 1\ . \tag{22}$$

By comparing Eqs. (3) and (6) with Eqs. (21)–(22) we note that for

$$W(\mathbf{r})=z,\quad W(\Delta\mathbf{r})=z\ ,\quad \text{independent of } \mathbf{r},\Delta\mathbf{r}\ , \tag{23}$$

the integrals over $\mathbf{r}_1,\ldots,\mathbf{r}_N$ and $\Delta\mathbf{r}_1,\ldots,\Delta\mathbf{r}_\nu$ from Eqs. (19) and (20) for $P^{(q)}(N)$ and $\varphi(\nu)$ are explicitly taken into account in expressions (3) and (6) for the generating functionals $\mathcal{G}_q$ and L; it follows that we have the following relationships between $\mathcal{G}_q(z),L(z)$ and $\mathcal{G}_q[W(\mathbf{r})],L[W(\Delta\mathbf{r})]$:

$$\mathcal{G}_q(z)=\mathcal{G}_q[W(\mathbf{r})=z\ \ \text{independent of } \mathbf{r}]\ , \tag{24}$$

$$L(z)=L[W(\Delta\mathbf{r})=z\ \text{independent of } \Delta\mathbf{r}]\ . \tag{25}$$

As $G_q[W(\mathbf{r})]$ is a particular case of $\mathcal{G}_q[W(\mathbf{r})]$, it follows that

$$G_q(z)=G_q[W(\mathbf{r})=z\ \text{independent of } \mathbf{r}] \tag{26}$$

is the generating function of the offspring at the qth generation of an individual from the zeroth generation. The above analysis shows that for $W(\mathbf{r})=z$ independent of $\mathbf{r}$, Eqs. (12)–(18) reduce to the basic relationships for space-independent branching processes presented in the literature [1]. In particular, for $W(\mathbf{r})=z$ Eqs. (16) and (17) become the classical forward and backward chain equations for space-independent branched chain processes in discrete time [1].

If the repeated iteration of the generating functional $L[W(\Delta\mathbf{r})]$ can be performed analytically then the Janossy densities $Q_N^{(q)}$ can be computed by expanding $\mathcal{G}_q[W(\mathbf{r})]$ in a functional Taylor series around $W(\mathbf{r})=0$, that is, by evaluating the functional derivatives

$$Q_N^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\left.\frac{\delta^N\mathcal{G}_q[W(\mathbf{r})]}{\delta W(\mathbf{r}_1)\cdots\delta W(\mathbf{r}_N)}\right|_{W(\mathbf{r})=0}\ . \tag{27}$$

By combining Eqs. (19) and (27) we can also compute the probability $P^{(q)}(N)$ of the total offspring number from the qth generation. Another function of interest depending

$\mathcal{E}(0) \rightarrow\!\!\rightarrow\!\!\rightarrow \mathcal{E}(1) \rightarrow\!\!\rightarrow\!\!\rightarrow \ldots\ldots\ldots \rightarrow\!\!\rightarrow\!\!\rightarrow \mathcal{E}(q-1) \rightarrow\!\!\rightarrow\!\!\rightarrow \mathcal{E}(q)$

PROBABILITIES $Q_{N_1}^{(1)} = Q_{N_o}^{(o)}*\{R_{\upsilon_1},\ldots,R_{\upsilon_{N_o}}\}\ldots\ Q_{N_q}^{(q)}= Q_{N_{q-1}}^{(q-1)}*\{R_{\upsilon_1},\ldots,R_{\upsilon_{N_{q-1}}}\}$

GENERATING FUNCTIONALS $\mathcal{G}_1 = \mathcal{G}_o[L]\ \ldots\ldots\ldots\ldots\ \mathcal{G}_q = \mathcal{G}_{q-1}[L]$

FIG. 1. Diagrammatic representation of the evolution of the population from generation to generation. $\mathcal{E}(q')$, $q'=0,1,\ldots,q$, represents the population from the q'th generation, $q'=0,1,\ldots,q$, and the arrow represents the process of replacement of the components from a generation by their direct offspring, expressed by the convolution product in the $(N,\mathbf{r}_1,\ldots,\mathbf{r}_N)$ space or by the process of functional iteration in the space of test functions $W(\mathbf{r})$.

on $Q_N^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ is the conditional probability density

$$\phi^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)d\mathbf{r}_1\cdots d\mathbf{r}_N \quad \text{with} \int\cdots\int \phi^{(q)}d\mathbf{r}_1\cdots d\mathbf{r}_N=1 \tag{28}$$

of the positions of the components from the qth generation provided the size of the population is N. We have

$$\phi^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)=Q_N^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)/[N!P^{(q)}(N)] . \tag{29}$$

Equations (19), (27), and (28) will be used in Sec. VI for the study of self-similar symmetric processes.

IV. PRODUCT DENSITIES AND POPULATION FLUCTUATIONS

In order to characterize population fluctuations we shall use a set of product densities of the numbers of components. In the literature of random point processes there are several different definitions for the product densities [21,27]. As far as we know, these definitions have not yet been used in the context of branched chain processes. In Appendix A we show how the definitions suggested by Stratonovich and Van Kampen [21] and by Carruthers [27] can be used for characterizing the population fluctuations in a branched chain process. We show that the two definitions are equivalent to each other and lead to the following expressions for the m-particle product densities at the qth generation:

$$f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)=\left.\frac{\delta^m \mathcal{G}_q[W(\mathbf{r})]}{\delta W(\mathbf{r}_1)\cdots\delta W(\mathbf{r}_m)}\right|_{W(\mathbf{r})=1} . \tag{30}$$

Similarly we can introduce a set of correlation functions $g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$, which can be computed by using a set of functional equations analogous to Eq. (30):

$$g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)=\left.\frac{\delta^m \ln \mathcal{G}_q[W(\mathbf{r})]}{\delta W(\mathbf{r}_1)\cdots\delta W(\mathbf{r}_m)}\right|_{W(\mathbf{r})=1} . \tag{30$'$}$$

The product density and the correlation function of the first order are equal to each other and with the average population density of individuals from the qth generation at position r, $\langle n(\mathbf{r})\rangle^{(q)}$, $f_1^{(q)}(\mathbf{r})=g_1^{(q)}(\mathbf{r})=\langle n(\mathbf{r})\rangle^{(q)}$. The superior moments of population density, although different from the corresponding product densities and correlation functions, can be expressed, however, in terms of $f_m^{(q)}$ and $g_m^{(q)}$. For details, see Appendix A.

V. RENORMALIZATION TRANSFORMATION

If for each generation there is a constant probability μ that the growth process takes place, then the probability χ_q that the growth stops after q generations is simply equal to

$$\chi_q=\mu^q(1-\mu) . \tag{31}$$

By combining Eqs. (7), (9), (15), (18), and (31) we get the following expression for the renormalized generating functional $\tilde{\mathcal{G}}[W(\mathbf{r})]$:

$$\tilde{\mathcal{G}}[W(\mathbf{r})]=(1-\mu)\sum_{q=1}^{\infty}\mu^q\mathcal{G}_0[L^{*(q)}[W(\mathbf{r})]] +(1-\mu)\mathcal{G}_0[W(\mathbf{r})] ; \tag{32}$$

by means of the substitutions

$$W(\mathbf{r})\rightarrow L[W(\mathbf{r})], \quad \tilde{\mathcal{G}}\rightarrow\mu\tilde{\mathcal{G}} , \tag{33}$$

Equation (32) becomes

$$\mu\tilde{\mathcal{G}}[L[W(\mathbf{r})]]=(1-\mu)\sum_{q=1}^{\infty}\mu^q\mathcal{G}_0[L^{*(q)}[W(\mathbf{r})]] . \tag{34}$$

Now we eliminate the sum from Eqs. (33) and (34); we obtain a functional equation for $\tilde{\mathcal{G}}[W(\mathbf{r})]$:

$$\tilde{\mathcal{G}}[W(\mathbf{r})]=(1-\mu)\mathcal{G}_0[W(\mathbf{r})]+\mu\tilde{\mathcal{G}}[L[W(\mathbf{r})]] . \tag{35}$$

Equation (35) has the typical form of a stochastic renormalization group equation [24,25]; such an equation was suggested for the first time almost thirty years ago by Novikov [28] for the modeling of vorticity cascades in turbulence.

If the solution $\tilde{\mathcal{G}}[W(\mathbf{r})]$ of the renormalization group equation (35) is known then we can compute the renormalized Jannosy densities $\tilde{Q}_N(\mathbf{r}_1,\ldots,\mathbf{r}_N)$, the renormalized product densities $\tilde{f}_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)$, and the renormalized correlation functions $\tilde{g}_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ by using functional equations of the type (27), (30), and (30′), where $Q_N^{(q)}$, $f_m^{(q)}$, $g_m^{(q)}$, and $\mathcal{G}_q[W(\mathbf{r})]$ are replaced by $\tilde{Q}_N$, $\tilde{f}_m$, $\tilde{g}_m$, and $\tilde{\mathcal{G}}[W(\mathbf{r})]$, respectively. In terms of the renormalized Janossy densities $\tilde{Q}_N(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ we can compute the renormalized probability

$$\tilde{P}(N)=\frac{1}{N!}\int\cdots\int\tilde{Q}_N(\mathbf{r}_1,\ldots,\mathbf{r}_N)d\mathbf{r}_1\cdots d\mathbf{r}_n \tag{36}$$

of the eventual size of the population and the conditional probability density

$$\tilde{\phi}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)d\mathbf{r}_1\cdots d\mathbf{r}_N =\tilde{Q}_N(\mathbf{r}_1,\ldots,\mathbf{r}_N)d\mathbf{r}_1\cdots d\mathbf{r}_N/[N!P(N)] \tag{37}$$

of the positions $\mathbf{r}_1,\ldots,\mathbf{r}_N$ of the components provided the final population size is N.

