Complex Trkalian fields and solutions to Euler's equations for the ideal fluid

P. R. Baldwin and G. M. Townsend
Department of Physics, The University of Akron, Akron, Ohio 44325-4001
(Received 22 December 1993; revised manuscript received 9 September 1994)

We consider solutions to the complex Trkalian equation $\nabla\times\mathbf{c}=\mathbf{c}$, where $\mathbf{c}$ is a three-component vector function with each component in the complex field and may be expressed in the form $\mathbf{c}=e^{ig}\nabla F$, with g real and F complex. We find that there are precisely two classes of solutions: one where g is a Cartesian variable and one where g is the spherical radial coordinate. We consider these flows to be the simplest of all exact three-dimensional solutions to Euler's equation for the ideal incompressible fluid. Pictures are presented. The approach we use in solving for these classes of solutions to these three-dimensional vector partial differential equations involves differential geometric techniques: one may employ the method to generate solutions to other classes of vector partial differential equations.

PACS number(s): 47.20.−k, 47.90.+a, 41.20.−q

I. INTRODUCTION

The Beltrami equation

$$\mathbf{v}\times(\nabla\times\mathbf{v})=\mathbf{0} \tag{1.1}$$

has received enormous attention in recent years. Reviews [1–3] note the prominent role that Beltrami fields play in the theory of exact, closed form solutions to the Euler and Navier Stokes equations and their relations to the electromagnetic wave equations. Moreover, Beltrami fields are related to minimum energy plasma fields and have therefore garnered much attention from the magnetohydrodynamics community [4–7]. In this paper, we would like to undertake the first steps in a systematic classification of these vector fields, using a differential geometric technique which we discuss below.

Any three-component vector function $\mathbf{v}$ in $\mathcal{R}^3$ may be written as the real part of a three-component complex vector field of the following form:

$$\mathbf{c}=e^{ig}\nabla F , \tag{1.2}$$

where g is a real function and F is a function from $\mathcal{R}^3$ to $\mathcal{C}^3$. Trkalian fields [8] are real vector fields which solve

$$\nabla\times\mathbf{v}=\mathbf{v} \tag{1.3}$$

and therefore satisfy (1.1). We solve the complex version of the Trkalian equation

$$\nabla\times\mathbf{c}=\mathbf{c} , \tag{1.4}$$

where $\mathbf{c}$ is also a vector of the form of Eq. (1.2). Notice that both the real and imaginary parts of $\mathbf{c}$ must solve the Trkalian equation, which is a linear equation. From solutions to the Trkalian equation, one may straightforwardly derive (nonzero helicity) three-dimensional (in a topological sense; see [9]) time-independent solutions to Euler's equations for the incompressible ideal fluid [10] and time-dependent solutions to the incompressible Navier-Stokes equations [11]. Therefore we consider Eqs. (1.2) and (1.4) to be the simplest possible three-dimensional hydrodynamical equation. Although we do not discuss the stability of these solutions, a discussion of boundary conditions is given in Sec. IV.

We completely categorize all solutions to the simultaneous set of equations given by (1.2) and (1.4). The two classes of solutions to this problem are (a) where the function g in (1.4) is identical to a Cartesian variable (for example, z) and the function F is a complex analytical function of the other two Cartesian variables [see (3.12) and (b) where g is the spherical radial coordinate r and the function F is an analytical function of a certain combination of the spherical angular variables [see (3.19)]. In both cases F and g each solve Laplace's equation. These solutions are closely related to the possible forms of linearly polarized TEM waves [12].

Our method of solution is very particular and we will employ it in further work. Finding the solution to Eqs. (1.2) and (1.4) is a first step in an attempt to solve three-dimensional hydrodynamical problems using potential methods. The procedure is as follows. The transformation from the three Cartesian variables to the functions $g,F,\bar{F}$ leads naturally to a metric function q_{ij},

$$ds^2=dx_1^2+dx_2^2+dx_3^2=q_{ij}dy^i dy^j , \tag{1.5}$$

where we have used $y_1=g$, $y_2=F$, and $y_3=\bar{F}$. First, the partial differential equation (PDE) (1.4) may be interpreted as a constraint on the metric tensor: the appropriate relations among the elements of the metric tensor q_{ij} are found. Next, with this metric one may calculate the Ricci tensor. In dimensions fewer than 4, the Riemannian curvature tensor and the Ricci tensor are linearly related to one another (see Ref. [13], for example). Indeed, the vanishing of the Ricci tensor is both a necessary and sufficient condition that the coordinate system $g,F,\bar{F}$ be a diffeomorphism of the Cartesian x_1,x_2,x_3 system. It is interesting that only in the very last step of the procedure, one makes the correspondence of the $g,F,\bar{F}$ system with the Cartesian system. That is, the original vector *PDE* is solved with respect to the natural coordinates of the problem, which are not in general the Cartesian variables.

In Sec. II we show how the solutions to (1.1) are related to solutions to Euler's equations. In Sec. III we completely characterize the solution to (1.1) by the method outlined above. Since the method is different, we show what the method might look like as applied in solving two-dimensional incompressible, potential flow problems in Appendix B. Of course, in the latter case, the solutions may be derived straightforwardly by other well known means; the discussion is included since it isolates the method particularily well (for some already familiar problems). The reader is strongly encouraged to read Appendix B before beginning Sec. III.

Section IV discusses what sort of boundary conditions the complex Trkalian fields may satisfy. Type A fields do not satisfy natural boundary conditions; however, type B (singular) fields may have no normal component to a bounding surface.

Section V discusses how the work presented here meshes with Bjørgum's [14] and Bjørgum and Godal's [15] work on Beltrami fields. Section VI is a discussion and conclusion. In ongoing work, we discuss solutions to other vector PDE's, including linearly polarized transverse electric field [16] and TEM [12] solutions to the electromagnetic wave equations. The main purpose of this article is to introduce a potential method capable of constructing truly three-dimensional vector field solutions to PDE's of mathematical physics.

A glossary of various terms is presented in Table I.

II. HYDRODYNAMICS AND THE COMPLEX TRKALIAN FIELD

Trkalian fields yield solutions to Euler's equations for an incompressible inviscid fluid. These latter equations may be written

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{\nabla p}{\rho}\ , \tag{2.1}$$

where the density ρ=const and $\nabla\cdot\mathbf{v}=0$. A pressure function may be found (at least locally) if and only if the vorticity equation is satisfied:

$$\frac{\partial \xi}{\partial t}+\nabla\times(\xi\times\mathbf{v})=\mathbf{0}\ , \tag{2.2}$$

where $\xi=\nabla\times\mathbf{v}$. Thus any time-independent vector field satisfying $\xi\times\mathbf{v}=\mathbf{0}$ is a solution to Euler's incompressible equations [17,18]. We shall call these flows Beltrami flows (in keeping with Bjørgum; see Table I). A subset of these flows are the Trkalian fields, satisfying $\xi=\lambda\mathbf{v}$, where λ is a constant, which under suitable scaling and inversion we can always take to be unity.

