Qualitative analysis of Cohen-Grossberg neural networks with multiple delays

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It is well known that a class of artificial neural networks with symmetric interconnections and without transmission delays, known as Cohen-Grossberg neural networks, possesses global stability (i.e., all trajectories tend to some equilibrium). We demonstrate in the present paper that many of the qualitative properties of Cohen-Grossberg networks will not be affected by the introduction of sufficiently small delays. Specifically, we establish some bound conditions for the time delays under which a given Cohen-Grossberg network with multiple delays is globally stable and possesses the same asymptotically stable equilibria as the corresponding network without delays. An effective method of determining the asymptotic stability of an equilibrium of a Cohen-Grossberg network with multiple delays is also presented. The present results are motivated by some of the authors earlier work [Phys. Rev. E **50**, 4206 (1994)] and by some of the work of Marcus and Westervelt [Phys. Rev. A **39**, 347 (1989)]. These works address qualitative analyses of Hopfield neural networks with one time delay. The present work generalizes these results to Cohen-Grossberg neural networks with multiple time delays. Hopfield neural networks constitute special cases of Cohen-Grossberg neural networks.

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I. INTRODUCTION

Cohen-Grossberg neural networks constitute a class of artificial feedback neural networks whose activation equations are of the form

$$\dot{x}_i=-a_i(x_i)\left[b_i(x_i)-\sum_{j=1}^{n} t_{ij}s_j(x_j)\right], \quad i=1,\ldots,n\ . \tag{1.1}$$

In (1.1) x_i denotes the state variable associated with the ith neuron, the function a_i represents an *amplification function,* and b_i is an arbitrary function; however, we will require that b_i be sufficiently well behaved to keep the solutions of Eq. (1.1) bounded. The matrix $T\triangleq[t_{ij}]_{n\times n}$ is a real and symmetric matrix and represents the neuron interconnections. The real function s_j is a sigmoidal nonlinearity representing the jth neuron. The neural dynamics model (1.1) has been widely studied (see, e.g., [1–10]). One of the useful qualitative properties of (1.1) is that it is globally stable [i.e., every trajectory of (1.1) converges to some equilibrium]. It is well known that (1.1) includes the Hopfield neural networks as a special case, a class of neural networks that has been studied widely (see, e.g., [4–10]). Hopfield neural networks are described by a system of equations of the form

$$\dot{x}_i=-c_ix_i+\sum_{j=1}^{n} t_{ij}s_j(x_j), \quad i=1,\ldots,n\ , \tag{1.2}$$

where $c_i\geq 0$ and x_i, t_{ij}, and s_j are the same as in (1.1). Since neural networks (1.1) and (1.2) have the potential of serving as associative memories and performing parallel computations, some electronic implementations of these neural networks in very-large-scale-integration technology have been realized (see, e.g., [11–13]). However, in the implementation process of artificial neural networks, time delays are unavoidably introduced and it is known that such delays can cause systems to oscillate (see, e.g., [14–16]). Therefore, it is important to take time delays into consideration and to investigate the qualitative properties of neural networks of the type (1.1) and (1.2) with delays. A class of Hopfield neural networks with identical delay for each state can be described by equations of the form

$$\dot{x}_i(t)=-c_ix_i(t)+\sum_{j=1}^{n} t_{ij}s_j(x_j(t-\tau))\ , i=1,\ldots,n\ , \tag{1.3}$$

where $\tau>0$ denotes the time delay. System (1.3) has recently been studied by several workers (see [14,17–20]). It is shown in [14,20] that system (1.3) is globally stable and that it possesses the same set of asymptotically stable equilibria as system (1.2), if the delay τ is less than a certain bound.

In the present paper, we extend the results of [20] to a significantly larger class of systems. Specifically, we consider Cohen-Grossberg neural networks (1.1) endowed with multiple delays, described by equations of the form

$$\dot{x}_i(t)=-a_i(x_i(t))\left[b_i(x_i(t))-\sum_{j=1}^{n} t_{ij}^{(0)}s_j(x_j(t)) -\sum_{k=1}^{K}\sum_{j=1}^{n} t_{ij}^{(k)}s_j(x_j(t-\tau_k))\right], \quad i=1,\ldots,n\ , \tag{1.4}$$

where the $t_{ij}^{(k)}$'s denote the interconnections which are associated with delay τ_k, τ_k denotes the kth time delay for $k=0,1,\ldots,K$ such that $0=\tau_0<\tau_1<\cdots<\tau_K$, and x_i,

b_i, and s_j are the same as the corresponding quantities in Eq. (1.1). We will establish bound conditions for the delays under which Cohen-Grossberg neural networks with multiple delays described by Eq. (1.4) will exhibit qualitative properties similar to the corresponding original Cohen-Grossberg neural networks without delays, described by Eq. (1.1) [i.e., conditions under which system (1.4) is globally stable and has the same local stability properties as system (1.1)]. More specifically, we will show that when the delays are small enough to satisfy the established bound conditions, then system (1.4) is globally stable and has the same set of (asymptotically) stable equilibria as system (1.1). From this result it is concluded that not only the global stability of system (1.4), but also the local stability of each equilibrium of system (1.4) will be unaffected by sufficiently small delays. Moreover, we will establish an effective criterion for the (asymptotic) stability of each equilibrium of system (1.4).

In the proofs of the preceding statements, we make use of an energy functional for system (1.4) and we show that this energy functional decreases along the solutions of (1.4), ultimately converging to some equilibrium of system (1.4). We will also show that any (asymptotically) stable equilibrium of (1.4) corresponds to a local minimum of the energy functional.

In the next section we provide the necessary notation used throughout this paper. In Sec. III we provide some preliminaries. In Sec. IV we establish our main result for the global stability of Cohen-Grossberg neural networks with multiple delays (1.4). In Sec. V we investigate the local stability properties of equilibria of system (1.4). Some concluding remarks are provided in (Sec. VI).