VI. SELF-SIMILAR SYMMETRIC PROCESSES: EVOLUTION EQUATIONS

For illustration we consider a self-similar symmetric model based on the following assumptions.

(i) The displacement vector $\Delta\mathbf{r}$ of a component from the position of its direct ancestor is a random variable selected from a symmetric probability law

$$p(\Delta\mathbf{r})d\Delta\mathbf{r} \quad \text{with} \int p(\Delta\mathbf{r})d\Delta\mathbf{r}=1, \quad p(\Delta\mathbf{r})=p(|\Delta\mathbf{r}|) , \tag{38}$$

regardless of the value of the direct offspring number and of the positions of the other components.

(ii) The probability $\varphi(\nu)$ of the direct offspring number ν of a component is independent of the spatial distribution of the population and is determined by a succession of probabilities $\lambda_1,\lambda_2,\ldots$; here λ_l is the probability that the lth new component is generated. We have

$$\varphi(0)=1-\lambda_1,\quad \varphi(\nu)=\lambda_1\cdots\lambda_\nu(1-\lambda_{\nu+1})\ . \tag{39}$$

Following Lotka [8], we make the assumption that all probabilities λ_l, $l=2,3,\ldots$, of generating the second component, the third component, etc., are equal to each other; the probability of generating the first component, however, can be different

$$\lambda_1=\beta,\quad \lambda_2=\lambda_3=\cdots=\lambda\ . \tag{40}$$

Equation (40) is consistent with the statistical data for human populations [8,29]; similar conditions have been used in studies of photon statistics [5].

If the above assumptions are valid, the Janossy densities $R_\nu(\Delta\mathbf{r}_1,\ldots,\Delta\mathbf{r}_\nu)$ are given by

$$\begin{aligned}&R_0=\varphi(0)\ ,\\ &R_\nu(\Delta\mathbf{r}_1,\ldots,\Delta\mathbf{r}_\nu)=\nu!\varphi(\nu)p(\Delta\mathbf{r}_1)\cdots p(\Delta\mathbf{r}_\nu)\ .\end{aligned} \tag{41}$$

By using Eqs. (41) the functional iteration in Eqs. (13)–(18) can be computed exactly. The calculations are rather long; their main steps are presented in Appendix B. Considering that at the zeroth generation the population is made up of a single individual at $\mathbf{r}=\mathbf{0}$, we can express $\mathcal{G}_q[W(\mathbf{r})]$ in terms of the m-fold space convolution product $p_q(\Delta\mathbf{r})=[p(\Delta\mathbf{r})\otimes]^{(q)}$ of the jump probability density $p(\Delta\mathbf{r})$. We obtain

$$\mathcal{G}_q[W(\mathbf{r})]=G_q[W(\mathbf{r})]=\frac{(1-\beta)[\beta^q-(1-\lambda)^q]-[(1-\beta)\beta^q-\lambda(1-\lambda)^q]\int W(\mathbf{r})p_q(\mathbf{r})d\mathbf{r}}{\lambda\beta^q-(1-\beta)(1-\lambda)^q-\lambda[\beta^q-(1-\lambda)^q]\int W(\mathbf{r})p_q(\mathbf{r})d\mathbf{r}}\ . \tag{42}$$

From Eqs. (19), (27), and (42) we can compute the Janossy densities $Q_N^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ of the population at the qth generation and the probability $P^{(q)}(N)$ of the total population size at the qth generation

$$Q_N^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)=N!P^{(q)}(N)p_q(\mathbf{r}_1)\cdots p_q(\mathbf{r}_N)\ , \tag{43}$$

$$P^{(q)}(N)=\frac{(\bar\nu-1)^2\bar\nu^q}{(\bar\nu_0\bar\nu^q+\bar\nu-\bar\nu_0-1)^2}\times\left[\frac{\bar\nu_0(\bar\nu^q-1)}{\bar\nu_0\bar\nu^q+\bar\nu-\bar\nu_0-1}\right]^{N-1},\quad N=1,2,\ldots \tag{44}$$

$$P^{(q)}(0)=(\bar\nu_0-\bar\nu+1)(\bar\nu^q-1)/(\bar\nu_0\bar\nu^q+\bar\nu-\bar\nu_0-1)\ . \tag{45}$$

The average direct offspring number of a component is

$$\bar\nu=\sum_{\nu=0}^{\infty}\nu\varphi(\nu)=\beta/(1-\lambda) \tag{46}$$

and the average offspring of a component if all new components have the same probability of being generated ($\lambda_1=\lambda_2=\cdots=\lambda$) is

$$\bar\nu_0=\bar\nu(\beta=\lambda)=\lambda/(1-\lambda)\ . \tag{47}$$

The conditional probability $\varphi^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)$ of the position of the components from the qth generation can be computed by combining Eqs. (29) and (43)–(45); we get

$$\phi^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)=\prod_{l=1}^{N}p_q(\mathbf{r}_l)\ . \tag{48}$$

From Eq. (48) we note that the positions of the different components from the qth generation are independent of each other and of the total size N of the population. This is a direct consequence of the first assumption introduced at the beginning of this section. We shall see later that the property of statistical independence of the positions of the different components is destroyed by the renormalization transformation introduced in Sec. V.

Now we investigate the asymptotic behavior of the population as the number of generations q tends to infinity. We restrict ourselves to the study of the supercritical regime for which the average direct offspring number of an individual is greater than unity

$$\bar\nu>1\ . \tag{49}$$

In this case only the branched chain process has a nontrivial behavior, which is due to the explosive increase of the population. From Eqs. (43)–(45) we get the following asymptotic expressions for the probability $P^{(q)}(N)$ of the population size at the qth generation:

$$P^{(q)}(N)\sim(\bar\nu/\bar\nu_0)^2\bar\nu^{-q}\exp(-N\bar\nu^{-q})\ ,\quad N>1,\ \bar\nu>1,\ q\to\infty\ , \tag{50}$$

$$P^{(q)}(0)\sim1-(\bar\nu-1)/\bar\nu_0,\quad \beta<1,\ \bar\nu>1,\ q\to\infty\ , \tag{51}$$

and

$$P^{(q)}(0)\sim0,\quad \beta=1,\ \bar\nu>1,\ q\to\infty\ . \tag{52}$$

For $\bar\nu>1$ both the average value and the dispersion of the population size at qth generation increase exponentially to infinity as $q\to\infty$ [see Eqs. (A33) and (A39) from Appendix A]; however, for $\beta<1$ there is always a positive probability $P^{(q)}(0)$ that the population becomes extinct. The physical explanation of this phenomenon is simple: for $\beta=1$, for each component from a given generation there is at least one offspring in the next generation; on the other hand, for $\beta<1$ there is always the possibility that all components from a generation have no offspring in the next generation.

For investigating the asymptotic behavior of the space distribution of the population we introduce the Fourier transform

$$\bar p(\mathbf{k})=\int\exp(i\mathbf{k}\cdot\Delta\mathbf{r})p(\Delta\mathbf{r})d\Delta\mathbf{r} \tag{53}$$

of the probability density of the displacement vector of a component from the position of its direct ancestor; by analogy with the random walk theory [19], we call $\bar{p}(\mathbf{k})$ the structure function for a generation. As $p(\Delta\mathbf{r})$ is symmetric [Eq. (38)], the average value of the displacement vector is always equal to zero. If the second moment of the absolute value of the displacement vector

$$\langle r_0^2\rangle=\langle|\Delta\mathbf{r}|^2\rangle \tag{54}$$

exists and is finite, then the structure function $\bar{p}(\mathbf{k})$ is analytic in $|\mathbf{k}|$ in the vicinity of $\mathbf{k}=\mathbf{0}$:

$$p(\mathbf{k})=1-(\langle r_0^2\rangle/2d_s)|\mathbf{k}|^2+o(|\mathbf{k}|^4) . \tag{55}$$