Consider therefore the Trkalian flows which solve the differential equation

$$\xi=\nabla\times\mathbf{v}=\mathbf{v}\ . \tag{2.3}$$

Trkalian fields are divergenceless and obey a linear superposition principle. Each component of the vector field satisfies the Helmholtz eigenvalue equation

$$\Delta v_i+v_i=0\ . \tag{2.4}$$

In the literature therefore, Trkalian fields are often represented as a sum over Fourier modes, using wave vectors with unit norm:

$$(\mathbf{a}_\mathbf{k}+i\mathbf{k}\times\mathbf{a}_\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}}\ , \tag{2.5}$$

where $\mathbf{k}\cdot\mathbf{a}_\mathbf{k}=0$ and $\mathbf{k}\cdot\mathbf{k}=1$. It is our contention that the

TABLE I. Some elementary fields. Note that the word complex has a different connotation in its usage in complex lamellar and complex Laplacian (where we mean "rescaled" or "conformally equivalent to) than it does in its usage in complex Trkalian. Complex Trkalian describes a vector field whose imaginary and real parts are both Trkalian.

Different fields	Definitions
Solenoidal	$\nabla\cdot\mathbf{v}=0$
Lamellar (potential, 1D)	$\nabla\times\mathbf{v}=\mathbf{0}$
Complex lamellar (2D)	$\mathbf{v}\cdot\nabla\times\mathbf{v}=0$
Laplacian	$\nabla\cdot\mathbf{v}=0$, $\nabla\times\mathbf{v}=\mathbf{0}$
Complex Laplacian	there exists an α such that $\mathbf{v}/\alpha$ is Laplacian
Beltrami	$\nabla\times\mathbf{v}=\Omega\mathbf{v}$ for some function Ω $\Longleftrightarrow \mathbf{v}\times(\nabla\times\mathbf{v})=\mathbf{0}$
Trkalian	$\nabla\times\mathbf{v}=k\mathbf{v}$ for some constant k
Complex Trkalian	complex vector field of the form $e^{ig}\nabla F$ satisfying Trkal's equation
Dual field	see Appendix A
$\boldsymbol{\omega}$	imaginary part of a complex vector field
Torsion function Ω	see Beltrami above
Torsion constant k	see Trkalian above
H	helicity of the vector field; square root of the Jacobian of the transformation from Clebsch to Cartesian coordinates
H'	$H'=-2iH$ the Jacobian of $(g,F,\bar{F})=>$ Cartesian

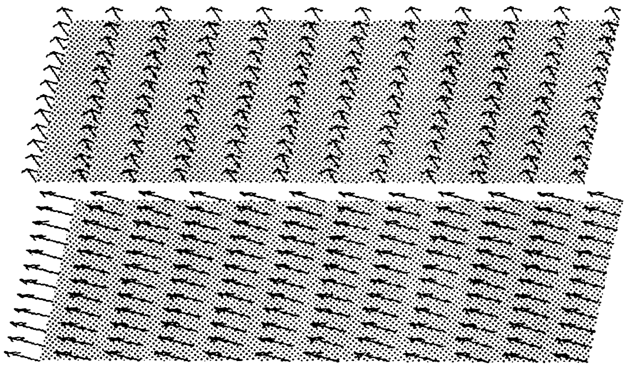

FIG. 1. Picture of the simplest Beltrami field (a uniplanar Trkalian field of the first kind). Here $g=z$ and $F=x+iy$. The patterns are reminiscent of the cholesteric mesophase of liquid crystals.

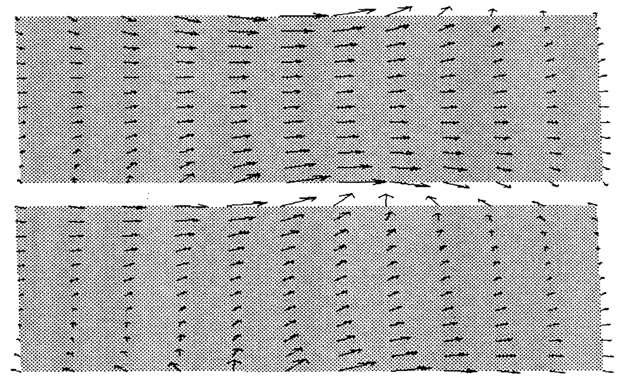

FIG. 2. Two planes of the flow $g=z$ and $F=\cos(x+iy)$. Each arrow from the lower plane to the upper plane is rotated the same amount around the screw axis (the z axis). In each plane, the vector field is potential flow (see Fig. 3).

Fourier type representation (2.5) is not necessarily the most useful representation of Trkalian fields.

Time-independent Trkalian fields (2.3) clearly solve the vorticity equation (2.2) for the incompressible Euler equations. Trkalian fields multiplied by the time decaying factor $e^{-\nu t}$ are solutions to the incompressible Navier-Stokes equations [Eq. (2.1) with a term $\nu\Delta\mathbf{v}$ included on the right-hand side, where ν represents the viscosity; see [11]].

For this paper we are interested in those vector fields which solve the Trkalian equation (2.3) and have a "topologically dual field" $\mathbf{w}$, which also solves the Trkalian equation. The relations between vector fields and their topological duals have recently been investigated by one of the authors [19]; some results are collected in Appendix A. In summary, a vector field $\mathbf{v}$ and a dual field $\mathbf{w}$ are related in such a way that

$$\mathbf{c}=\mathbf{v}+i\mathbf{w}=e^{ig}\nabla(F_1+iF_2) \tag{2.6}$$

for some g, F, and $\bar{F}$, where g,F_1,F_2 are all real functions. As pointed out in Appendix A, *any real vector field may be represented as the real part of a complex vector field, written as (2.6).* That is, $\mathbf{c}$ (as well as $\mathbf{w}$) may be expressed with precisely the same three functions with which one may express $\mathbf{v}$. Among other properties, a vector field is always orthogonal to the curl of a dual field, and a field and a dual field both have precisely the same value of the helicity $H=\mathbf{v}\cdot\nabla\times\mathbf{v}=\mathbf{w}\cdot\nabla\times\mathbf{w}$. A given positive helicity vector field does not in general have a unique dual field (although a suitable gauge fixing condition could probably be defined).