II. NOTATION

Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}^n$ denote real n space. If $x \in \mathbb{R}^n$, then $x^T=(x_1, \ldots, x_n)$ denotes the transpose of x. Let $\mathbb{R}^{n\times m}$ denote the set of $n \times m$ real matrices. If $B=[b_{ij}]_{n\times m} \in \mathbb{R}^{n\times m}$, then B^T denotes the transpose of B. For $x \in \mathbb{R}^n$, let $\|x\|$ denote the Euclidean vector norm $\|x\|=(x^Tx)^{1/2}$ and for $A \in \mathbb{R}^{n\times n}$, let $\|A\|$ denote the norm of A induced by the Euclidean vector norm, i.e., $\|A\|=[\lambda_{\max}(A^TA)]^{1/2}$. I denotes the identity $n\times n$ matrix.

Let $\mathbb{R}^+$ denote the set of non-negative real numbers, i.e., $\mathbb{R}^+=[0,+\infty)$. Let X be a subset of $\mathbb{R}^n$ and let Y be a subset of $\mathbb{R}^m$. We denote by $C(X,Y)$ the set of all continuous functions from X to Y and we denote by $C^k(X,Y)$ the set of all functions from X to Y which have continuous derivatives up to order k. With $\tau>0$, $x \in C([-\tau,+\infty),\mathbb{R}^n)$, and with $t>0$, we define $x_t \in C([-\tau,0],\mathbb{R}^n)$ as $x_t(s)=x(t+s)$ for $s\in[-\tau,0]$. For any $\phi \in C([-\tau,0],\mathbb{R}^n)$, the norm of ϕ, denoted by $|\phi|$, is defined as $|\phi|=\max\{\|\phi(t)\|: t\in[-\tau,0]\}$.

The system (1.4) is said to be *globally stable* if for any solution $x(t)$,$\lim_{t\to\infty}x(t)$ exists. For the definitions of stability and asymptotic stability of an equilibrium of (1.4), refer to any of several standard texts (see, e.g., [21]).

III. PRELIMINARIES

In the present paper, we assume that the Cohen-Grossberg neural networks (1.1) and (1.4) satisfy the following assumptions.

Assumption A. (i) The function a_i is bounded, positive, and continuous; (ii) the function b_i is continuous; (iii) $T \triangleq [t_{ij}]_{n\times n}$ is a symmetric matrix; (iv) $s_j \in C^1(\mathbb{R},\mathbb{R})$ is a sigmoidal function [so that $s_j'(x_j) \triangleq ds_j(x_j)/dx_j>0$, $\lim_{x_j\to+\infty}s_j(x_j)=1$, $\lim_{x_j\to-\infty}s_j(x_j)=-1$, and $\lim_{|x_j|\to\infty}s_j'(x_j)=0$); and (v) $\lim_{x_i\to+\infty}b_i(x_i)=+\infty$ and $\lim_{x_i\to-\infty}b_i(x_i)=-\infty$. ■

Remark 1. Assumption A is hypothesized in many references dealing with feedback artificial neural networks without delays (see, e.g., [3]). We note in particular that part (v) of Assumption A ensures the boundedness of the solutions of the neural network (1.1) (*without delays*). In the following, we show in Fact 1 that part (v) of Assumption A ensures also the boundedness of the solutions of the neural network (1.4) *(with delays)*.

More generally, we could replace part (v) of Assumption A by some other hypothesis which ensures the boundedness of solutions of (1.4). For example, it can be shown that parts (e) and (f) of Theorem 1 in [1] ensure the boundedness of the solutions of neural network (1.1) and also of neural network (1.4). ■

Fact 1. If Assumption A is satisfied for systems (1.1) and (1.4), then any solution of (1.1) and (1.4) is bounded.

Proof. We only need to consider system (1.4) since (1.1) is a special case of (1.4). We know by Assumption A that the terms $s_j(x_j(t))$ and $s_j(x_j(t-\tau_k))$ are bounded for all $j=1,\ldots,n$. Furthermore, since $\lim_{x_i\to+\infty}b_i(x_i)=+\infty$ and $\lim_{x_i\to-\infty}b_i(x_i)=-\infty$, there must exist an $M>0$ such that

$$b_i(x_i(t))-\sum_{j=1}^{n} t_{ij}^{(0)}s_j(x_j(t))-\sum_{k=1}^{K}\sum_{j=1}^{n} t_{ij}^{(k)}s_j(x_j(t-\tau_k))>0$$

whenever $x_i(t)\ge M$ and

$$b_i(x_i(t))-\sum_{j=1}^{n} t_{ij}^{(0)}s_j(x_j(t))-\sum_{k=1}^{K}\sum_{j=1}^{n} t_{ij}^{(k)}s_j(x_j(t-\tau_k))<0$$

whenever $x_i(t)\ge -M$ for all $i=1,\ldots,n$. Since $a_i(x_i(t))$ is positive by Assumption A, it can be concluded that, for any solution $x(t)$ of (1.4), $\dot{x}_i(t)<0$ whenever $x_i(t)\ge M$ and $\dot{x}_i(t)>0$ whenever $x_i(t)\le -M$ for all $i=1,\ldots,n$. We may assume that for the initial condition $x_0(\ )$ (x_0 is a function) $|x_0|<M$, for otherwise we just pick a larger M. Thus we can conclude that $\|x_i(t)\|<M$ for all $t\ge 0$ and all $i=1,\ldots,n$. ■

If we let $x=(x_1,\ldots,x_n)^T \in \mathbb{R}^n$, $A(x)=\text{diag}\{a_1(x_1),\ldots,a_n(x_n)\}\in\mathbb{R}^{n\times n}$, $B(x)=(b_1(x_1),\ldots,b_n(x_n))^T\in\mathbb{R}^n$, $T=[t_{ij}]_{n\times n}$, and $S(x)=(s_1(x_1),\ldots,s_n(x_n))^T$, then Eq. (1.1) can be rewritten as

$$\dot{x}=-A(x)[B(x)-TS(x)]\ . \tag{3.1}$$

When delays are present in the Cohen-Grossberg neural networks, we need to modify (3.1) as

$$\dot{x}(t) = -A(x(t))\left[B(x(t)) - T_0 S(x(t)) - \sum_{k=1}^{K} T_k S(x(t-\tau_k))\right], \tag{3.2}$$

where the T_k make up the interconnections associated with delay τ_k, $k=0,1,\ldots,K$, so that $T=T_0+T_1+\cdots+T_K$, $0=\tau_0<\tau_1<\cdots<\tau_K$, and $A(x)$, $B(x)$, T, and $S(x)$ are the same as in Eq. (3.1). Clearly, Eq. (3.2) is equivalent to Eq. (1.4).