If $\langle \mathbf{r}_0^2\rangle$ is infinite the structure function $p(\mathbf{k})$ is nonanalytic in $|\mathbf{k}|$ in the vicinity of $\mathbf{k}=\mathbf{0}$, a situation that can be represented by the function [19]

$$\bar{p}(\mathbf{k})\sim 1-(b|\mathbf{k}|)^{\varepsilon},\quad \mathbf{k}\to\mathbf{0},\ 2>\varepsilon>0 , \tag{56}$$

where ε is a fractal exponent and b is a characteristic length. By taking the Fourier transform of the conditional probability $\phi^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)$ the q-fold convolution products $p_q(\mathbf{r}_l)$, $l=1,\ldots,N$, can be expressed in terms of the powers of the structure function $\bar{p}(\mathbf{k})$; returning to the real space representation by means of an inverse Fourier transformation, we get

$$\phi^{(1)}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)=(2\pi)^{-Nd_s}\int\cdots\int\exp\left[-i\sum\mathbf{k}_l\cdot\mathbf{r}_l\right]\times[\bar{p}(\mathbf{k}_1)\cdots\bar{p}(\mathbf{k}_N)]^q\times d\mathbf{k}_1\cdots d\mathbf{k}_N . \tag{57}$$

If the second moment $\langle r_0^2\rangle$ exists and is finite then Eqs. (55) and (57) lead to

$$\phi^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)=\left[\frac{d_s}{2\pi q\langle r_0^2\rangle}\right]^{Nd_s/2}\times\exp\left[-\frac{d_s}{2q\langle r_0^2\rangle}\sum|\mathbf{r}_l|^2\right]\quad\text{as } q\to\infty , \tag{58}$$

that is, $\phi^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)$ is a multivariate Gaussian density, the corresponding second moments

$$\langle\mathbf{r}_l\cdot\mathbf{r}_{l'}\rangle=\delta_{ll'}q\langle r_0^2\rangle , \tag{59}$$

increasing linearly with the generation number q. If the structure function $\bar{p}(\mathbf{k})$ is nonanalytic near $\mathbf{k}=\mathbf{0}$ Eqs. (56) and (57) give

$$\phi^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)=q^{-Nd_s/\varepsilon}b^{-Nd_s}\prod_{l=1}^{N}\mathcal{L}_{\varepsilon}^{(d_s)}\left[\frac{|\mathbf{r}_l|}{q^{1/\varepsilon}b}\right] , \tag{60}$$

where

$$\mathcal{L}_{\varepsilon}^{(d_s)}(|\mathbf{x}|)=(2\pi)^{-d_s}\int\exp(-i\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|^{\varepsilon})d\mathbf{k} \tag{61}$$

is the Lévy-type symmetric stable probability density in d_s-dimensional Euclidean space [26].

From Eqs. (43)–(48) it follows that the Janossy densities $Q_N^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ are proportional to the product of $\phi^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)$ and $P^{(q)}(N)$

$$Q_N^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)=N!\phi^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)P^{(q)}(N) \tag{62}$$

and thus the asymptotic behavior of the Janossy densities as $q\to\infty$ is completely determined by Eqs. (50), (51), (58), (60), and (62).

VII. SELF-SIMILAR SYMMETRIC PROCESSES: POPULATION FLUCTUATIONS

The product densities $f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ can be computed from Eqs. (37) and (39); the functional differentiation of $\mathcal{G}_q[W(\mathbf{r})]$ gives

$$f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)=m!(\bar{\nu})^q\left[\frac{\bar{\nu}_0(\bar{\nu}^q-1)}{\bar{\nu}-1}\right]^{m-1}\times p_q(\mathbf{r}_1)\cdots p_q(\mathbf{r}_m) , \tag{63}$$

from which, by integrating over $\mathbf{r}_1,\ldots,\mathbf{r}_m$, we can compute the factorial moments $F_m^{(q)}$ of the total population size from the qth generation (see Appendix A)

$$F_m^{(q)}=\langle N(N-1)\cdots(N-m+1)\rangle^{(q)}=m!(\bar{\nu})^q\left[\frac{\bar{\nu}_0(\bar{\nu}^q-1)}{\bar{\nu}-1}\right]^{m-1} . \tag{64}$$

In principle, the moments of the population density $n(\mathbf{r})$ can be computed from the product densities $f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ by using the expressions derived in Appendix A; the method is rather tedious. For the particular model considered in this section it is more advantageous to evaluate directly the moments of the population density. We express the instantaneous value of the population density $n(\mathbf{r})$ as a sum of delta functions

$$n(\mathbf{r})=\delta(\mathbf{r}-\mathbf{r}_1)+\cdots+\delta(\mathbf{r}-\mathbf{r}_N) , \tag{65}$$

where the stochastic properties of the total population size N and of the position vectors $\mathbf{r}_1,\ldots,\mathbf{r}_N$ are characterized by the set of Janossy densities derived in Sec. VI [Eq. (43)]. Starting from Eqs. (43) and (65), all moments and cumulants of the population density can be expressed in terms of the generating functional $\mathcal{G}_q[W(\mathbf{r})]$ [Eq. (42)]. The computations are presented in Appendix C. In particular we obtain the following expressions for the central moments and cumulants of the first and the second order of the qth generation:

$$\langle\langle n(\mathbf{r})\rangle\rangle^{(q)}=\langle n(\mathbf{r})\rangle^{(q)}=(\bar{\nu})^q p_q(\mathbf{r}) , \tag{66}$$

$$\langle\langle n(\mathbf{r}_1)n(\mathbf{r}_2)\rangle\rangle^{(q)}=\langle n(\mathbf{r}_1)n(\mathbf{r}_2)\rangle^{(q)}-\langle n(\mathbf{r}_1)\rangle^{(q)}\langle n(\mathbf{r}_2)\rangle^{(q)}$$
$$=p_q(\mathbf{r}_1)p_q(\mathbf{r}_2)\bar{\nu}^{2q}\left[\frac{2\bar{\nu}_0(1-\bar{\nu}^{-q})}{\bar{\nu}-1}-1\right]+p_q(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2)\bar{\nu}^q . \quad (67)$$

The relative fluctuation of the population density is given by

$$\rho_q(\mathbf{r}_1,\mathbf{r}_2)=\left[\frac{\langle\langle n(\mathbf{r}_1)n(\mathbf{r}_2)\rangle\rangle^{(q)}}{\langle\langle n(\mathbf{r}_1)\rangle\rangle^{(q)}\langle\langle n(\mathbf{r}_2)\rangle\rangle^{(q)}}\right]^{1/2}$$
$$=\left[\frac{2\bar{\nu}_0(1-\bar{\nu}^{-q})}{\bar{\nu}-1}-1+[p_q(\mathbf{r}_1)]^{-1}\delta(\mathbf{r}_1-\mathbf{r}_2)\bar{\nu}^{-q}\right]^{1/2} . \quad (68)$$

As $q\to\infty$ the asymptotic behavior of Eqs. (63) and (66)–(68) is

$$f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)\sim m!(\bar{\nu})^{mq}[\bar{\nu}_0/(\bar{\nu}-1)]^{m-1}\times p_q(\mathbf{r}_1)\cdots p_q(\mathbf{r}_m) , \quad \bar{\nu}>1, \ q\to\infty \quad (69)$$

$$\langle n(\mathbf{r})\rangle^{(q)}\sim(\bar{\nu})^q p_q(\mathbf{r}) , \quad (70)$$

$$\langle\langle n(\mathbf{r}_1)n(\mathbf{r}_2)\rangle\rangle^{(q)}\sim p_q(\mathbf{r}_1)p_q(\mathbf{r}_2)\bar{\nu}^{2q}\left[\frac{2\bar{\nu}_0}{\bar{\nu}-1}-1\right]+p_q(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2)\bar{\nu}^q , \quad \bar{\nu}>1, \ q\to\infty \quad (71)$$

$$\rho_q(\mathbf{r}_1,\mathbf{r}_2)\sim\left[\frac{2\bar{\nu}_0}{\bar{\nu}-1}-1\right]^{1/2} , \quad \mathbf{r}_1\neq\mathbf{r}_2, \ \bar{\nu}>1, \ q\to\infty . \quad (72)$$

Equations (71) and (72) show that for the supercritical regime ($\bar{\nu}>1$) the fluctuations of population density have an intermittent behavior; indeed, both the first and second cumulants increase with the generation index q, the rate of increase of the second cumulant being too times larger than the rate of increase of the first cumulant. As a result, for $q\to\infty$ the importance of fluctuations increases; the relative fluctuation does not decrease to zero, but rather tends towards a constant value; for $q\to\infty$, from generation to generation the multiplicative mechanism of the branching process amplifies the average size of not only the population but also the fluctuations.

The intermittent behavior of the fluctuations is also displayed by the cumulants of the total population size and by the relative fluctuation ρ_q^* of the total population

$$\langle\langle N\rangle\rangle^{(q)}=(\bar{\nu})^q , \quad (73)$$

$$\langle\langle N^2\rangle\rangle^{(q)}=\bar{\nu}^q(\bar{\nu}^q-1)\left[\frac{2\bar{\nu}_0}{\bar{\nu}-1}-1\right] , \quad (74)$$

$$\rho_q^*=(\langle\langle N^2\rangle\rangle^{(q)})^{1/2}/\langle\langle N\rangle\rangle^{(q)}$$
$$=(1-\bar{\nu}^{-q})^{1/2}\left[\frac{2\bar{\nu}_0}{\bar{\nu}-1}-1\right]^{1/2} ,$$
$$\sim\left[\frac{2\bar{\nu}_0}{\bar{\nu}-1}-1\right]^{1/2} , \quad \bar{\nu}>1, \quad q\to\infty . \quad (75)$$

By comparing Eqs. (68) and (76) we note that both relative fluctuations have exactly the same type of asymptotic behavior.