We call the simplest possible solution to (1.1) and (1.2) "the simple helical field":

$$\mathbf{v}=(\sin z,\cos z,0)\ , \tag{2.7}$$

with

$$\mathbf{w}=(-\cos z,\sin z,0)\ , \tag{2.8}$$

$$\mathbf{c}=\mathbf{v}+i\mathbf{w}=e^{iz}(-i,1,0)=e^{iz}\nabla(x+iy)(-i)\ . \tag{2.9}$$

Equation (2.9) is obviously of the form of (1.1). Notice that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal and both have unit helicity (implying $\mathbf{c}\cdot\mathbf{c}=0$ where the dot is the usual scalar product). For the Trkalian vector fields, we may very roughly think of a dual field as a rotation of each vector around the screw axis (in this case the z axis), where the screw axis is along the ∇g direction (see Fig. 1).

III. SOLUTION TO THE COMPLEX TRKALIAN EQUATION

The reader is urged to read Appendix B so that the logic used in our approach will be completely clear. We begin with the complex Trkalian equation

$$\nabla\times\mathbf{c}=\mathbf{c}\ . \tag{3.1}$$

Suppose in addition that

$$\nabla\times\mathbf{c}=i\nabla g\times\mathbf{c}\ , \tag{3.2}$$

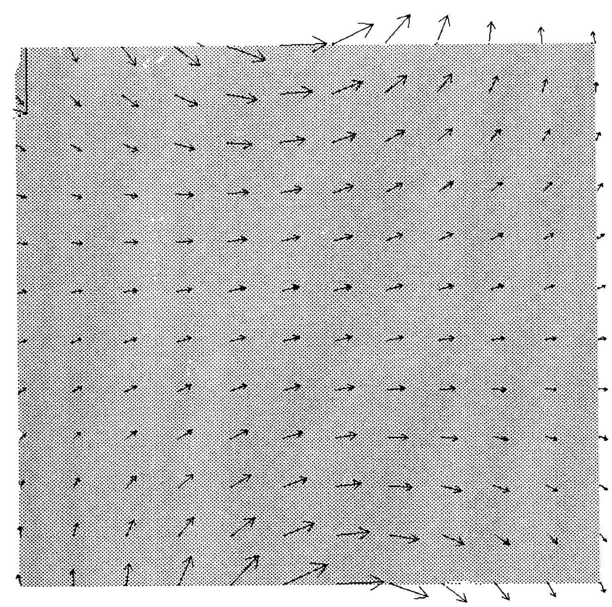

FIG. 3. Bottom plane of Fig. 2.

where g is some real function. Equation (3.2) implies that $\mathbf{c}=e^{ig}\nabla F$ for some complex valued F. That is, $\mathbf{c}$ is a zero-helicity field of a very special type. (In general, $\mathbf{c}=G\nabla F$, where F,G are complex, yields $\mathbf{c}\cdot\nabla\times\mathbf{c}=0$.) Substituting the relation $\mathbf{c}=e^{ig}\nabla F$ into (3.1) yields

$$\nabla g\times\nabla F=-i\nabla F\ , \tag{3.3}$$

$$\nabla g\times\nabla\bar{F}=+i\nabla\bar{F}\ , \tag{3.4}$$

where $\bar{F}$ is the complex conjugate of F. We wish to find the components of the tensor $q_{ij}\equiv(\partial x^p/\partial y^i)(\partial x^p/\partial y^j)$, where we define $y_1\equiv g$, $y_2\equiv F$, and $y_3\equiv\bar{F}$. First we note that this tensor and the tensor $q^{lk}\equiv(\partial y^l/\partial x^p)(\partial y^k/\partial x^p)$ are matrix inverses. Next, from (3.3), for example, one immediately notes that $\nabla F\cdot\nabla F=0$. Thus $q^{22}=0$. Similarly we find $q^{33}=0$, $q^{13}=q^{12}=0$, and $q^{11}=1$. The last result comes from taking the inner product of (3.3) and (3.4) and simplifying. Finally, define $H'\equiv\nabla y^1\cdot\nabla y^2\times\nabla y^3$, which turns out to be a pure complex entity. As is always true, we have $\det q^{ab}=(H')^2$. Putting these together, we may write

$$q^{ab}=\begin{bmatrix}1&0&0\\0&0&iH'\\0&iH'&0\end{bmatrix}\ , \tag{3.5}$$

with the matrix inverse

$$q_{ab}=\begin{bmatrix}1&0&0\\0&0&\dfrac{1}{iH'}\\0&\dfrac{1}{iH'}&0\end{bmatrix}\ . \tag{3.6}$$

This is equivalent to

$$ds^2=dg^2+2e^{2J}dFd\bar{F}\ , \tag{3.7}$$

if we define e^{2J} (as yet undetermined) to be $1/iH'$. The nonvanishing components of the metric tensor have been identified. This completes the first step of the procedure.

At this point we have just calculated the metric tensor. We must ensure, however, that the various coordinate relations encoded into the metric tensor are consistent with a flat (Euclidean) three-dimensional space. We begin with the line element 3.6 [or (3.7)] and calculate the Ricci tensor. First

$$\Gamma^i_{kl}\equiv\frac{1}{2}q^{im}\left[\frac{\partial q_{mk}}{\partial y_l}+\frac{\partial q_{ml}}{\partial y_k}-\frac{\partial q_{kl}}{\partial y_m}\right]\ . \tag{3.8}$$

Then

$$R_{bc}\equiv\Gamma^a_{bc,a}-\Gamma^a_{ba,c}+\Gamma^r_{bc}\Gamma^a_{ra}-\Gamma^a_{br}\Gamma^r_{ca}$$
$$=\begin{bmatrix}-2(J_1^2+J_{11}) & -J_{12} & -J_{13}\\ & 0 & -e^{2J}(J_{11}+2J_1^2)-2J_{33}\\ & & 0\end{bmatrix}\ , \tag{3.9}$$

where the subscripts following commas denote derivatives with respect to g, F, and $\bar{F}$, respectively. The Ricci tensor R is symmetric; thus we have written down only the upper triangle in (3.9). Fortunately, there are symbolic manipulation programs available that allow one to calculate the Ricci tensor from the metric tensor in a matter of moments.

The curvature tensor vanishes if and only if the $g,F,\bar{F}$ system is a diffeomorphism of the Cartesian system. In three dimensions this condition is satisfied if and only if the Ricci tensor vanishes [20]. Thus, since $J_{12}=J_{13}=0$, we need

$$J=\ln G(g)+\ln M(F,\bar{F})\ . \tag{3.10}$$

Since $J_1^2+J_{11}=0$, then up to scaling and translation we have, for conditions A or B, respectively,

$$G=1/\sqrt{2}\quad\text{or}\quad G=g/\sqrt{2}\ . \tag{3.11}$$

Conditions A or B ensure that all the components of the Ricci tensor (3.11) vanish except for the (2,3) component.