In order to ensure that the Cohen-Grossberg neural networks (3.1) [or equivalently (1.1)] possess global stability, we will require that the set of all equilibria of (3.1) be a discrete set. This requirement can be ensured by almost all choices (in the sense of Lebesgue measure) of the interconnection matrix T (see [1]). For this reason, we assume throughout this paper that system (3.1) [or equivalently (1.1)] satisfies the following assumption.

Assumption B. For any equilibrium x_e of (3.1), the matrix $J(x_e)$ is nonsingular, where

$$J(x) = -T + \operatorname{diag}\left\{\frac{b_1'(x_1)}{s_1'(x_1)}, \ldots, \frac{b_n'(x_n)}{s_1'(x_1)}\right\}, \tag{3.3}$$

and $b_i'(x_i) \triangleq db_i(x_i)/dx_i$ for $i=1,\ldots,n$. ■

It can be proved using Sard's theorem (see Lemma 3.3 of [8]) that the following result is true.

Lemma 1. For almost all $T\in\mathbb{R}^{n\times n}$ (except a set with Lebesgue measure 0), system (3.1) satisfies Assumption B. ■

Furthermore, by using the inverse function theorem (see Remark 3.4 of [8]), we can establish the following result.

Lemma 2. When system (3.1) satisfies Assumption B, the set of equilibria of system (3.1) is a discrete set. ■

IV. GLOBAL STABILITY ANALYSIS OF COHEN-GROSSBERG NEURAL NETWORKS WITH MULTIPLE DELAYS

In the present action, we address the global stability properties of Cohen-Grossberg neural networks with multiple delays described by the retarded type differential-difference equation (3.2), or equivalently by Eq. (1.4). To establish the main results of the present section, we require the following properties of system (3.2).

Lemma 3. If system (3.2) satisfies Assumption B, then the set of equilibria of system (3.2) is a discrete set. ■

Remark 2. The set of equilibria of system (3.2), which is identical to the set of equilibria of system (3.1), is a discrete set, by Lemma 2. Furthermore, it follows from Lemma 1 that for almost all $T\in\mathbb{R}^{n\times n}$ (except on a set of Lebesgue measure 0), system (3.2) satisfies Assumption B. ■

We are now in a position to establish the main result of this section.

Theorem 1. Suppose that for system (3.2), Assumptions A and B are satisfied, and suppose that

$$\sum_{k=1}^{K} (\tau_k \beta \|T_k\|) < 1, \tag{4.1}$$

where $\beta = \max_{x\in\mathbb{R}^n}\|A(x)S'(x)\|$ and $S'(x) \triangleq \operatorname{diag}\{s_1'(x_1),\ldots,s_n'(x_n)\}$. Then system (3.2) is *globally stable.*

Proof. Since inequality (4.1) is satisfied, there must exist a sequence of positive numbers $(\alpha_1,\ldots,\alpha_K)$, such that

$$\sum_{k=1}^{K} \alpha_k = 1, \quad \tau_k\beta\|T_k\| < \alpha_k \quad \text{for } k=1,\ldots,K. \tag{4.2}$$

To prove the present result, we define for any $x_t\in C([-\tau_K,0],\mathbb{R}^n)$ an energy functional $E(x_t)$ associated with (3.2) by

$$E(x_t) = -S^T(x_t(0))TS(x_t(0)) + 2\sum_{i=1}^{n}\int_0^{[x_t(0)]_i} b_i(\sigma)s_i'(\sigma)\,d\sigma + \sum_{k=1}^{K}\frac{1}{\alpha_k}\int_{-\tau_k}^{0}[S(x_t(\theta))-S(x_t(0))]^T T_k^T f_k(\theta) T_k [S(x_t(\theta))-S(x_t(0))]\,d\theta, \tag{4.3}$$

where $(\alpha_1,\ldots,\alpha_K)$ is a sequence of positive numbers such that condition (4.2) is satisfied and $f_k(\theta)\in C^{-1}([-\tau_k,0],\mathbb{R}^n)$, $k=1,\ldots,K$, will be specified later. After changing integration variables, (4.3) can be written as

$$E(x_t) = -S^T(x(t))TS(x(t)) + 2\sum_{i=1}^{n}\int_0^{x_i(t)} b_i(\sigma)s_i'(\sigma)\,d\sigma + \sum_{k=1}^{K}\frac{1}{\alpha_k}\int_{t-\tau_k}^{t}[S(x(w))-S(x(t))]^T T_k^T f_k(w-t) T_k [S(x(w))-S(x(t))]\,dw. \tag{4.4}$$

The derivative of $E(x_t)$ with respect to t along any solution of (3.2) can be computed as

$$\frac{dE(x_t)}{dt} = -2S^T(x(t))TS'(x(t))A(x(t))\left[-B(x(t))+T_0S(x(t))+\sum_{k=1}^{K}T_kS(x(t-\tau_k))\right]$$
$$+2x^T(t)B(x(t))S'(x(t))A(x(t))\left[-B(x(t))+T_0S(x(t))+\sum_{k=1}^{K}T_kS(x(t-\tau_k))\right]$$
$$-\sum_{k=1}^{K}\frac{1}{\alpha_k}\left\{[S(x(t-\tau_k))-S(x(t))]^TT_k^Tf_k(-\tau_k)T_k[S(x(t-\tau_k))-S(x(t))]\right.$$
$$+\int_{t-\tau_k}^{t}[S(x(w))-S(x(t))]^TT_k^Tf_k'(w-t)T_k[S(x(w))-S(x(t))]dw$$
$$+\int_{t-\tau_k}^{t}\left[-B(x(t))+T_0S(x(t))+\sum_{k=1}^{K}T_kS(x(t-\tau_k))\right]^T$$
$$\times A(x(t))S'(x(t))T_k^Tf_k(w-t)T_k[S(x(w))-S(x(t))]dw$$
$$+\int_{t-\tau_k}^{t}[S(x(w))-S(x(t))]^TT_k^Tf_k(w-t)T_kS'(x(t))A(x(t))$$
$$\left.\times\left[-B(x(t))+T_0S(x(t))+\sum_{k=1}^{K}T_kS(x(t-\tau_k))\right]dw\right\}, \quad (4.5)$$