VIII. SELF-SIMILAR SYMMETRIC PROCESSES: RENORMALIZED APPROACH

By applying the general equations derived in Sec. V to the self-similar symmetric model studied in Secs. VI and VII we get the following expressions for the renormalized Janossy and product densities:

$$\tilde{Q}_N(\mathbf{r}_1,\ldots,\mathbf{r}_N)=N!(1-\mu)\sum_{q=0}^{\infty}\mu^q P^{(q)}(N)p_q(\mathbf{r}_1)\cdots p_q(\mathbf{r}_N) , \quad (76)$$

and

$$\tilde{f}_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)=(1-\mu)\sum_{q=0}^{\infty}\mu^q F_m^{(q)}p_q(\mathbf{r}_1)\cdots p_q(\mathbf{r}_N) , \quad (77)$$

where $P^{(q)}(N)$ and $F_m^{(q)}$ are given by Eqs. (44) and (45), and (64), respectively. From these equations we can compute the renormalized probability $\tilde{P}(N)$ of the total size of the population and the renormalized conditional probability density $\tilde{\phi}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)$ of the positions of the components:

$$\tilde{P}(N)=(1-\mu)\sum_{q=0}^{\infty}\mu^q P^{(q)}(N) \quad (78)$$

and

$$\tilde{\phi}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)=(1-\mu)\sum_{q=0}^{\infty}\mu^q P^{(q)}(N)p_q(\mathbf{r}_1)\cdots p_q(\mathbf{r}_N)/\tilde{P}(N) . \quad (79)$$

In order to clarify the nature of the population fluctuations for the renormalized process we should evaluate the sums over q in Eqs. (76)–(79). We start out by investigating Eq. (77) for the product densities $\tilde{f}_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)$. By taking the Fourier transform of Eq. (77), the sum over q reduces to the summation of a finite number of geometric series depending on the structure function $\bar{p}(\mathbf{k})$; by evaluating these series and returning to the real space variables $\mathbf{r}_1,\ldots,\mathbf{r}_m$ we obtain

$$\tilde{f}_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)=m[(m-1)!]^2[\bar{v}_0/(\bar{v}-1)]^{m-1}(1-\mu)\times\sum_{j=0}^{m-1}\frac{(-1)^{m-1-j}}{j!(m-1-j)!}(2\pi)^{-md_s}\int\cdots\int\exp\left[-i\sum\mathbf{k}_l\cdot\mathbf{r}_l\right]\times[1-\mu\bar{v}^{j+1}\bar{p}(\mathbf{k}_1)\cdots\bar{p}(\mathbf{k}_m)]^{-1}d\mathbf{k}_1\cdots d\mathbf{k}_m,\quad m<H \tag{80}$$

and

$$f_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)=\infty,\quad m\geq H\ , \tag{81}$$

where

$$H=\ln(1/\mu)/\ln\bar{v} \tag{82}$$

is a positive fractal exponent that expresses the balance between the average rate of increase of the population and the probability that the growth process stops. For $m<H$ the asymptotic form of the product densities can be evaluated in the limit of large $|\mathbf{r}_l|$, $l=1,\ldots,m$; we have

$$f_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)\cong m[(m-1)!]^2[\bar{v}_0/(\bar{v}-1)]^{m-1}(1-\mu)\times\sum_{j=1}^{m-1}\frac{(-1)^{m-1-j}}{j!(m-1-j)!}(1-\mu\bar{v}^{j+1})^{-1}\psi_j(\mathbf{r}_1,\ldots,\mathbf{r}_m),\quad H<m,\ |\mathbf{r}_l|\to\infty,\ l=1,\ldots,m\ , \tag{83}$$

where the functions $\psi_j(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ depend on the behavior of the structure function $\bar{p}(\mathbf{k})$. If $\bar{p}(\mathbf{k})$ is analytic near $\mathbf{k}=\mathbf{0}$, $\psi_j(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ are given by

$$\psi_j(\mathbf{r}_1,\ldots,\mathbf{r}_m)=\left[\frac{d_s(1-\mu\bar{v}^{j+1})}{2\pi\mu\bar{v}^{j+1}\langle r_0^2\rangle}\right]^{md_s/2}\times\exp\left[-\frac{d_s(1-\mu\bar{v}^{j+1})}{2\mu\bar{v}^{j+1}\langle r_0^2\rangle}\sum_{l=1}^{m}|\mathbf{r}_l|^2\right]; \tag{84}$$

if $\bar{p}(\mathbf{k})$ is nonanalytic near $\mathbf{k}=\mathbf{0}$ we have

$$\psi_j(\mathbf{r}_1,\ldots,\mathbf{r}_m)=b^{-md_s}\left[\frac{1-\mu\bar{v}^{j+1}}{\mu\bar{v}^{j+1}}\right]^{md_s/\varepsilon}\times\prod_{l=1}^{m}\mathcal{L}_\varepsilon^{(d_s)}\left\{\frac{|\mathbf{r}_l|}{b}\left[\frac{1-\mu\bar{v}^{j+1}}{\mu\bar{v}^{j+1}}\right]^{1/\varepsilon}\right\}. \tag{85}$$

The asymptotic behavior of the Janossy densities $Q_N(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ can be analyzed in a similar way, by taking the Fourier transform over $\mathbf{r}_1,\ldots,\mathbf{r}_N$, evaluating the sum over q, and returning to the real space variables $\mathbf{r}_1,\ldots,\mathbf{r}_N$. In this case the sum over q can be evaluated by using the Poisson summation formula [30]. After lengthy computations we get

$$\tilde{Q}_N(\mathbf{r}_1,\ldots,\mathbf{r}_N)=N!\tilde{P}(N)\tilde{\phi}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)\ , \tag{86}$$

where

$$\tilde{P}(N)\sim N^{-(1+H)}\mathcal{B}(\ln N),\quad N\to\infty \tag{87}$$

is the probability of the population size N as $N\to\infty$,

$$\mathcal{B}(\ln N)=\left[\frac{\bar{v}}{\bar{v}_0}\right]^2\frac{1-\mu}{\ln\bar{v}}\left[\Gamma(1+H)+2\sum_{m=1}^{\infty}\{F^+(1+H,2\pi m/\ln(\bar{v}))\cos[2\pi m(\ln N)/\ln(\bar{v})]+F^-(1+H,2\pi m/\ln(\bar{v}))\sin[2\pi m(\ln N)/\ln(\bar{v})]\}\right] \tag{88}$$

is a periodic function of $\ln N$ with period $\ln\bar{v}$,

$$\Gamma(x)=\int_0^\infty t^{x-1}\exp(-x)dx \tag{89}$$

is the complete gamma function,

$$F^{\pm}(a,b)=[\mathrm{Re},\mathrm{Im}]\Gamma(x=a+bi) \tag{90}$$

are the real and imaginary parts of the complete gamma function of complex argument, respectively, and $\tilde{\phi}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)$ is the probability density of the positions of the components for large N and $|\mathbf{r}_l|$, $l=1,\ldots,N$. If the structure function $\bar{p}(\mathbf{k})$ is analytic near $\mathbf{k}=\mathbf{0}$ we have

$$\tilde{\phi}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)=\left[\frac{d_s\ln\bar{v}}{2\pi\langle r_0^2\rangle\ln N}\right]^{Nd_s/2}\times\exp\left[-\frac{d_s\ln\bar{v}}{2\langle r_0^2\rangle\ln N}\sum_{l=1}^{N}|\mathbf{r}_l|^2\right], \tag{91}$$

whereas for a nonanalytic structure function

$$\tilde{\phi}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)=b^{-Nd_s}\left[\frac{\ln\bar{\nu}}{\ln N}\right]^{Nd_s/\varepsilon}\times\prod_{l=1}^{N}\mathcal{L}_\varepsilon^{(d_s)}\left[\frac{|\mathbf{r}_l|}{b}\left[\frac{\ln\bar{\nu}}{\ln N}\right]^{1/\varepsilon}\right] . \quad (92)$$

In the case of analytic behavior the distribution of the positions of the components is Gaussian, the corresponding second moments increasing linearly with the logarithm of the population size

$$\langle \mathbf{r}_l\cdot\mathbf{r}_{l'}\rangle \sim \delta_{ll'}\langle r_0^2\rangle(\ln N)/\ln(\bar{\nu}) . \quad (93)$$