Case A leads to $(\ln M)_{23}=0$ [since $J_1=0$ in (3.11)], implying $2J=-\ln 2+\ln W'(F)+\ln\bar{W}'(\bar{F})$ for some $W,\bar{W}$ which are arbitrary functions of $F,\bar{F}$, respectively (the prime denotes a derivative). Then $2e^{2J}=W'(F)\bar{W}'(\bar{F})$. Thus the line element may be written as $ds^2=dg^2+dWd\bar{W}$. Thus we may define x and y by $W=x+iy$ so that we identify g with the Cartesian variable z. That is, we may pick axes so that $g=z$ and F is any complex analytic function of $W=x+iy$. Thus for case A, $ds^2=dz^2+dx^2+dy^2$ and

$$\mathbf{c}=e^{iz}\nabla[F(x+iy)]=e^{iz}F'\cdot(1,i,0)\ . \tag{3.12}$$

For the simple helical field mentioned in Sec. II, we would have $F(x+iy)=-i(x+iy)$ and $\mathbf{c}=e^{iz}(-i,1,0)$. Curiously it seems to have been Beltrami [21] who first wrote down that the Trkal equation was solved by the real part of (3.12); unfortunately this line of thinking has never been developed thoroughly until this article. Examples are given in Figs. 1–3.

For case B we wish the 23 components of the Ricci tensor to vanish:

$$e^{2J}(J_{11}+2J_1^2)+2J_{23}=0\ . \tag{3.13}$$

We substitute $J=-\frac{1}{2}\ln 2+\ln g+\ln M(F,\bar{F})$ (this ensures that all the other components of the Ricci tensor vanish) into (3.7) and (3.15) giving

$$ds^2=dg^2+g^2[M(F,\bar{F})]^2dFd\bar{F}\ , \tag{3.14}$$

with

$$-M^2=4\frac{\partial^2\ln M}{\partial F\partial\bar{F}}\ . \tag{3.15}$$

As one might expect from the line element (3.14), we can relate case B to the study of the spherical coordinates.

Equation (3.15) has fascinating properties [22–24]. It arises in the study of heat propagation movement of plasmas and traces its origins back to Emden, Darboux [25], and Liouville [26] as well. In different contexts, it bears the names Emden-Fowler, Arrhenius, and Thomas-Fermi equations. First we note that it is equivalent to the two-dimensional equation $\Delta\Psi=-e^{\psi}$ [using $M=(1/\sqrt{2})e^{\psi/2}$]. Second we note that if $M(F,\bar{F})$ is a solution to (3.15), then $|A'(F)|M(A,\bar{A})$ is also a solution to (3.15) for any arbitrary complex analytic function $A(F)$ [or $\psi(F,\bar{F})$ is a solution implies that $\rightarrow\psi(A(F),\bar{A}(\bar{F}))+\ln[A(F)]+\ln[\bar{A}(\bar{F})]$ also is a solution].

Thus introduce $A,\bar{A}$ so that

$$M^2=A'(F)\bar{A}'(\bar{F})/\cosh^2\left[\frac{A+\bar{A}}{2}\right] . \tag{3.16}$$

This solves (3.15) for arbitrary complex analytic functions $A,\bar{A}$. That any solution to (3.15) may be expressed in the form of (3.16) was originally given by Liouville [26] (who expressed his solution with respect to a variable corresponding to the exponential of the function A). For example, setting $A=b(\ln c+\ln F)$, where b and c are real constants, yields $M=2b/\{R[(cR)^b+(cR)^{-b}]\}$, where $R^2=F\bar{F}$. Solutions of this type correspond to the much discussed radial solutions in flame propagation [22]. On the other hand, setting $A=-ib(\ln c+\ln F)$ yields $M=1/[R\cosh(\arg F)]$, where arg takes the argument of a complex number.

Next define $\cos\theta=\tanh[(A+\bar{A})/2]$ and $\phi=i(A-\bar{A})/2$. Then it is not difficult to show that $M^2dFd\bar{F}=d\theta^2+\sin^2\theta d\phi^2$. Thus

$$ds^2=dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2) \tag{3.17}$$

and we recover the spherical coordinate system. Since A was a completely arbitrary function of F, then conversely F is a completely arbitrary function of A. So due to the definitions of θ, ϕ we have

$$F_{\text{spherical}}=F[-\text{arc}\tanh(\cos\theta)+i\phi] . \tag{3.18}$$

We should require also that either F be a 2π-periodic function of ϕ or a (complex) scalar multiple of the identity function. This ensures that the resulting vector fields are single valued. Notice that F is annihilated by the operator $\partial_\theta+i(1/\sin\theta)\partial\phi$, which may be used to factor the angular momentum operator. Thus

$$\mathbf{c}=\frac{e^{ir}}{r\sin\theta}[\hat{\theta}+i\hat{\phi}]F' . \tag{3.19}$$

A typical case is given in Figs. 4 and 5. Choosing $F(a)=\cos(-ia)$ and using (3.18) and (3.19) yields

$$\mathbf{v}=\frac{-1}{r\sin^2\theta}\{\hat{\theta}[\cos r\cos\phi\cos\theta+\sin r\sin\phi] + \hat{\phi}[\cos r\sin\phi-\sin r\cos\phi\cos\theta]\} .$$

Thus even a very simple function chosen for F may yield a rather complicated expression for $\mathbf{v}$.

In summary, there are precisely two cases: A, the Cartesian case (3.12), where the screw direction is in the direction of a Cartesian variable and F is an analytic function of the other two Cartesian variables, and B, the spherical case (3.19), where the screw direction is in the radial direction of a spherical coordinate system and F is an analytic function of the variable $[-\text{arc}\tanh(\cos\theta)+i\phi]$, with the additional requirement that the resulting vector field be single valued.