where $f'(\theta)=df(\theta)/d\theta$. If we adopt the notation

$$H_0=-B(x(t))+T_0S(x(t))+\sum_{k=1}^{K}T_kS(x(t-\tau_k)), \quad (4.6)$$

$$H_k=T_k[S(x(t-\tau_k))-S(x(t))], \quad k=1,\ldots,K \quad (4.7)$$

$$G_k=T_k[S(x(w))-S(x(t))], \quad k=1,\ldots,K \quad (4.8)$$

$$Q=A(x(t))S'(x(t))=S'(x(t))A(x(t)), \quad (4.9)$$

Eq. (4.5) can be rewritten as

$$\frac{dE(x_t)}{dt}=-2S^T(x(t))TQH_0+2x^T(t)B(x(t))QH_0$$
$$-\sum_{k=1}^{K}\frac{1}{\alpha_k}\left\{H_k^Tf_k(-\tau_k)H_k+\int_{t-\tau_k}^{t}[G_k^Tf_k'(w-t)G_k+H_0^TQT_k^Tf_k(w-t)G_k+G_k^Tf_k(w-t)T_kQH_0]dw\right\}$$
$$=-2H_0^TQH_0+2\sum_{k=1}^{K}H_k^TQH_0$$
$$-\sum_{k=1}^{K}\frac{1}{\alpha_k}\left\{H_k^Tf_k(-\tau_k)H_k+\int_{t-\tau_k}^{t}[G_k^Tf_k'(w-t)G_k+H_0^TQT_k^Tf_k(w-t)G_k+G_k^Tf_k(w-t)T_kQH_0]dw\right\}$$
$$=\sum_{k=1}^{K}\left[2H_k^TQH_0-\frac{1}{\alpha_k}\left\{2H_0^TQH_0+H_k^Tf_k(-\tau_k)H_k\right.\right.$$
$$\left.\left.+\int_{t-\tau_k}^{t}[G_k^Tf_k'(w-t)G_k+H_0^TQT_k^Tf_k(w-t)G_k+G_k^Tf_k(w-t)T_kQH_0]dw\right\}\right]$$
$$=-\sum_{k=1}^{K}\int_{-\tau_k}^{0}[\eta_k(x_t,\theta)]^TM_k(x_t,\theta)\eta_k(x_t,\theta)d\theta, \quad (4.10)$$

where $[\eta_k(x_t,\theta)]^T=[H_0^T,H_k^T,\tilde{G}_k^T]^T$ with H_0 and H_k given by (4.6) and (4.7),

$$\tilde{G}_k=T_k[S(x(t+\theta))-S(x(t))],\quad k=1,\ldots,K \tag{4.11}$$

$$M_k(x_t,\theta)=\begin{bmatrix} \dfrac{2\alpha_k Q}{\tau_k} & -\dfrac{Q}{\tau_k} & \dfrac{QT_k^T f_k(\theta)}{\alpha_k} \\ -\dfrac{Q}{\tau_k} & \dfrac{f_k(-\tau_k)}{\tau_k\alpha_k}I & 0 \\ \dfrac{f_k(\theta)T_k Q}{\alpha_k} & 0 & \dfrac{f_k'(\theta)I}{\alpha_k} \end{bmatrix}, \tag{4.12}$$

and I denotes the $n\times n$ identity matrix. To obtain the last expression of (4.10), we changed the integration variables from w to θ.

We will now show that if the hypotheses of Theorem 1 are satisfied, then $M_k(x_t,\theta)$ is positive definite for all $\theta\in[-\tau_k,0]$ and all x_t which satisfy Eq. (3.2), for $k=1,\ldots,K$. In doing so, we let $U=U_3U_2U_1$, where

$$U_1=\begin{bmatrix} \alpha_k^{-1/2}I & 0 & 0 \\ (2\alpha_k^{1/2})^{-1}I & \alpha_k^{1/2}I & 0 \\ 0 & 0 & \alpha_k^{1/2}I \end{bmatrix},$$

$$U_2=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\dfrac{\tau_k}{2}f_k(\theta)\dfrac{T_k}{\alpha_k} & 0 & I \end{bmatrix},$$

and

$$U_3=\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -\dfrac{1}{2}f_k(\theta)\dfrac{T_k}{\alpha_k}Q\left[\dfrac{f_k(-\tau_k)}{\tau_k}I-\dfrac{Q}{2\tau_k}\right]^{-1} & I \end{bmatrix}.$$

It is not difficult to verify that $\tilde{M}_k=UM_k(x_t,\theta)U^T$ is a diagonal matrix. In fact,

$$\tilde{M}_k=\mathrm{diag}\{M_{k,1},M_{k,2},M_{k,3}\}, \tag{4.13}$$

where

$$M_{k,1}=\frac{2Q}{\tau_k}, \tag{4.14}$$

$$M_{k,2}=\frac{f_k(-\tau_k)}{\tau_k}I-\frac{Q}{2\tau_k}, \tag{4.15}$$

and

$$M_{k,3}=f_k'(\theta)I-\frac{f_k(\theta)T_kQ}{2\alpha_k}\left[\left[\frac{f_k(-\tau_k)}{\tau_k}I-\frac{Q}{2\tau_k}\right]^{-1}+2\tau_kQ^{-1}\right]\frac{QT_k^Tf_k(\theta)}{2\alpha_k}. \tag{4.16}$$

It follows that $M_k(x_t,\theta)$ is positive definite if and only if $\tilde{M}_k$ is positive definite and if and only if $M_{k,1}$, $M_{k,2}$, and $M_{k,3}$ are all positive definite.