In the nonanalytic case the distribution of the positions of the individuals is a superposition of Lévy laws with a characteristic length obeying a logarithmic scaling law similar to Eq. (93):

$$b^*=b\,[(\ln N)/\ln(\bar{\nu})]^{1/\varepsilon} . \quad (94)$$

The probability density (92) has a long tail of the inverse power law type. By using the asymptotic expansion of the Lévy laws [26] for large values of the random variables we obtain

$$\tilde{\phi}(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)\sim b^{\varepsilon N}\left\{\mathcal{M}(\varepsilon,d_s)\left[\frac{\ln N}{\ln\bar{\nu}}\right]\right\}^N\times\prod_l\left\{|\mathbf{r}_l|^{-(d_s+\varepsilon)}\right\},\quad |\mathbf{r}_l|\to\infty , \quad (95)$$

where $\mathcal{M}(\varepsilon,d_s)$ is a positive function depending on the fractal exponent ε and on the space dimension d_s. Thus we have

$$\mathcal{M}(\varepsilon,d_s=1)=\frac{1}{\pi}\Gamma(1+\varepsilon)\sin(\tfrac{1}{2}\pi\varepsilon) ,$$

$$\mathcal{M}(\varepsilon,d_s=3)=\frac{1+\varepsilon}{2\pi^2}\Gamma(1+\varepsilon)\sin(\tfrac{1}{2}\pi\varepsilon) .$$

In the analytic case there is a very simple physical explanation for the logarithmic scaling law (93). In the supercritical regime the space occupied by the population increases from generation to generation. The average of the absolute value of the mean square displacement vector is a measure of the increase of the occupied space for a new generation. For q generations the dispersion of the position of a component can be computed by following a genealogical path from $q=0$ to q. As the displacement vectors corresponding to different generations are independent random variables, the total dispersion is q times larger than $\langle r_0^2\rangle$:

$$\langle |\mathbf{r}|^2\rangle \sim q\langle r_0^2\rangle . \quad (96)$$

On the other hand, the typical size of the population increases exponentially with the number of generations

$$N\sim\bar{\nu}^q,\quad \text{i.e.,}\quad q\cong(\ln N)/\ln(\bar{\nu}) . \quad (97)$$

By eliminating the generation index q from Eqs. (93)–(97) we recover Eq. (93) for the particular case when $l=l'$,

$$\langle |\mathbf{r}|^2\rangle \sim \langle r_0^2\rangle(\ln N)/\ln(\bar{\nu}) . \quad (98)$$

The second scaling condition in terms of $\ln N$, Eq. (94), also reflects a relationship of balancing between the processes of population increase and of spatial dispersion; however, since in this case the dispersion of displacement for a generation is infinite $\langle r_0^2\rangle=\infty$, there is no alternative physical proof for Eq. (94).

By comparing Eq. (48) for the probability density of the positions of the components at the qth generation with Eqs. (91) and (92) of the same probability density for the final population described by the renormalized approach, we notice an important difference. For the qth generation the position of a component is independent of the size of the population and of the positions of the other components. For the final population, even though the probability density $\phi(\mathbf{r}_1,\ldots,\mathbf{r}_N|N)$ is the product of N probability densities corresponding to the different components, there is, however, an interaction of the mean field type among the different components; due to this mean field interaction the width of the probability distribution of the position of each component increases logarithmically with the total number of components.

IX. CONCLUSION

In this article we have suggested a model for space-dependent random growth processes in terms of a generalized branching process that incorporates the positions of the components as supplementary random variables. Our approach is a synthesis of the theories of branching processes [1], random point processes [21], and random walks [19]. The mathematical formalism relies upon the use of four different types of generating functionals; the discrete nature of the process allows us to define and compute the generating functionals of the theory without introducing an integration measure over the space of the field of random population density. The mathematical approach used is a functional generalization of the classical theory of branching processes. The generating functional of the number and the positions of the components from a given generation is a multiple functional iterate of the generating functional for the direct offspring of a component. We have derived formal expressions for the Janossy and product densities and for the moments and the cumulants of the population density corresponding to a given generation. A renormalization group approach has been introduced by assuming that the growth of the population comes to an end after a random number of generations. We outline that our renormalization group equation, unlike most similar equations used in the literature [24,25], is a consequence of the model rather than an *ad hoc* assumption; it expresses the statistical compensation between the exponential increase of the population from generation to generation and the exponential decrease of the probability that the growth process continues.

An exactly solvable model has been suggested; it was inspired by two different problems from biological population dynamics [8,29] and the physical kinetics of photons [5]. Such a model may also describe the growth of an *in vitro* bacterial colony on a large Petri dish or a fractal growth process of the type of diffusion limited aggregation [31]. For this solvable model there is the possibility of the simultaneous occurrence of two different types of statistical fractals; one fractal may be generated by the random walk that governs the spatial dispersion of the population and the second fractal is due to the balancing between the population growth and extinction. This second statistical fractal displays oscillations in the logarithm of the population size that are caused by the discrete nature of the population growth.

For concrete physical and biological applications it is desirable to increase the complexity of the model by incorporating some features characteristic for real growth processes. Such a feature is the overlapping of generations. If the generations are overlapping, components from different generations may exist at the same time. This is a common situation that occurs in both physical and biological systems; to describe it the growth process should be analyzed both in terms of continuous time and in terms of the discrete generation index. Work on this problem is in progress and is planned to be presented elsewhere.

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APPENDIX A

In this appendix we try to establish a relationship between the definitions for the product densities suggested by Carruthers for the statistical study of multiplicity distributions in particle physics [27] and the classical definition of Stratonovich and Van Kampen [21]. We are interested in the possibility of the occurrence of m particles, the first at a position between $\mathbf{r}_1$ and $\mathbf{r}_1+d\mathbf{r}_1, \ldots$ and the mth at a position between $\mathbf{r}_m$ and $\mathbf{r}_m+d\mathbf{r}_m$, regardless of the number of components that may exist at other positions. Given a realization of $N \geq m$ individuals at positions between $\mathbf{r}'_l$ and $\mathbf{r}'_l+d\mathbf{r}'_l$, $l=1,\ldots,N$, the instantaneous value of the mth product density is a product of $m\,\delta$ functions

$$\delta(\mathbf{r}_1-\mathbf{r}'_{\alpha_1})\cdots\delta(\mathbf{r}_m-\mathbf{r}'_{\alpha_m}), \alpha_1,\ldots,\alpha_m=1,\ldots,m\ , \qquad \alpha_k \neq \alpha_j,\ k,j=1,\ldots,m\ . \tag{A1}$$

Since the components are distinct, $\alpha_1,\ldots,\alpha_m$ should be different from each other. The product density of order m is an average of the instantaneous value (A1) over all distinct values of $\alpha_1,\ldots,\alpha_m$, over the positions $\mathbf{r}'_1,\ldots,\mathbf{r}'_N$, and over the total number N of individuals:

$$f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)=\langle\delta(\mathbf{r}_1-\mathbf{r}'_{\alpha_1})\cdots\delta(\mathbf{r}_m-\mathbf{r}'_{\alpha_m})\rangle = \sum_{N=0}^{\infty}\frac{1}{N!}\int\cdots\int d\mathbf{r}'_1\cdots d\mathbf{r}'_N Q_N^{(g)}(\mathbf{r}'_1,\ldots,\mathbf{r}'_N)\sum_{\alpha_1,\ldots,\alpha_m=1}^{N}{}' \delta(\mathbf{r}_1-\mathbf{r}'_{\alpha_1})\cdots\delta(\mathbf{r}_m-\mathbf{r}'_{\alpha_m})\ , \tag{A2}$$

where the prime shows that in the sum over $\alpha_1,\ldots,\alpha_m$ all labels should be distinct. In order to compute the product densities we introduce the generating functional

$$\Xi^{(q)}[W(\mathbf{r})]=1+\sum_{m=1}^{\infty}\frac{1}{m!}\int\cdots\int f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)W(\mathbf{r}_1)\cdots W(\mathbf{r}_m)d\mathbf{r}_1\cdots d\mathbf{r}_m\ , \tag{A3}$$

and insert Eq. (A2) into Eq. (A3). By getting rid of the δ functions through integration over $\mathbf{r}_1,..,\mathbf{r}_N$ we obtain

$$\Xi^{(q)}[W(\mathbf{r})]=\sum_{N=0}^{\infty}\frac{1}{N!}\int\cdots\int Q_N^{(g)}(\mathbf{r}'_1,\ldots,\mathbf{r}'_N)\sum_{m=0}^{N}\frac{1}{m!}\sum_{\alpha_1,\ldots,\alpha_m}{}'\prod_{\rho=1}^{m}W(\mathbf{r}'_{\alpha_\rho})d\mathbf{r}'_1\cdots d\mathbf{r}'_N\ . \tag{A4}$$

Now we use the identity

$$1+\sum_{m=1}^{N}\frac{1}{m!}\sum_{\alpha_1,\ldots,\alpha_m}{}'\prod_{\rho=1}^{m}W(\mathbf{r}'_{\alpha_\rho})=\prod_{q=1}^{N}[1+W(r'_l)]\ , \tag{A5}$$

which can be easily proved by direct expansion of the product over l. By making use of Eqs. (6) and (A5) we can express the right-hand side of Eq. (A4) in terms of the generating functional $\mathcal{G}_q[W(r)]$:

$$\Xi^{(q)}[W(\mathbf{r})]=\mathcal{G}^{(q)}[1+W(\mathbf{r})]\ . \tag{A6}$$

By expanding $\Xi^{(q)}[W(\mathbf{r})]$ given by Eq. (A6) in a functional Taylor series around $W(\mathbf{r})=0$ and comparing the result with Eq. (A3), we can express the product densities $f_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ in terms of the generating functional $\mathcal{G}^{(q)}[W(\mathbf{r})]$:

$$f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)=\left.\frac{\delta^m\mathcal{G}_q[W(\mathbf{r})]}{\delta W(\mathbf{r}_1)\cdots\delta W(\mathbf{r}_m)}\right|_{W(\mathbf{r})=1}\ . \tag{A7}$$