It is straightforward to calculate the helicity for either case A or B. We find, respectively,

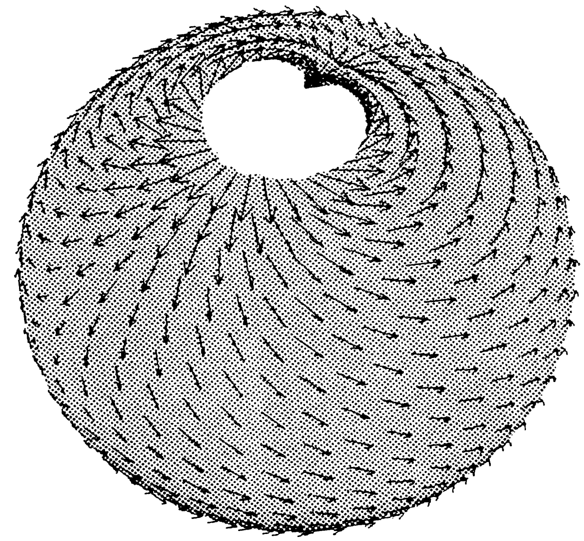

FIG. 4. A total of 714 vectors are plotted on the surface of the sphere showing the complex Trkalian flows for $g=r$ and $F=\cos[\phi+i\,\text{arc}\tanh(\cos\theta)]$. Here r is fixed to be 0.5. The vector magnitudes are not drawn to scale. The true vector magnitudes are the fourth power of those represented in the figure. This is to keep the arrows at the poles from dominating the figure (where the vectors become singularly large).

$$\begin{aligned} h&=v^2=\tfrac{1}{2}\nabla F\cdot\nabla\bar{F}=F'\bar{F}' , \\ h&=v^2=\tfrac{1}{2}\nabla F\cdot\nabla\bar{F}=F'\bar{F}'/(r^2\sin^2\theta) , \end{aligned} \tag{3.20}$$

where F' depends on either A, $x+iy$, or B, $-\text{arc}\tanh(\cos\theta)+i\phi$. In both cases $\Delta F=0$, the three-dimensional Laplacian of F is zero for either class. In

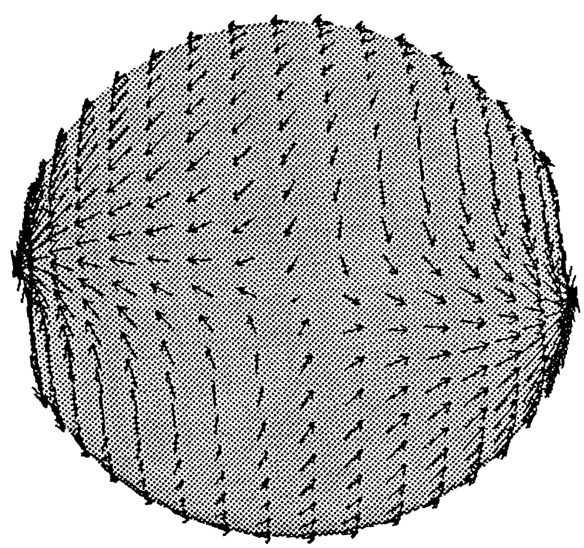

FIG. 5. A better look at one of the singularities of the complex Trkalian flow of Fig. 4. There are a total of four singularities. Two singularities lie at the poles (on the z axis) and two more singularities lie on opposite sides of the sphere (at the same latitude).

case A, actually the two-dimensional Laplacian is of course zero and, in case B, the "angular momentum part" of the Laplace operator acting on F is zero.

As a final remark, notice from (3.20) that there are no oscillations in the helicity (equivalently velocity magnitude) in the direction of increasing g (that is, $\hat{z}$ or $\hat{r}$). This is related to the fact that $\mathbf{c}$ is a complex null vector $\mathbf{c}\cdot\mathbf{c}=0$, so that only the cross terms $\mathbf{v}^2=0$ survive in calculating v^2.

From (3.20), we notice that nothing precludes that the helicity should vanish at exceptional points. Note also that the integrated helicity over any infinite domain of R^3 (for the complex Trkalian flows) will always be infinite. In that sense, these solutions lack a certain natural global character.

IV. BOUNDARY CONDITIONS

Bjørgum has shown that in regions free from singularities, if either $\mathbf{v}\cdot\mathbf{n}=0$ or $\mathbf{v}\times\mathbf{n}=\mathbf{0}$, where $\mathbf{v}$ is a solenoidal Beltrami field and $\mathbf{n}$ is the normal to some bounding surface, it necessarily follows that the vector field $\mathbf{v}$ must vanish identically. We are then able to discuss which types of boundary conditions the *complex Trkalian fields* may obey.

Since the type A complex Trkalian fields (with screw direction along a Cartesian axis) are solenoidal, they cannot satisfy the aforementioned natural boundary conditions. The type B complex Trkalian fields always have a singularity interior to any region containing the origin so that the rigorous result of Bjørgum does not apply. Moreover, these fields always have a vanishing component in the radial direction, so that one may pick a boundary (r is a constant) that encloses a simply connected region in such a way that the vector field lies tangent to such a surface. However, there is a line singularity passing through the center of the sphere, leading to an infinite value of the integrated helicity. In conclusion, then, we see that no finite amplitude complex Trkalian field in a simply connected domain may satisfy the physically interesting boundary conditions $\mathbf{v}\cdot\mathbf{n}=0$ on some surface. In multiply connected regions, Trkalian fields satisfying physically interesting boundary conditions may exist.

V. BJØRGUM AND GODAL'S WORK ON BELTRAMI FIELDS

In this section, we discuss the work of Bjørgum and co-workers and interpret his results in the same framework as ours. In his first long paper on Beltrami fields [14], Bjørgum writes that the Beltrami vector fields are precisely those vector fields which may be written as

$$\mathbf{v}=\frac{\nabla q_1\times\nabla q_2}{\Omega}=q_1\nabla q_2+\nabla q_3\ , \tag{5.1}$$

where Ω is the torsion function. This leads to

$$\frac{\partial\mathbf{r}}{\partial q_a}\cdot\frac{\partial\mathbf{r}}{\partial q_b}=\begin{bmatrix} R & S & 0\\ S & T & q_1\dfrac{\Omega}{H}\\ 0 & q_1\dfrac{\Omega}{H} & \dfrac{\Omega}{H}\end{bmatrix}_{ab}\ , \tag{5.2}$$

where R,S,T,M are themselves functions and T may be found from

$$T=S^2/R+\frac{q_1^2\Omega}{H}+\frac{1}{\Omega HR}\ . \tag{5.3}$$

(The determinant of the matrix 5.2 is $1/H^2$, where $H=\nabla q_1\cdot\nabla q_2\times\nabla q_3$ is the Jacobian of the transformation from the q's to the Cartesian variables.) One may demonstrate Eq. (5.2), using the identity

$$\frac{\partial\mathbf{r}}{\partial q_3}=\frac{\nabla q_1\times\nabla q_2}{\nabla q_1\cdot\nabla q_2\times\nabla q_3}\ , \tag{5.4}$$

which holds for any three functionally unrelated functions q_1,q_2,q_3. Using (5.4), the Beltrami equation [Eq. (5.1)] may be written

$$\frac{\partial\mathbf{r}}{\partial q_3}=\Omega\left[q_1\frac{\partial\mathbf{r}}{\partial q_3}\times\frac{\partial\mathbf{r}}{\partial q_1}+\frac{\partial\mathbf{r}}{\partial q_1}\times\frac{\partial\mathbf{r}}{\partial q_2}\right]. \tag{5.5}$$

Equation (5.2) then follows directly. Bjørgum, however, does not ensure that the above metric tensor given by (5.2) yields a zero curvature tensor. This *must* be the case if the q_1, q_2, and q_3 coordinates are to be themselves functions of Cartesian coordinates, as we argue in our paper.