We now show that if the condition $\tau_k\beta\|T_k\|<\alpha_k$ is satisfied, where

$$\beta=\max_{x\in\mathbb{R}}\|A(x)S'(x)\|=\max_{x\in\mathbb{R}}\|Q\|,$$

then we can always find a suitable $f_k(\theta)\in C([-\tau_k,0],\mathbb{R}^+)$ such that $M_{k,1}$, $M_{k,2}$, and $M_{k,3}$ are positive definite for all x_t which satisfy Eq. (3.2) and for all $\theta\in[-\tau_k,0]$. From this it follows that $M_k(x_t,\theta)$ is positive definite for all $k=1,\ldots,K$ and therefore $dE(x_t)/dt\le 0$ along any solution x_t of (3.2).

By the assumptions that $s_i'(x_i)>0$ and $a_i(x_i)>0$ for all $x_i\in\mathbb{R}$, the matrix $M_{k,1}$ is automatically positive definite. The matrix $M_{k,2}$ will always be positive definite if condition

$$2f_k(-\tau_k)-\beta>0 \tag{4.17}$$

is satisfied. For $M_{k,3}$, it is easily shown that if

$$f_k'(\theta)>\frac{1}{4}f_k^2(\theta)\frac{\|T_k\|^2}{\alpha_k^2}\times\left\|\left\{Q\left[\left[\frac{f_k(-\tau_k)}{\tau_k}I-\frac{Q}{2\tau_k}\right]^{-1}+2\tau_kQ^{-1}\right]Q\right\}\right\| \tag{4.18}$$

is true, then $M_{k,3}$ is also positive definite. Notice that the matrix

$$D\triangleq Q\left[\left[\frac{f_k(-\tau_k)}{\tau_k}I-\frac{Q}{2\tau_k}\right]^{-1}+2\tau_kQ^{-1}\right]Q$$

is a diagonal matrix, i.e., $D=\mathrm{diag}\{d_1,\ldots,d_n\}$. If we denote $Q=\mathrm{diag}\{q_1,\ldots,q_n\}$, then it is easy to show that

$$d_i=\frac{4f_k(-\tau_k)q_i\tau_k}{2f_k(-\tau_k)-q_i}\quad\text{for } i=1,\ldots,n.$$

Since $q_i<\beta$ by the definitions of β and Q, we have, in view of (4.17), that

$$d_i\le\frac{4f_k(-\tau_k)\beta\tau_k}{2f_k(-\tau_k)-\beta}.$$

Therefore, we obtain

$$\|D\|\le\frac{4f_k(-\tau_k)\beta\tau_k}{2f_k(-\tau_k)-\beta}$$

and, furthermore, condition (4.18) will be satisfied if (4.17) is satisfied and

$$f_k'(\theta)>\frac{1}{4}f_k^2(\theta)\frac{\|T_k\|^2}{\alpha_k^2}\frac{4f_k(-\tau_k)\beta\tau_k}{2f_k(-\tau_k)-\beta} \tag{4.19}$$

is satisfied.

Next, we need to show that there is an $f_k\in C^1([-\tau_k,0],\mathbb{R})$ such that conditions (4.17) and

(4.19) are satisfied. We choose

$$f_k(-\tau_k)=\left[\beta\tau_k^2\frac{\|T_k\|^2}{\alpha_k^2}\right]^{-1} . \tag{4.20}$$

Condition (4.17) is satisfied by the choice (4.20). Furthermore,

$$\left[f_k(-\tau_k)\frac{\|T_k\|}{\alpha_k}-\frac{\alpha_k}{\beta\tau_k\|\tau_k\|}\right]^2+1-\frac{\alpha_k^2}{\beta^2\tau_k^2\|T_k\|^2}=1-\frac{\alpha_k^2}{\beta^2\tau_k^2\|T_k\|^2}<0 \tag{4.21}$$

is true because $\beta\tau_k\|T_k\|<\alpha_k$. It follows from (4.21) that

$$\delta f_k(-\tau_k)\tau_k<1 , \tag{4.22}$$

where

$$\delta=\frac{\|T_k\|^2 f_k(-\tau_k)\beta\tau_k}{\alpha_k^2[2f_k(-\tau_k)-\beta]} . \tag{4.23}$$

Since $\delta f_k(-\tau_k)\tau_k<1$, we can always find an l such that $0<l<1$ and $\delta f_k(-\tau_k)\tau_k<l$. Therefore, we will always have $\gamma>0$ where γ is given by

$$\gamma=\frac{l}{\delta f_k(-\tau_k)}-\tau_k . \tag{4.24}$$

We now choose $f_k(\theta)$ on $[-\tau_k,0]$ as

$$f_k(\theta)=\frac{l}{\delta(\gamma-\theta)} . \tag{4.25}$$

It is easily verified that this choice is consistent with (4.20). Clearly, $f_k\in C([-\tau_k,0],\mathbb{R}^+)$ since $\gamma>0$. The derivative of $f_k(\theta)$ is given by

$$f_k'(\theta)=\frac{l}{\delta(\gamma-\theta)^2}=\frac{\delta}{l}f_k^2(\theta)>\delta f_k^2(\theta) \tag{4.26}$$

since $l<1$. Combining (4.23) and (4.26), we can verify that $f_k(\theta)$ satisfies condition (4.19).

Therefore, we have shown that if $\beta\tau_k\|T_k\|<\alpha_k$, then there exists an $f_k(\theta)$ [given by (4.25), where γ, δ, and $f_k(-\tau_k)$ are given by (4.24), (4.23), and (4.20), respectively] such that conditions (4.17) and (4.19) are satisfied. Thus $M_k(x_t,\theta)$ is positive definite for all x_t satisfying Eq. (3.2) and all $\theta\in[-\tau_k,0]$ for $k=1,\ldots,K$. We have shown

$$\frac{dE(x_t)}{dt}\le 0 \tag{4.27}$$

along any solution x_t of Eq. (3.2), where $E(x_t)$ is the energy functional given by (4.3).