This equation is identical to Eq. (30) in the text. To express the product densities $f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ in terms of the Janossy densities $Q_N^{(g)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)$, we expand both the right- and the left-hand sides of Eq. (A6) in a functional Taylor series and equate the coefficients of $W(\mathbf{r}_1)\cdots W(\mathbf{r}_N)$, obtaining

$$f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)=\sum_{N=0}^{\infty}\frac{1}{N!}\int\cdots\int Q_{N+m}^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_{N+m})\times d\mathbf{r}_{m+1}\cdots d\mathbf{r}_{m+N}\ . \tag{A8}$$

Conversely, in order to express the Janossy densities $Q_N^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ in terms of the product densities, we write Eq. (A6) in the form

$$\mathcal{G}^{(q)}[W'(\mathbf{r})]=\Xi^{(q)}[W'(\mathbf{r})-1]\quad \text{with } W'(\mathbf{r})=1+W(\mathbf{r}) \tag{A9}$$

and expand both terms of Eq. (A9) in a functional Taylor series, thus arriving at

$$Q_N^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_N)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int\cdots\int f_{m+N}^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_{m+N})\times d\mathbf{r}_{N+1}\cdots d\mathbf{r}_{N+m}\ . \tag{A10}$$

From Eqs. (A9) and (A10) it follows that the Carruthers definition (A2) of the product densities is equivalent to the definition given by Stratonovich and Van Kampen [21].

Concerning the physical interpretation of the product densities, we note that for $m=1$ the Carruthers definition (A2) shows that $f_1^{(q)}(\mathbf{r})$ is the average of the population density $n^{(q)}(\mathbf{r})$ at position $\mathbf{r}$

$$f_1^{(q)}(\mathbf{r})=\langle n^{(q)}(\mathbf{r})\rangle\ . \tag{A11}$$

The other product densities $f_m^{(q)}$, $m=2,3,\ldots,$ however, are different from the corresponding moments of the density

$$\langle n^{(q)}(\mathbf{r}_1)\cdots n^{(q)}(\mathbf{r}_m)\rangle=\sum_{N=0}^{\infty}\frac{1}{N!}\int\cdots\int d\mathbf{r}'_1\cdots d\mathbf{r}'_N\, Q_N^{(q)}(\mathbf{r}'_1,\ldots,\mathbf{r}'_N)\sum_{\alpha_1,\ldots,\alpha_m=1}^{N}\delta(\mathbf{r}_1-\mathbf{r}'_{\alpha_1})\cdots\delta(\mathbf{r}_m-\mathbf{r}'_{\alpha_m})\ . \tag{A12}$$

The difference is due to the fact that in the definition (A12) of the moments of population density the sum over $\alpha_1,\ldots,\alpha_m$ runs over all possible values of $\alpha_1,\ldots,\alpha_m$. The relation between the moments of population density and the product densities is of the same nature as the relation between the moments of the total population number N

$$\langle N^m\rangle^{(q)}=\sum_{N=0}^{\infty}N^mP^{(q)}(N) \tag{A13}$$

and the factorial moments

$$\begin{aligned}F_m^{(q)}&=\langle N(N-1)\cdots(N-m+1)\rangle^{(q)}\\&=\sum_{N=m}^{\infty}N(N-1)\cdots(N-m+1)P^{(q)}(N)\\&=d^m\mathcal{G}_q(z)/dz^m|_{z=1}\ .\end{aligned} \tag{A14}$$

Equation (A7) is the space-dependent analog of the relationships (A14) for the factorial moments. By remembering that the generating function $\mathcal{G}_q(z)$ can be obtained from the generating functional $\mathcal{G}_q[W(\mathbf{r})]$ for $W(\mathbf{r})=z$ and comparing the Taylor expansions of $\mathcal{G}_q(z)$ and $\mathcal{G}_q[W(\mathbf{r})]$ around $W(\mathbf{r})=z=1$, we can express the factorial moments $F_m^{(q)}$ in terms of the product densities

$$\begin{aligned}F_m^{(q)}&=\int\cdots\int d\mathbf{r}_1\cdots d\mathbf{r}_m\left.\frac{\delta^m\mathcal{G}_q[W(\mathbf{r})]}{\delta W(\mathbf{r}_1)\cdots\delta W(\mathbf{r}_m)}\right|_{W(\mathbf{r})=1}\\&=\int\cdots\int f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_1)d\mathbf{r}_1\cdots d\mathbf{r}_m\ .\end{aligned} \tag{A15}$$

In order to compare the difference between the actual spatial population distribution and the Poissonian law that corresponds to a perfect random distribution of individuals we introduce the correlation functions

$$g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m),\quad m=1,2,\ldots\ . \tag{A16}$$

The correlation functions $q_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ are defined by a cumulant expansion of the generating functional $\Xi_q[W(\mathbf{r})]$,

$$\Xi_q[W(\mathbf{r})]=\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\int\cdots\int g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)\times W(\mathbf{r}_1)\cdots W(\mathbf{r}_m)\times d\mathbf{r}_1\cdots d\mathbf{r}_m\right\}\ . \tag{A17}$$

From Eq. (A17) it follows that the correlation functions $g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ are equal to the functional derivatives of $\ln\Xi_q[W(\mathbf{r})]$ and $\ln\mathcal{G}_q[W(\mathbf{r})]$ at the points $W(\mathbf{r})=0$ and 1, respectively,

$$\begin{aligned}g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)&=\left.\frac{\delta^m\ln\Xi_q[W(\mathbf{r})]}{\delta W(\mathbf{r}_1)\cdots\delta W(\mathbf{r}_m)}\right|_{W(\mathbf{r})=0}\\&=\left.\frac{\delta^m\ln\mathcal{G}_q[W(\mathbf{r})]}{\delta W(\mathbf{r}_1)\cdots\delta W(\mathbf{r}_m)}\right|_{W(\mathbf{r})=1}\ .\end{aligned} \tag{A18}$$

The correlation functions $g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ are the space-dependent analogs of the factorial cumulants $\mathcal{E}_m^{(q)}$ of the total population

$$\mathcal{E}_m^{(q)}=d^m\ln\mathcal{G}_q(z)/dz^m|_{z=1}\ . \tag{A19}$$

Between the correlation functions $g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ and the factorial cumulants $\mathcal{E}_m^{(q)}$ there is the relationship

$$\begin{aligned}\mathcal{E}_m^{(q)}&=\int\cdots\int\left.\frac{\delta^m\ln\mathcal{G}_q[W(\mathbf{r})]}{\delta W(\mathbf{r}_1)\cdots\delta W(\mathbf{r}_m)}\right|_{W(\mathbf{r})=1}d\mathbf{r}_1\cdots d\mathbf{r}_m\\&=\int\cdots\int g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)d\mathbf{r}_1\cdots d\mathbf{r}_m\ ,\end{aligned} \tag{A20}$$

which can be proved in the same way as Eq. (A15).

Similar product densities and correlation functions can be defined for the direct offspring of a component characterized by the Janossy densities $R_\nu(\Delta\mathbf{r}_1,\ldots,\Delta\mathbf{r}_\nu)$ or by the generating functional $L[W(\Delta\mathbf{r})]$. These functions obey exactly the same type of relationships as the corresponding functions attached to the offspring from the qth generation; to save space we shall not write all the equations here. We mention only that the product densities of the direct offspring of a component

$$\eta_m(\Delta\mathbf{r}_1,\ldots,\Delta\mathbf{r}_m)=\langle\delta(\Delta\mathbf{r}_1-\Delta\mathbf{r}'_{\alpha_1})\cdots\delta(\Delta\mathbf{r}_m-\Delta\mathbf{r}'_{\alpha_m})\rangle$$
$$=\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\int\cdots\int d\Delta\mathbf{r}'_1\cdots d\Delta\mathbf{r}'_\nu R_\nu(\Delta\mathbf{r}'_1,\ldots,\Delta\mathbf{r}'_\nu)\sum_{\alpha_1,\ldots,\alpha_m=1}^{\nu}{}'\,\delta(\Delta\mathbf{r}_1-\Delta\mathbf{r}'_{\alpha_1})\cdots\delta(\Delta\mathbf{r}_m-\Delta\mathbf{r}'_{\alpha_m}) \tag{A21}$$

can be expressed in terms of the generating functional $L[W(\Delta\mathbf{r})]$ by a relationship similar to Eq. (A7):

$$\eta_m(\Delta\mathbf{r}_1,\ldots,\Delta\mathbf{r}_m)=\left.\frac{\delta^m L[W(\Delta\mathbf{r})]}{\delta W(\Delta\mathbf{r}_1)\cdots\delta W(\Delta\mathbf{r}_m)}\right|_{W(\Delta\mathbf{r})=1}. \tag{A22}$$