If one uses the representation given in Eq. (A7)

$$\mathbf{v}=\tfrac{1}{2}(e^{iy_1}\nabla y_2+e^{-iy_1}\nabla y_3)\ , \tag{5.6}$$

where $y_1=q_2$ is real and y_2 and y_3 are complex conjugates, then Bjørgum's statement is equivalent to the following. The Beltrami fields are precisely those vector fields where

$$\frac{\partial\mathbf{r}}{\partial y_a}\cdot\frac{\partial\mathbf{r}}{\partial y_b}=\begin{bmatrix} B & -\dfrac{i}{2}e^{iy_1}A & \dfrac{i}{2}e^{-iy_1}A\\ -\dfrac{i}{2}e^{iy_1}A & e^{2iy_1}D & P\\ \dfrac{i}{2}e^{-iy_1}A & P & e^{-2iy_1}D\end{bmatrix}_{ab}\ , \tag{5.7}$$

where

$$B=\frac{1+A^2\Omega H}{H\Omega R}\ ,$$
$$D=\frac{\Omega-RH}{4H}\ , \tag{5.8}$$
$$P=\frac{\Omega+RH}{4H}$$

are real functions. Equation (5.7) may be used as a starting point for the Ricci tensor calculation. Given Ω and H, there are two functions (say, R and S) that characterize the matrix (5.2). Similarly, two functions characterize the matrix given by (5.7) (say, R and A). The function A is related to S by

$$A = S + Rq_3 \; . \tag{5.9}$$

The determinant of the matrix (5.7) is $[D+P][B(D-P)+a^2/2] = -1/4H^2 = 1/H'^2$. This is because $H' = \nabla y_1 \cdot \nabla y_2 \times \nabla y_3 = -2i \nabla q_1 \cdot \nabla q_2 \times \nabla q_3 = -2iH$, as may be found from calculating $\mathbf{v}\cdot\nabla\times\mathbf{v}$ from (5.1) and (5.6). To proceed from (5.2) to (5.7), one needs

$$\frac{\partial \mathbf{r}}{\partial y_a}\cdot\frac{\partial \mathbf{r}}{\partial y_b} = \frac{\partial q_i}{\partial y_a}\frac{\partial \mathbf{r}}{\partial q_i}\cdot\frac{\partial \mathbf{r}}{\partial q_j}\frac{\partial q_j}{\partial y_b} = Z_{ia}\frac{\partial \mathbf{r}}{\partial q_i}\cdot\frac{\partial \mathbf{r}}{\partial q_j}Z_{jb} \; , \tag{5.10}$$

with

$$Z_{ia} = \frac{\partial q_i}{\partial y_a} = \begin{bmatrix} q_3 & 1 & -q_1 \\ \frac{-i}{2}e^{iy_1} & 0 & \frac{1}{2}e^{iy_1} \\ \frac{i}{2}e^{-iy_1} & 0 & \frac{1}{2}e^{-iy_1} \end{bmatrix} . \tag{5.11}$$

The above matrix $[Z]$ may be found straightforwardly by using $y_1 = q_2$, $y_2 = (q_3 + iq_1)e^{-iq_2}$, and $y_3 = (q_3 - iq_1)e^{+iq_2}$. The Beltrami equation in terms of the y variables then may be written

$$e^{iy_1} i \frac{\partial \mathbf{r}}{\partial y_3} + i e^{-iy_1}\frac{\partial \mathbf{r}}{\partial y_2} = \Omega\left[e^{iy_1}\frac{\partial \mathbf{r}}{\partial y_3}\times\frac{\partial \mathbf{r}}{\partial y_1} + e^{-iy_1}\frac{\partial \mathbf{r}}{\partial y_1}\times\frac{\partial \mathbf{r}}{\partial y_2}\right] . \tag{5.12}$$

For complex Trkalian fields it is straightforward to show in that case that one must have $A = D = 0$ and $B = 1$ (which specifies the torsion function). This matches the coefficients of e^{iy_1} and e^{-iy_1} above in Eq. (5.12) or, equivalently, discards the oscillating terms of the matrix (5.7). At that point one is left with the matrix described below Eq. (3.6):

$$\frac{\partial \mathbf{r}}{\partial y_a}\cdot\frac{\partial \mathbf{r}}{\partial y_b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{iH'} \\ 0 & \frac{1}{iH'} & 0 \end{bmatrix}_{ab} . \tag{5.13}$$

The curvature condition then ensures a complete solution. In future work we plan to examine less restrictive cases than the complex Trkalian case and calculate curvature more directly from Eq. (5.7). One might suspect that the complex Trkalian fields hold a very prominent place in the theory of the Beltrami fields since they seem to take the elements of the matrix (5.7), which are most robust in some sense. As we plan to show in later work, the complex Trkalian fields may be shown to yield precisely the linearly polarized TEM solutions [12] of electromagnetic wave theory.

VI. CONCLUSIONS AND DISCUSSION

In this paper we have introduced a method for constructing true three-dimensional vector field solutions to vector PDE's of mathematical physics. As opposed to axisymmetric flows, which are the dominant solutions to fluid mechanical equations listed in texts, true three-dimensional flows should have a finite value of helicity $\mathbf{v}\cdot\nabla\times\mathbf{v}$. It is clear that the Clebsch representation is then the most convenient: $\mathbf{v} = f\nabla g + \nabla h$. That is because nonzero helicity fields may always be expressed in such a manner [27] and can never be expressed with fewer than three Clebsch functions. Given the representation, however, the question remains how to construct these functions (f, g, h).

Our approach then is to represent the appropriate vector field with the appropriate Clebsch functions and then to construct the relations between the elements of a metric tensor q_{ij} using the original PDE. After having constructed a metric tensor consistent with the equations one wishes to solve, one computes the Ricci tensor. If the Clebsch functions are to be functions of the Cartesian variables, then (in three dimensions) the Ricci tensor must necessarily vanish. After solving the partial differential equations for the vanishing of the Ricci tensor, one finally searches for the class of transformations of the metric tensor back to Cartesian coordinates. By this means, the integrity of the good coordinates—the Clebsch functions—remains intact.