We know from (4.10) that if $dE(x_t)/dt=0$, then $H_0=0$, $H_k=0$, and $\tilde{G}_k=0$ for $k=1,\ldots,K$, where H_0, H_k, and $\tilde{G}_k$ are given by (4.6), (4.7), and (4.11), respectively. For any $\phi\in C([-\tau_K,0],\mathbb{R}^n)$, we denote

$$\dot{E}_\phi=0$$

if

$$-B(\phi(0))+T_0S(\phi(0))+\sum_{k=1}^{K}T_kS(\phi(-\tau_k))=0 , \tag{4.28}$$

$$T_k[S(\phi(-\tau_k))-S(\phi(0))]=0,\quad k=1,\ldots,K \tag{4.29}$$

$$T_k[S(\phi(-\theta))-S(\phi(0))]=0 \quad \text{for all } \theta\in[-\tau_K,0],\ k=1,\ldots,K . \tag{4.30}$$

It is obvious that for any solution x_t of (3.2), $dE(x_t)/dt=0$ if and only if $\dot{E}_{x_t}=0$.

Since for any x_t satisfying Eq. (3.2), x_t is bounded (Fact 1) and since $dE(x_t)/dt\le 0$, it follows from the invariance theory (see Chap. 4, Lemmas 1.4 and 2.1 of [21]) that the limit set of x_t as $t\to\infty$ is the invariant subset of the set $\Lambda=\{\phi\in C([-\tau_K,0],\mathbb{R}^n)\text{: } \dot{E}_\phi=0\}$. Therefore, we have $|x_t-\phi|\to 0$ as $t\to\infty$ for some $\phi\in\Lambda$. In particular, we have $x_t(0)\to\phi(0)$ and $x_t(-\tau_k)\to\phi(-\tau_k)$ as $t\to\infty$, $k=1,\ldots,K$. Combining this with (4.28) and (4.29), we conclude that

$$-B(x_t(0))+T_0S(x_t(0))+\sum_{k=1}^{K}T_kS(x_t(-\tau_k))\to 0$$

and

$$T_k[S(x_t(\phi(-\tau_k)))-S(x_t(0))]\to 0,\quad k=1,\ldots,K$$

as $t\to\infty$. It follows that $-B(x_t(0))+TS(x_t(0))\to 0$ or $-B(x(t))+TS(x(t))\to 0$, as t approaches ∞. Now since x_t is bounded (Fact 1), we conclude that any point in the limit set of $x(t)$ as $t\to\infty$ is an equilibrium of system (3.2) [or, equivalently, an equilibrium of system (3.1)]. Furthermore, since the set of equilibria of system (3.2) is a discrete set (Lemma 3), it follows that $x(t)$ approaches some equilibrium of system (3.2) as t tends to ∞. ■

Remark 3. Consider a special case of (3.2), a Hopfield neural network with the same delay τ, described by the equation of the form

$$\dot{x}(t)=-Cx+T_0S(x(t))+T_1S(x(t-\tau)) , \tag{4.31}$$

where $C=\text{diag}\{c_1,\ldots,c_n\}$ with $c_i>0$ for $i=1,\ldots,n$, and T_0, T_1, and $S(x)$ are the same as in (3.2). System (4.31) has been addressed by several workers. Applying Theorem 1, we obtain that system (4.31) is globally stable if

$$\tau\beta\|T_1\|<1 , \tag{4.32}$$

where β is defined in the same way as in Theorem 1, noticing, however, that $A(x)$ equals the identity matrix in this case. The bound condition (4.32) for the global stability of system (4.31) is identical to the result reported in [20]. ■

V. LOCAL STABILITY ANALYSIS OF COHEN-GROSSBERG NEURAL NETWORKS WITH MULTIPLE DELAYS

In the preceding section we showed that if $\sum_{k=1}^{K}\tau_k\beta\|T_k\|<1$, the Cohen-Grossberg neural networks with multiple delays described by Eq. (3.2) [or, equivalently, by Eq. (1.4)] possess global stability. Since in the implementation of neural networks as associative memories information is stored in specific asymptotically stable equilibria (called stable memories), good criteria which ensure the asymptotic stability of an equilibrium of (3.2) are of great interest. We address this issue in the present section.

At the present time, there are no known general results which provide necessary and sufficient conditions for the asymptotic stability of an equilibrium for Cohen-Grossberg neural networks with multiple delays (3.2). Even for special cases of (3.2), a class of Hopfield neural networks with one delay, given by (4.31), there are only results which provide sufficient conditions for the asymptotic stability of an equilibrium. Some of these results are obtained by linearizing (4.31) about an equilibrium of interest (see, e.g., [14]). Other results, which make use of sector conditions for nonlinearities, have been obtained by Lyapunov's second method (see, e.g., [19]). In the present section we will show that if the conditions of Theorem 1 are satisfied, then the asymptotic stability of an equilibrium of (3.2) can be deduced from the asymptotic stability of the same corresponding equilibrium of system (3.1). In other words, if $\sum_{k=1}^{K}\tau_k\beta\|T_k\|<1$, then (as shown in the preceding section), Cohen-Grossberg neural networks (3.1) and Cohen-Grossberg neural networks with multiple delays (3.2) are both globally stable and, furthermore (as will be shown in the present section), both have similar local stability properties at an asymptotically stable equilibrium. This enables us to verify the asymptotic stability of the equilibria of system (3.2) by ascertaining the asymptotic stability of corresponding equilibria of system (3.1).

In order to proceed further, we require the following.

Definition. An element $\phi\in C([-\tau,0],\mathbb{R}^n)$ is called a *local minimum* of the energy functional E defined by (4.3) if there exists a $\delta>0$ such that for any $\tilde{\phi}\in C([-\tau,0],\mathbb{R}^n)$, $E(\phi)\le E(\tilde{\phi})$ whenever $|\phi-\tilde{\phi}|<\delta$. ■

We are now able to establish the following results.