At least in principle all product densities $f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ and the correlation functions $g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ for the qth generation can be expressed in terms of the product densities $\eta_m(\Delta\mathbf{r}_1,\ldots,\Delta\mathbf{r}_m)$ for the direct offspring of a component by means of repeated functional differentiation of the forward evolution equations (12), (16), and (18) or of the backward equation (17) and (18). By functional differentiation the two sets of equations lead to two different chains of equations that relate the product densities or the correlation functions corresponding to two successive generations; by solving these equations generation by generation we can compute all functions $f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$ and $g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)$. For illustration we compute the first two functions $f_{1,2}^{(q)}$ and $g_{1,2}^{(q)}$ by using the forward equations. The functional differentiation of Eq. (12) gives

$$\frac{\delta\mathcal{G}_q[W]}{\delta W(\mathbf{r})}=\int\frac{\delta\mathcal{G}_{q-1}[L]}{\delta L(\mathbf{r}')}\frac{\delta L[W]}{\delta W(\mathbf{r}-\mathbf{r}')}d\mathbf{r}' \tag{A23}$$

and

$$\frac{\delta^2\mathcal{G}_q[W]}{\delta W(\mathbf{r}_1)\delta W(\mathbf{r}_2)}=\int\int\frac{\delta^2\mathcal{G}_{q-1}[L]}{\delta L(\mathbf{r}'_1)\delta L(\mathbf{r}'_2)}\frac{\delta L[W]}{\delta W(\mathbf{r}_1-\mathbf{r}'_1)}\frac{\delta L[W]}{\delta W(\mathbf{r}_2-\mathbf{r}'_2)}d\mathbf{r}'_1 d\mathbf{r}'_2+\int\frac{\delta\mathcal{G}_{q-1}[L]}{\delta L(\mathbf{r}')}\frac{\delta^2 L[W]}{\delta W(\mathbf{r}_1-\mathbf{r}')\delta W(\mathbf{r}_2-\mathbf{r}')}d\mathbf{r}' \tag{A24}$$

from which, by using Eqs. (A7) and (A22), we get the integral equations

$$f_1^{(q)}(\mathbf{r})=\int f_1^{(q-1)}(\mathbf{r}')\eta_1(\mathbf{r}-\mathbf{r}')d\mathbf{r}' \tag{A25}$$

and

$$f_2^{(q)}(\mathbf{r}_1,\mathbf{r}_2)=\int f_1^{(q-1)}(\mathbf{r}')\eta_2(\mathbf{r}_1-\mathbf{r}',\mathbf{r}_2-\mathbf{r}')d\mathbf{r}'+\int\int f_2^{(q-1)}(\mathbf{r}'_1,\mathbf{r}'_2)\eta_1(\mathbf{r}_1-\mathbf{r}'_1)\times\eta_1(\mathbf{r}_2-\mathbf{r}'_2)d\mathbf{r}'_1 d\mathbf{r}'_2\,. \tag{A26}$$

Similarly, by differentiating the logarithm of $\mathcal{G}_q[W(\mathbf{r})]$ and using Eq. (A18) we can derive a system of two integral equations for the first two correlation functions $g_1^{(q)}$ and $g_2^{(q)}$:

$$g_1^{(q)}(\mathbf{r})=\int g_1^{(q-1)}(\mathbf{r}')\eta_1(\mathbf{r}-\mathbf{r}')d\mathbf{r}' \tag{A27}$$

and

$$g_2^{(q)}(\mathbf{r}_1,\mathbf{r}_2)=\int g_1^{(q-1)}(\mathbf{r}')\eta_2(\mathbf{r}_1-\mathbf{r}',\mathbf{r}_2-\mathbf{r}')d\mathbf{r}'+\int\int g_2^{(q-1)}(\mathbf{r}'_1,\mathbf{r}'_2)\eta_1(\mathbf{r}_1-\mathbf{r}'_1)\times\eta_1(\mathbf{r}_2-\mathbf{r}'_2)d\mathbf{r}_1 d\mathbf{r}'_2\,. \tag{A28}$$

Equations (A25) and (A26) and Eqs. (A27) and (A28) have exactly the same structure and thus the expressions for $g_{1,2}^{(q)}$ can be derived from the expressions for $f_{1,2}^{(q)}$ and thus we solve only Eqs. (A25) and (A26). By introducing the Fourier transforms

$$\bar{f}_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)=\int\cdots\int\exp\left[\sum i\mathbf{k}_l\cdot\mathbf{r}_l\right]\times f_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)d\mathbf{r}_1\cdots d\mathbf{r}_m \tag{A29}$$

and

$$\bar{\eta}_m(\mathbf{k}_1,\ldots,\mathbf{k}_m)=\int\cdots\int\exp\left[\sum i\mathbf{k}_l\cdot\mathbf{r}_l\right]\times\eta_m(\mathbf{r}_1,\ldots,\mathbf{r}_m)\times d\mathbf{r}_1\cdots d\mathbf{r}_m\,, \tag{A30}$$

Eqs. (A25) and (A26) lead to a system of difference equations in $\bar{f}_{1,2}^{(q)}$:

$$\bar{f}_1^{(q)}(\mathbf{k})=\bar{f}_1^{(q-1)}(\mathbf{k})\bar{\eta}_1(\mathbf{k}) \tag{A31}$$

and

$$\bar{f}_2^{(q)}(\mathbf{k}_1,\mathbf{k}_2)=\bar{f}_2^{(q-1)}(\mathbf{k}_1,\mathbf{k}_2)\bar{\eta}_1(\mathbf{k}_1)\bar{\eta}_1(\mathbf{k}_2)+\bar{f}_1^{(q-1)}(\mathbf{k}_1+\mathbf{k}_2)\bar{\eta}_2(\mathbf{k}_1,\mathbf{k}_2)\,. \tag{A32}$$

By solving Eqs. (A31) and (A32) and performing two inverse Fourier transformations we get the following formal expressions for the first two product densities attached to the qth generation:

$$f_1^{(q)}(\mathbf{r})=(2\pi)^{-d_s}\int\exp(-i\mathbf{k}\cdot\mathbf{r})\bar{f}_1^{(0)}(\mathbf{k})\times[\bar{\eta}_1(\mathbf{k})]^q d\mathbf{k}\,, \tag{A33}$$

and

$$f_2^{(q)}(\mathbf{r}_1,\mathbf{r}_2)=(2\pi)^{-2d_s}\int\int \exp(-i\mathbf{k}_1\cdot\mathbf{r}_1-i\mathbf{k}_2\cdot\mathbf{r}_2)\times\Bigg\{\bar{f}_2^{(0)}(\mathbf{k}_1,\mathbf{k}_2)[\bar{\eta}_1(\mathbf{k}_1)\bar{\eta}_1(\mathbf{k}_2)]^q+\bar{\eta}_2(\mathbf{k}_1,\mathbf{k}_2)\bar{f}_1^{(0)}(\mathbf{k}_1+\mathbf{k}_2)\frac{[\bar{\eta}_1(\mathbf{k}_1+\mathbf{k}_2)]^q-[\bar{\eta}_1(\mathbf{k}_1)\bar{\eta}_1(\mathbf{k}_2)]^q}{\bar{\eta}_1(\mathbf{k}_1+\mathbf{k}_2)-\bar{\eta}_1(\mathbf{k}_1)\bar{\eta}_1(\mathbf{k}_2)}\Bigg\}d\mathbf{k}_1 d\mathbf{k}_2\ , \quad \text{(A34)}$$

where $d_s=1,2,3$ is the space dimension. The solutions of Eqs. (A27) and (A28) for $g_{1,2}^{(q)}$ can be derived from Eqs. (62) and (63) by means of the substitutions

$$f_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m)\to g_m^{(q)}(\mathbf{r}_1,\ldots,\mathbf{r}_m),\quad m=1,2$$
$$\bar{f}_m^{(0)}(\mathbf{k}_1,\ldots,\mathbf{k}_m)\to \bar{g}_m^{(0)}(\mathbf{k}_1,\ldots,\mathbf{k}_m),\ m=1,2\ . \quad \text{(A35)}$$

For evaluating the product densities $f_{1,2}^{(q)}$ from Eqs. (A33) and (A34) we need to know the expressions of the functions $\eta_{1,2}$ and $f_{1,2}^{(0)}$; the results of an explicit computation have been given in Sec. VII for the particular case of self-similar symmetric processes. Here we mention only that Eqs. (A33) and (A34) are consistent with the space-independent theory of branched chain processes [1]. From Eq. (A15) it follows that the factorial moments of the total population from the qth generation and the factorial moments of the direct offspring number of a component can be expressed in terms of the Fourier transforms of the product densities

$$\langle N(N-1)\cdots(N-m+1)\rangle^{(q)}=\bar{f}_m^{(q)}(\mathbf{k}_1=\mathbf{0},\ldots,\mathbf{k}_m=\mathbf{0})\ , \quad \text{(A36)}$$

$$\langle(\nu-1)\cdots(\nu-m+1)\rangle=\bar{\eta}_m(\mathbf{k}_1=\mathbf{0},\ldots,\mathbf{k}_m=\mathbf{0})\ . \quad \text{(A37)}$$

By inserting Eqs. (A36) and (A37) and expressing the second factorial moments $\langle N(N-1)\rangle^{(q)}$ and $\langle \nu(\nu-1)\rangle$ in terms of the dispersions $\langle \Delta N^2\rangle^{(q)}=\langle (N-\langle N\rangle)^2\rangle^{(q)}$ and $\langle \Delta\nu^2\rangle=\langle(\nu-\langle\nu\rangle)^2\rangle$ we obtain

$$\langle N\rangle^{(q)}=\langle N\rangle^{(0)}[\langle\nu\rangle]^q \quad \text{(A38)}$$

and

$$\langle \Delta N^2\rangle^{(q)}=\langle\Delta N^2\rangle^{(0)}[\langle\nu\rangle]^{2q}+\langle N\rangle^{(0)}[\langle\nu\rangle]^{q-1}\left[\frac{[\langle\nu\rangle]^q-1}{\langle\nu\rangle-1}\right]\ . \quad \text{(A39)}$$

We have recovered the well-known expressions for the average value and the dispersion of the population number at qth generation for space-independent branched chain processes [1].