In particular in this manuscript we have undertaken steps in a systematic classification of Beltrami [28,29], and Trkalian fields which are thought to be important constituents of high Reynolds number flow fields. For example, the so-called Arnold-Beltrami-Childress (Trkalian) flow has been extensively studied [30] particularily for its chaotic properties [31–34]. Note that, although it is always possible to write down a formal solution to the Trkalian equation as in Eq. (2.5) in terms of Fourier modes, the use of Clebsch functions, which we advocate here, has many advantages. For example, when one Fourier decomposes a problem one implicitly assumes that the Cartesian coordinates are a natural coordinate system for the problem. As we have discussed above, this is not necessarily the case.

In this article the analysis proceeds by considering the Beltrami fields as real parts of complex vector fields and then considering complex vector fields of a certain type. Complex vector fields play a prominent role in many areas of mathematical physics including electromagnetism (see Ref. [35]) and minimal surfaces (see Ref. [36] or [37]). In this paper, we examine complex vector fields, which satisfy the Trkalian equation (1.3), and may be written in the form $e^{ig}\nabla F$, with g real and F complex (that is, we restrict the problem enough in order to give a complete solution). Both the real and imaginary parts of such a vector field will then be Trkalian. The two classes of solutions to the complex Trkalian equation with the complex vector field of the above form are given in Sec. III. The class of solutions where $\mathbf{v}$ is a two-dimensional (2D) potential field with a screw direction in a Cartesian direction has been studied before (Bjørgum's uniplanar

Trkalian flows). The class of solutions where **v** is a zero angular momentum field with a screw direction in the spherical radial direction does not appear in the Beltrami literature. Examples of both types of fields are presented in the figures.

Although the Clebsch representation of the vector field has found fruitful application in fluid mechanics [38–45], in thermodynamics, and more generally in variational principles [46–49], the authors still maintain that this representation has been underexploited. We demonstrate in this article that the method we have used employing Clebsch functions has much practical merit in producing solutions to partial differential equations. The same method will be used in further articles on fluid mechanics and wave equations.

ACKNOWLEDGMENTS

The author would like to express his deep gratitude to R. M. Kiehn at the University of Houston for teaching him about differential forms, the intrinsic merit of Clebsch functions, and a topological approach to physics and for sharing his results prior to publication [50]. The author also thanks Professor Giles Auchmuty of the University of Houston for discussions and for pointing out several references.

APPENDIX A: VECTOR FIELDS AND THEIR TOPOLOGICAL DUALS

All the following arguments are local for a region where the helicity is of constant sign (nonzero).

Proposition 1. Every vector field **v** may be written as the real part of a complex vector field of the form

$$\mathbf{c}=e^{ig}\nabla F \,, \tag{A1}$$

where g is a function with domain $\mathcal{R}^3$ and range $\mathcal{R}$, and F has domain $\mathcal{R}^3$ and range of the complex numbers.

Proof. One knows that it is always possible to construct the Monge potentials f,g,h so that a vector field **v** may be expressed in the decomposition due to Clebsch [27]

$$\mathbf{v}=f\nabla g+\nabla h \,. \tag{A2}$$

Now define

$$F\equiv(h+if)e^{-ig} \,, \tag{A3}$$

$$\bar{F}\equiv(h-if)e^{+ig} \,, \tag{A4}$$

or equivalently

$$f=\frac{-i}{2}(Fe^{ig}-\bar{F}e^{-ig}) \,, \tag{A5}$$

$$h=\tfrac{1}{2}(Fe^{ig}+\bar{F}e^{-ig}) \,. \tag{A6}$$

Now we may write **v** as

$$\mathbf{v}=\tfrac{1}{2}(e^{ig}\nabla F+e^{-ig}\nabla\bar{F}) \,, \tag{A7}$$

from which Proposition 1 follows.

Proposition 2. The real part of c,**v** and the imaginary part of c,**w** satisfy

$$\nabla\times\mathbf{v}+\nabla g\times\mathbf{w}=\mathbf{0} \,, \tag{A8}$$

$$\nabla\times\mathbf{w}-\nabla g\times\mathbf{v}=\mathbf{0} \,, \tag{A9}$$

$$\mathbf{v}\times\mathbf{w}\neq\mathbf{0} \,. \tag{A10}$$

We say that any two vector fields that satisfy (A8)–(A10) for some function g are said to be *dual* to one another with respect to g.

Equations (A8)–(A10) are easy to show beginning from

$$\mathbf{w}=-h\nabla g+\nabla f \,. \tag{A11}$$

In particular it may be shown from the above that dual vector fields satisfy

$$\mathbf{w}\cdot(\nabla\times\mathbf{v})=0 \,, \tag{A12}$$

$$\mathbf{v}\cdot(\nabla\times\mathbf{w})=0 \,, \tag{A13}$$

$$\mathbf{v}\cdot(\nabla\times\mathbf{v})=\mathbf{w}\cdot(\nabla\times\mathbf{w}) \,, \tag{A14}$$

$$\nabla\times\left[\frac{(\mathbf{v}\times\mathbf{w})\cdot(\nabla\times\mathbf{v})}{(\mathbf{v}\times\mathbf{w})\cdot(\mathbf{v}\times\mathbf{w})}\mathbf{v}-\frac{(\mathbf{w}\times\mathbf{v})\cdot(\nabla\times\mathbf{w})}{(\mathbf{v}\times\mathbf{w})\cdot(\mathbf{v}\times\mathbf{w})}\mathbf{w}-\frac{\mathbf{v}\cdot(\nabla\times\mathbf{v})}{(\mathbf{v}\times\mathbf{w})\cdot(\mathbf{v}\times\mathbf{w})}\mathbf{v}\times\mathbf{w}\right]=0 \,. \tag{A15}$$

The terms in large parentheses in (A15) constitute a solution for $-\nabla g$ from (A8) and (A9).

A complete classification of vector fields in this fashion does not seem to be known. For example, one can also take a vector field in $\mathcal{R}^4$ and write it as the real part of a zero helicity vector field with domain the complex numbers. One can also write a vector field in $\mathcal{R}^8$ as the real part of a zero helicity vector field with domain the quaternions. For the dimensions 5–7, however, no obvious standard form exists, although clearly these vector fields may be thought of as the real part of zero helicity quaternionic fields of a restricted type (the same way that any real vector field may be thought of as the real part of a complex vector field satisfying $\nabla\times\mathbf{c}=i\nabla g\times\mathbf{c}$).

Dual concepts are prevalent in mathematical literature. Typically one hopes to represent a function (or vector of functions) as the real part of a complex function with slightly nicer properties. The complex function uses the same functions as the real part as building blocks. Thus one writes a harmonic function as the real part of an analytic function (which depends on a single variable, but where this variable belongs to the complex field). Similarily one writes the panharmonic function as the real part of a μ regular function (see Ref. [51]).