Theorem 2. Suppose that the conditions of Theorem 1 are satisfied. If x_e is an equilibrium of (3.2), then the following statements are equivalent: (1) x_e is a stable equilibrium of (3.2); (2) x_e is an asymptotically stable equilibrium of (3.2); (3) ϕ_{x_e} is a local minimum of the energy functional E given by (4.3), where $\phi_{x_e}\in C([-\tau,0],\mathbb{R}^n)$ such that $\phi_{x_e}\equiv x_e$; and (4) $J(x_e)$ is positive definite, where $J(x)$ is given in Eq. (3.3) in Assumption B.

Proof.

(a) (1) ⟹ (2). Since Assumption B is satisfied, the set of equilibria of system (3.2) is a discrete set by Lemma 3. Therefore, when $\epsilon>0$ is sufficiently small, there is no other equilibrium in $U(x_e,\epsilon)$, a neighborhood of x_e, given by

$$U(x_e,\epsilon)\triangleq\{x\in\mathbb{R}^n\colon \|x-x_e\|<\epsilon\}\ . \tag{5.1}$$

Since x_e is a stable equilibrium of (3.2), there exists an $\eta>0$ such that for any $\phi\in C([-\tau,0],\mathbb{R}^n)$ satisfying $|\phi-x_e|<\eta$, $|x_t-x_e|<\epsilon$ for all $t>0$, where x_t is the solution of (3.2) with initial condition ϕ. Thus $x_t\in C([-\tau,0],U(x_e,\epsilon))$ for all t. In view of Theorem 1, x_t will converge to some equilibrium of system (3.2). Since x_e is the only equilibrium of (3.2) in $U(x_e,\epsilon)$, it follows that x_t converges to x_e. Thus we have shown that x_e is an attractive equilibrium of system (3.2). Therefore, the stable equilibrium x_e of (3.2) is an asymptotically stable equilibrium of system (3.2).

(b) (2) ⟹ (3). Since x_e is an asymptotically stable equilibrium of system (3.2), there exists an $\eta>0$ such that for any $\phi\in C([-\tau,0],\mathbb{R}^n)$ satisfying $|\phi-x_e|<\eta$, x_t converges to x_e, where x_t is the solution of (3.2) with initial condition ϕ. Therefore $E(\phi_{x_e})\le E(x_t)\le E(\phi)$ for any $\phi\in C([-\tau,0],\mathbb{R}^n)$ satisfying $|\phi-x_e|<\eta$. Therefore, ϕ_{x_e} is a local minimum of the energy functional E.

(c) (3) ⟹ (4). Let $\tilde{E}$ be a function from $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\tilde{E}(x)\triangleq -S^T(x)TS(x)+2\sum_{i=1}^{n}\int_0^{x_i}b_i(\sigma)s_i'(\sigma)d\sigma\ . \tag{5.2}$$

Comparing E with $\tilde{E}$, we note that $\tilde{E}$ is a function defined on $\mathbb{R}^n$, while E is a functional defined on $C([-\tau,0],\mathbb{R}^n)$. Since ϕ_{x_e} is a local minimum of E, x_e must be a local minimum of $\tilde{E}$. Otherwise there would exist a sequence $\{x_n\}\subset\mathbb{R}^n$ such that $x_n\to x_e$ as $n\to\infty$ and $\tilde{E}(x_n)<\tilde{E}(x_e)$. Let ϕ_{x_n} denote the constant function $\phi_{x_n}\equiv x_n$ in $C([-\tau,0],\mathbb{R}^n)$. Then $|\phi_{x_n}-\phi_{x_e}|\to 0$ as $n\to\infty$ and

$$E(\phi_{x_n})=\tilde{E}(x_n)<\tilde{E}(x_e)=E(\phi_{x_e})\ .$$

This contradicts the fact the ϕ_{x_e} is a local minimum of E. Therefore, x_e is a local minimum of $\tilde{E}$. Hence $\tilde{J}(x_e)$ is positive semidefinite (see, e.g., Theorem 3.6 of [22]), where $\tilde{J}(x)$ is the Hessian matrix of $\tilde{E}$ given by

$$\tilde{J}(x)=\left[\frac{\partial^2\tilde{E}}{\partial x_i\partial x_j}\right]_{n\times n}\ . \tag{5.3}$$

It can be shown that

$$\tilde{J}(x)=2S'(x)J(x)S'(x)\ ,$$

where $S'(x)=\mathrm{diag}\{s_1'(x_1),\ldots,s_n'(x_n)\}$ and $J(x)$ is given by Eq. (3.3) in Assumption B. Therefore, $J(x_e)$ is also positive semidefinite. By Assumption B, $J(x_e)$ is a nonsingular matrix. Thus we have shown that $J(x_e)$ is positive definite.

(d) (4) ⟹ (1). We need to prove that x_e is a stable equilibrium of system (3.2), i.e., for any $\epsilon>0$, there exists a $\delta>0$ such that for any $\phi\in C([-\tau,0],\mathbb{R}^n)$, if $|\phi-x_e|<\delta$, then $|x_t-x_e|<\epsilon$, where x_t is the solution of (3.2) with initial condition ϕ.