APPENDIX B

In the particular case of self-similar symmetric systems introduced in Sec. VI, the repeated functional iteration (13)–(18) of the functional $L[W(\mathbf{r})]$ can be carried out explicitly, i.e., it can be reduced to the repeated iteration of a homographic function. By combining Eqs. (3) and (A39)–(A41) we get the following expression for the generating functional $L[W(\Delta\mathbf{r})]$ of the direct offspring of a component:

$$L[W(\Delta\mathbf{r})]=1-\beta+\beta(1-\lambda)\times\sum_{\nu=1}^{\infty}\lambda^{\nu-1}\left[\int p(\Delta\mathbf{r})W(\Delta\mathbf{r})d\Delta\mathbf{r}\right]^{\nu}=c(Z_1[W(\Delta\mathbf{r})])\ , \quad \text{(B1)}$$

where

$$c(z)=1-\beta+\beta(1-\lambda)z/(1-\lambda z) \quad \text{(B2)}$$

and

$$Z_1[W(\Delta\mathbf{r})]=\int W(\Delta\mathbf{r})p(\Delta\mathbf{r})d\Delta\mathbf{r}\ . \quad \text{(B3)}$$

From Eqs. (B1)–(B3) it follows that the qth iterate of $L[W(\Delta r)]$, $L^{*(q)}[W(\Delta\mathbf{r})]$, is equal to:

$$L^{*(q)}[W(\Delta\mathbf{r})]=c^{*(q)}(Z_q[W(\Delta\mathbf{r})])\ , \quad \text{(B4)}$$

where the functionals $Z_q[W(\Delta\mathbf{r})]$, $q=2,3,\ldots$, obey the chain of recursive equations

$$Z_q[W(\Delta r)]=\int Z_{q-1}[W(\Delta\mathbf{r}+\Delta\mathbf{r}')]p(\Delta\mathbf{r}')d\Delta\mathbf{r}'\ ,\quad q=2,3,\ldots \quad \text{(B5)}$$

and $c^{*(q)}(z)$ is the qth iterate of the function $c(z)$ given by Eq. (B2).

By applying Eqs. (B3) and (B5) recursively we can compute the functionals $Z_q[W(\Delta\mathbf{r})], q=1,2,\ldots,$ step by step to obtain

$$Z_q[W(\Delta\mathbf{r})]=\int\cdots\int W(\Delta\mathbf{r}+\Delta\mathbf{r}_1'+\cdots+\Delta\mathbf{r}_q')\times p(\Delta\mathbf{r}_1')\cdots p(\Delta\mathbf{r}_q')\times d\Delta\mathbf{r}_1'\cdots d\Delta\mathbf{r}_q'\ ,\quad q=1,2,\ldots\ . \quad \text{(B6)}$$

By introducing the new integration variable

$$\Delta r'=\Delta\mathbf{r}_1'+\cdots+\Delta\mathbf{r}_q'\ , \quad \text{(B7)}$$

Eq. (88) can be written in a more convenient form

$$Z_q[W(\Delta\mathbf{r})]=\int W(\Delta\mathbf{r}+\Delta\mathbf{r}')p_q(\Delta\mathbf{r}')d\Delta\mathbf{r}'\ , \quad \text{(B8)}$$

where

$$p_q(\Delta\mathbf{r}')=\int\cdots\int\delta(\Delta\mathbf{r}'-\Delta\mathbf{r}_1'-\cdots-\Delta\mathbf{r}_q')\times p(\Delta\mathbf{r}_1')\cdots p(\Delta\mathbf{r}_q')d\Delta\mathbf{r}_1'\cdots d\Delta\mathbf{r}_q'=[p(\Delta\mathbf{r})\otimes]^{(q)} \quad \text{(B9)}$$

is the m-fold convolution product of the probability density $p(\Delta\mathbf{r})$.

We have reduced the process of functional iteration of the generating functional $L[W(\Delta\mathbf{r})]$ to the iteration of the function $c(z)$ given by Eq. (B2). The qth iterate of $c(z)$, $c^{*(q)}(z)$, is the solution of a nonlinear difference equation

$$(1-\beta-c^{*(q)})(1-\lambda c^{*(q-1)})+\beta(1-\lambda)c^{*(q-1)}=0 , \tag{B10}$$

with the initial condition

$$c^{*(0)}=z . \tag{B11}$$

By means of the substitution

$$c^{*(q)}=1-(\beta/\lambda)+(A_q/A_{q+1}) , \tag{B12}$$

the nonlinear equation (B12) can be reduced to a second-order linear difference equation

$$\beta(1-\lambda)A_{q+1}-\lambda(1-\lambda+\beta)A_q+\lambda^2 A_{q-1}=0 , \tag{B13}$$

with the initial condition

$$z=1-(\beta/\lambda)+(A_0/A_1) . \tag{B14}$$

By solving Eq. (B13) with the initial condition (B14) we can express A_q as a function of A_0; the qth iterate $c^{*(q)}$ can be computed by inserting the resulting expression of A_q into Eq. (B12). After lengthy manipulations we arrive at

$$c^{*(q)}(z)=\frac{(1-\beta)[\beta^q-(1-\lambda)^q]-[(1-\beta)\beta^q-\lambda(1-\lambda)^q]z}{\lambda\beta^q-(1-\beta)(1-\lambda)^q-\lambda[\beta^q-(1-\lambda)^q]z} . \tag{B15}$$

Combining Eqs. (13)–(18), (B4), and (B15) we arrive at Eq. (42).

APPENDIX C

We introduce the characteristic functional $\Lambda_q[K(\mathbf{r})]$ for the population density $n(\mathbf{r})$ at the qth generation

$$\Lambda_q[K(\mathbf{r})]=\left\langle\exp\left[i\int K(\mathbf{r})n(\mathbf{r})d\mathbf{r}\right]\right\rangle , \tag{C1}$$

where $K(\mathbf{r})$ is an appropriate test function, $n(\mathbf{r})$ is given by Eq. (65), and the stochastic properties of the total population size and of the position vectors of the different individuals are characterized by Eq. (42). By inserting Eq. (65) into Eq. (C1) and evaluating the average with the help of Eq. (42) we obtain

$$\begin{aligned}\Lambda_q[K(\mathbf{r})]&=\sum_{N=0}^{\infty}P^{(q)}(N)\int\cdots\int\exp\left[i\sum_l K(\mathbf{r}_l)\right]p_q(\mathbf{r}_1)\cdots p_q(\mathbf{r}_N)d\mathbf{r}_1\cdots d\mathbf{r}_N\\&=\sum_{N=0}^{\infty}P^{(q)}(N)\left[\int p_q(\mathbf{r})\exp(iK(\mathbf{r}))d\mathbf{r}\right]^N=\mathcal{G}_q\left[W(\mathbf{r})=\int p_q(\mathbf{r})\exp[iK(\mathbf{r})]d\mathbf{r}\right] .\end{aligned} \tag{C2}$$

The central moments and the cumulants of the density field $n(\mathbf{r})$ may be evaluated from Eq. (C2) by computing the functional derivatives

$$\langle n(\mathbf{r}_1)\cdots n(\mathbf{r}_m)\rangle^{(q)}=(-i)^m\left.\frac{\delta^m\Lambda_q[K(\mathbf{r})]}{\delta K(\mathbf{r}_1)\cdots\delta K(\mathbf{r}_m)}\right|_{K(\mathbf{r})=0} , \tag{C3}$$

$$\langle\langle n(\mathbf{r}_1)\cdots n(\mathbf{r}_m)\rangle\rangle^{(q)}=(-i)^m\left.\frac{\delta^m\ln\Lambda_q[K(\mathbf{r})]}{\delta K(\mathbf{r}_1)\cdots\delta K(\mathbf{r}_m)}\right|_{K(\mathbf{r})=0} . \tag{C4}$$

By applying Eqs. (C2)–(C4) for $m=1,2$ we get Eqs. (66)–(67).

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