APPENDIX B: THE PROPOSED METHOD EMPLOYED IN A FAMILIAR PROBLEM

We give a very simple example of the general procedure we are outlining. Consider the case of incompressible, time-independent, irrotational flow past an obstacle in two dimensions. We wish to study a potential function ϕ, a stream function ψ, and a complex potential $W=\phi+i\psi$. Let us try to write a differential element of length in terms of the functions ϕ and ψ. Since ϕ and ψ are, respectively, the real and the imaginary part of the complex potential W, they satisfy the Cauchy-Riemann relations $\partial\phi/\partial x=-\partial\psi/\partial y$, $\partial\phi/\partial y=\partial\psi/\partial x$, or, more compactly,

$$\nabla\phi=\hat{\mathbf{z}}\times\nabla\psi \ . \tag{B1}$$

Now one may straightforwardly calculate

$$\nabla\phi\cdot\nabla\phi=\nabla\psi\cdot\nabla\psi \ , \tag{B2}$$

$$\nabla\phi\cdot\nabla\psi=0 \ . \tag{B3}$$

For shorthand, let $y^1=\phi$ and $y^2=\psi$. Then

$$g^{lk}=\frac{\partial y^l}{\partial x^p}\frac{\partial y^k}{\partial x^p}=\begin{bmatrix}1/a^2 & 0\\ 0 & 1/a^2\end{bmatrix} .$$

The matrix

$$g_{ij}=\frac{\partial x^p}{\partial y^i}\frac{\partial x^p}{\partial y^j}=\begin{bmatrix}a^2 & 0\\ 0 & a^2\end{bmatrix} \tag{B4}$$

is the matrix inverse of g^{lk}.

The first step of the calculational procedure is then complete: for the equations of irrotational, incompressible flow (Cauchy-Riemann conditions), *the appropriate nonvanishing components of the metric tensor* are identified. We are then left with a line element (or metric tensor) given by

$$ds^2=a^2(d\phi^2+d\psi^2) \ , \tag{B5}$$

where $a=a(\phi,\psi)$. Of course, (B1) immediately implies (B5) due to conformality of a complex analytic transformation; one needs virtually no calculation. In the approach taken here, *flow equations such as (B1) are interpreted as geometric constraints (on the metric function).* Note that in his papers Bjørgum calculated correctly the nonvanishing components of the metric tensor. He never ensured, however, that the curvature was zero. Applying this spirit to the current problem, Bjørgum would have halted at this point.

Now we calculate *the Ricci tensor* given the metric function (B4):

$$g_{ij}=\begin{bmatrix}a^2 & 0\\ 0 & a^2\end{bmatrix}_{ij} . \tag{B6}$$

Let $A=\ln a$. Then

$$\Gamma^i_{kl}\equiv\frac{1}{2}g^{im}\left[\frac{\partial g_{mk}}{\partial y^l}+\frac{\partial g_{ml}}{\partial y^k}-\frac{\partial g_{kl}}{\partial y^m}\right] \tag{B7}$$

$$=A_{,l}\delta_{ik}+A_{,k}\delta_{il}-A_{,k}\delta_{lk} \ , \tag{B8}$$

where the indices following commas denote derivatives. Now the Ricci tensor is given by

$$R_{bc}\equiv\Gamma^a_{bc,a}-\Gamma^a_{ba,c}+\Gamma^r_{bc}\Gamma^a_{ra}-\Gamma^a_{br}\Gamma^r_{ca}$$

$$=-(A_{,11}+A_{,22})\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}_{bc} . \tag{B9}$$

Now we may express the functions ϕ and ψ in terms of the Cartesian variables if and only if the Ricci tensor vanishes identically. Thus we need the function $\ln a$ to be harmonic with respect to the functions ϕ and ψ. Actually this is quite obvious since

$$ds^2=dzd\bar{z}=\frac{dz}{dW}\frac{d\bar{z}}{d\bar{W}}dWd\bar{W} \ , \tag{B10}$$

where $z=x+iy$, so that $(\partial^2/\partial W\partial\bar{W})\ln[(dz/dW)(d\bar{z}/d\bar{W})]=0$. The above A corresponds to $\ln(dz/dW)$.

The actual choice for $\ln a$ must be determined by the *boundary conditions* of an actual two-dimensional flow problem. For flow past an obstacle, we wish to map (part of) the real axis of the variable $W=\phi+i\psi$ to the boundary of the obstacle. The upper half plane is then mapped to the exterior of the obstacle in the upper half plane. Once $a(\phi,\psi)$ is determined, then we may try to calculate the Cartesian variables by means of the eikonal-like equations

$$\left[\frac{\partial x}{\partial\phi}\right]^2+\left[\frac{\partial x}{\partial\psi}\right]^2=\left[\frac{\partial y}{\partial\phi}\right]^2+\left[\frac{\partial y}{\partial\psi}\right]^2=a^2(\phi,\psi) \ . \tag{B11}$$

There is no new result outlined in the approach above; we merely demonstrate the method, which we use to construct Clebsch functions in this paper, to a more familiar problem.

An example of such a 2D flow is symmetrical flow past a unit disk with unit speed far downstream. We need a function mapping (part of) the real line to the boundary of a unit disc. This is well known: $z\equiv x+iy=e^{i\arccos W/2}=(W+\sqrt{W^2-4})/2$. We then calculate $a^2=|dz/dW|^2=\frac{1}{4}|1+W/\sqrt{W^2-4}|^2$. So $ds^2=a^2(d\phi^2+d\psi^2)$, where ϕ,ψ are the real and imaginary parts of W.

For symmetrical flow past a plate of width 4 aligned perpendicular to the flow along the x direction, we have $z=\sqrt{W^2-4}$ and we calculate $a^2=|dz/dW|^2=(\phi^2+\psi^2)/\sqrt{(\phi^2+\psi^2-4)^2+16\psi^2}$. For both the plate and the disk, there is a singularity in a when $\phi=2$ and $\psi=0$. Conversely given the function a^2 (where $\ln A$ is harmonic in ψ,ϕ), then one may use (B11) to reconstruct the boundary. Also note that since the streamlines correspond to contours of constant ψ, then roughly the function ψ plays the role of an energy for a one degree of freedom Hamiltonian system, whereas the function ϕ plays the role of a phase (or time) variable.

As is well known, the harmonic functions are precisely those functions which serve as nice coordinates of orthogonal coordinate systems in two dimensions. They are the "isothermal parameters" [36] of the plane. In a similar sense, the (Clebsch) functions we solve for in this paper are also the good coordinates for the problem at hand: the building blocks of the complex Trkalian flows.

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