Since $J(x_e)$ is positive definite, then $\tilde{J}(x_e)$ must also be positive definite where $\tilde{J}(x)$ is the Hessian matrix of $\tilde{E}$ given by (5.4). Furthermore,

$$\nabla_x\tilde{E}(x)=2[-TS(x)+B(x)-b]^TS'(x)\ ,$$

where $S'(x)$ is given in part (b). Therefore, $\nabla_x\tilde{E}(x_e)=0$ since x_e is an equilibrium of (3.2). It follows (by Theorem

3.6 of [22]) that x_e is a local minimum of $\tilde{E}$, i.e., there exists a $\delta_1 > 0$, $\delta_1 < \epsilon$ such that whenever $0 < \|x - x_e\| \le \delta_1$, $\tilde{E}(x_e) < \tilde{E}(x)$. Let $r = \min\{\tilde{E}(x): \|x - x_e\| = \delta_1\}$. Then it is true that $r > \tilde{E}(x_e)$. Since $E(\phi_{x_e}) = \tilde{E}(x_e)$, it follows that $r > E(\phi_{x_e})$. Note that E is a continuous functional. Therefore, there exists a $\delta \in (0, \delta_1)$ such that whenever $|\phi - x_e| < \delta$, where $\phi \in C([-\tau, 0], \mathbb{R}^n)$, we have $E(\phi) < r$. Suppose x_t is any solution of (3.2) with the initial condition ϕ such that $|\phi - x_e| < \delta$. We will show that $|x_t - x_e| < \delta_1 < \epsilon$. Otherwise there would exist a $t_0 > 0$ such that $\|x_{t_0}(0) - x_e\| = \delta_1$, i.e., $\|x(t_0) - x_e\| = \delta_1$. By the definition of E and $\tilde{E}$, we have $E(x_{t_0}) \ge \tilde{E}(x(t_0)) \ge r$. Therefore, we obtain $E(x_{t_0}) > E(\phi)$, which contradicts the fact that E is monotonically decreasing along any solution of (3.2). Thus we have shown that x_e is an asymptotically stable equilibrium of system (3.2). ■

Remark 4. We note that statement (4) in Theorem 2 is independent of the delays τ_k, $k = 1, \ldots, K$. Therefore, if system (3.2) satisfies Assumptions A and B and if the condition $\sum_{k=1}^{K} \tau_k \beta \|T_k\| < 1$ is satisfied, then the locations of the (asymptotically) stable equilibria of system (3.2) will not depend on the delays τ_k for $k = 1, \ldots, K$. This is true if, in particular, $\tau_k = 0$, $k = 1, \ldots, K$. Therefore, if $\sum_{k=1}^{K} \tau_k \beta \|T_k\| < 1$, then systems (3.2) and (3.1) [obtained by letting $\tau_k = 0$ for $k = 1, \ldots, K$ in (3.2)] will have identical (asymptotically) stable equilibria. We state this in the form of a corollary. ■

Corollary 1. Under the conditions of Theorem 1, x_e is an (asymptotically) stable equilibrium of system (3.2) if and only if x_e is an (asymptotically) stable equilibrium of system (3.1). This is true if and only if $J(x_e)$ is positive definite, where $J(x)$ is given in Eq. (3.3) (in Assumption B). ■

Remark 5. Corollary 1 provides an effective criterion for testing the (asymptotic) stability of any equilibrium of Cohen-Grossberg neural networks with multiple delays described by (3.2). This criterion constitutes necessary and sufficient conditions, as long as $\sum_{k=1}^{K} \tau_k \beta \|T_k\| < 1$. ■

VI. CONCLUDING REMARKS

In this paper we considered the local stability as well as the global stability of Cohen-Grossberg neural networks with multiple delays given by system (3.2). We showed that if the condition $\sum_{k=1}^{K} \tau_k \beta \|T_k\| < 1$ is satisfied, then Cohen-Grossberg neural networks with multiple delays and corresponding Cohen-Grossberg neural networks without delays given by system (3.1) have identical asymptotically stable equilibria and both networks are globally stable [i.e., any trajectory of system (3.2) approaches some equilibrium of (3.2)]. In addition, we proved that if the above bound is satisfied, then any equilibrium x_e of system (3.2) is asymptotically stable if and only if $J(x_e)$ is positive definite.

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[1] M. Cohen and S. Grossberg, IEEE Trans. Syst. Man. Cybern. **SMC-13**, 815 (1983).

[2] G. Carpenter, M. Cohen, and S. Grossberg, Science **37**, 54 (1987).

[3] B. Kosko, *Neural Networks and Fuzzy Systems* (Prentice-Hall, Englewood Cliffs, NJ, 1992).

[4] J. J. Hopfield, Proc. Natl. Acad. Sci. U.S.A. **79**, 2554 (1982).

[5] J. J. Hopfield, Proc. Natl. Acad. Sci. U.S.A. **81**, 3088 (1984).

[6] L. Personnaz, I. Guyon, and G. Dreyfus, Phys. Rev. A **34**, 4217 (1986).

[7] L. Personnaz, I. Guyon, and G. Dreyfus, J. Phys. Lett. **46**, L359 (1985).

[8] J. H. Li, A. N. Michel, and W. Porod, IEEE Trans. Circuits Syst. **CAS-35**, 976 (1988).

[9] A. N. Michel, J. A. Farrell, and W. Porod, IEEE Trans. Circuits Syst. **CAS-36**, 229 (1989).

[10] L. O. Chua and L. Yang, IEEE Trans. Circuits Syst. **35**, 1273 (1988).

[11] J. J. Hopfield and D. Tank, Science **2**, 625 (1986).

[12] D. Tank and J. J. Hopfield, Sci. Am. **257** (6), 104 (1984).

[13] *Neural Networks for Computing,* edited by John S. Denker, AIP Conf. Proc. No. 151 (AIP, New York, 1986).

[14] C. M. Marcus and R. M. Westervelt, Phys. Rev. A **39**, 347 (1989).

[15] B. D. Coleman and G. H. Renninger, SIAM J. Appl. Math. **31**, 111 (1976).

[16] U. van der Heiden, *Analysis of Neural Networks* (Springer, New York, 1980).

[17] T. A. Burton, Neural Networks **6**, 677 (1993).

[18] P. P. Civalleri, M. Gilli, and L. Pandolfi, IEEE Trans. Circuits Syst. **40**, 157 (1993).

[19] H. Ye, A. N. Michel, and K. Wang (unpublished).

[20] H. Ye, A. N. Michel, and K. Wang, Phys. Rev. E **50**, 4206 (1994).

[21] J. Hale, *Theory of Functional Differential Equations* (Springer-Verlag, Berlin, 1977).

[22] C. Goffman, *Calculus of Several Variables* (Harper & Row, New York, 1965).