Symbolic dynamics analysis of topological entropy and its multifractal structure

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By means of symbolic dynamics, we establish an expression determining the topological entropy of a Derrida-Gervois-Pomeau (DGP) compound sequence, the concept of an equal topological entropy class, and we find a global regularity in interval dynamics—the devil's staircase of topological entropy, which possesses a fractal structure and a rigorous subinterval similarity. An important result is that the bifurcation processes (period-doubling or period-n-tupling) are all processes preserving topological entropy. We also discuss the uniform compressibility of the operator $\mathbf{Q}_*$ and the nonuniform compressibility in a series of stair-climbing sequences. Finally, we suggest a classification of all admissible words in the complete topological space Σ_2 of two symbols.

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I. INTRODUCTION

It is known that all the quantitative universalities in chaotic dynamics of one-dimensional (1D) iterative systems, such as convergent rates $\delta(\mathbf{W})$, scaling factors $\alpha(\mathbf{W})$ [1], and generalized fractal dimensions $D_q(\mathbf{W})$ or singularity spectra $f(\alpha)$ for the critical attractors in the Feigenbaum scenario, depend rigorously on the words $\mathbf{W}$ of symbolic dynamics in the complete topological space Σ_2 of two symbols, or equivalently on the values λ of the parameter. However, a global regularity which does not depend on the values λ of the parameter is quite important for a *bona fide* thermodynamic formalism of chaotic dynamics in the whole topological space. Not long ago, we found a global regularity β [2] which belongs to the whole parameter space and leads to a thermodynamic formalism of the whole topological space. Recently, we found a new global regularity—a devil's staircase of topological entropy in interval dynamics [3]. The devil's staircase exhibited in the multifractal entropy curve $h(\mathbf{W})$ versus λ has a perfect subinterval similarity and an infinite number of scales for each subinterval, and is entirely different from the Jensen-Bak-Bohr devil's staircase in the mode-locking structure for the circle map [4]. The complementary set to the entropy staircase indicates chaos in both coarse grain and fine grain. The purpose of this paper is to establish the devil's staircase of topological entropy, reveal its fractal structure by means of symbolic dynamics, and discuss the classification of all admissible words in Σ_2.

In Sec. II, we study the topological entropy of a compound kneading sequence, extract the conception of the equal topological entropy class (ETEC), and find that there is an ETEC in h-λ space for each primitive word as well as a single point for an infinite word. In Sec. III, we indicate that, as a whole, all ETEC's and single points of the admissible words form a complete devil's staircase of topological entropy in h-λ space, which possesses a multifractal structure and a rigorous subinterval similarity. The devil's staircase shows a global regularity of 1D unimodal maps. Furthermore, in Sec. IV, we analyze the relation between various universalities and their corresponding compressibilities, and point out that there is a stair-climbing effect. The results reveal that all the Feigenbaum bifurcation processes (period doubling and period-n-tupling, main and associated) preserve topological entropy, and that various universalities found so far belong to sequences which are all located at the same ETEC and originate from the uniform compressibility of the operator $\mathbf{Q}*$. Finally, in Sec. V, we obtain a classification of all admissible words in complete space Σ_2.

II. EQUAL TOPOLOGICAL ENTROPY CLASS

A. The topological entropy of a compound kneading sequence

The topological entropy is an important characteristic quantity for describing the complexity of systems. The topological entropy is obtained from the smallest positive root of the Milnor-Thurston characteristic polynomial [5] or the largest eigenvalue of the Stefan matrix [6], which can be constructed from the space transition order on a trajectory. Peng and Luo [7] found the recursive relation between points on the elementary periodic trajectory and points on the n-tupling periodic trajectory. This implies that we can find the relation between topological entropies of any two Metropolis-Stein-Stein (MSS) sequences and the topological entropy of their $*$ compound sequence.

As usual, a superstable kneading sequence of the map $f(x,\lambda)$ is $K=K_1K_2\cdots K_kC$:

$$K_i=R\ \ (C,L)\ \ \text{for } f^i(0)>\ (=,<)\ 0,\quad i=1,2,\ldots .$$

For each symbol the character $\varepsilon_i=+(-)$ is given if $K_i=L\ (R)$ and $\varepsilon_{k+1}=\prod_{j=1}^{k}\varepsilon_j$ if $K_{k+1}=C$, and the invariant coordinate of $K_1K_2\cdots K_n$ in the kneading sequence is defined as the following:

$$\Theta(K_1K_2\cdots K_n)=\Theta^n\equiv\prod_{i=1}^{n}\varepsilon_i \quad (\text{convention } \Theta^0=1)\ . \tag{2.1}$$

According to Milnor-Thurston's theory the characteristic polynomial (or kneading determinant) for the kneading sequence $\mathbf{K}$ reads

$$P_{\mathbf{K}}(\tau)=\sum_{n=0}^{\infty}\Theta^n\tau^n\ ; \tag{2.2}$$

the topological entropy h of the kneading sequence $\mathbf{K}$ is

$$h(\mathbf{K})=-\ln\tau(\mathbf{K})\ , \tag{2.3}$$

where $\tau(\mathbf{K})$ is the smallest positive root of $P_{\mathbf{K}}(\tau)$.

Let two kneading sequences be $\mathbf{Q}=Q_1Q_2\cdots Q_qC$ and $\mathbf{S}=S_1S_2\cdots S_sC$; the ith symbol of compound sequence $\mathbf{Q}*\mathbf{S}$ can be written as [7]

$$(\mathbf{Q}*\mathbf{S})_i=\begin{cases} Q_j & \text{if } i \bmod (q+1)=j\neq 0\ ,\\ S_k^{t(Q)} & \text{if } i=k(q+1)\ ,\\ C & \text{if } i=(q+1)(s+1)\ . \end{cases} \tag{2.4}$$

Here

$$t(\mathbf{Q})=t^{J(\mathbf{Q})}\ ,$$

$J(\mathbf{Q})$ is the R parity of $\mathbf{Q}$ (the number of appearances of the symbol R in the sequences $\mathbf{Q}$), and t is the parity inverse operator

$$R^t=L,\quad C^t=C,\quad L^t=R\ , \tag{2.5}$$

$$t^{2n+1}=t,\quad t^{2n}=I,\quad n\in\mathbb{Z}^+\ . \tag{2.6}$$

I denotes the identity operator. Using expression (2.4), we can establish a relation to determine the topological entropy of the compound sequence constructed from two kneading sequences by DGP $*$ composition rule [6].

As a convention, the MSS sequence $\mathbf{K}=(K_1K_2\cdots K_kC)^\infty$ with period $\|\mathbf{K}\|=k+1$, with the repeating byte $K_1K_2\cdots K_kC$ of finite length as its characteristic, is called a finite word. Employing the polynomial with finite terms

$$\bar{P}_{\mathbf{K}}(\tau)=\sum_{n=0}^{k}\Theta^n\tau^n\ , \tag{2.7}$$

the characteristic polynomial (2.2) for the finite word can be formalized to

$$P_{\mathbf{K}}(\tau)=\lim_{n\to\infty}\bar{P}_{K^n}(\tau)=\frac{\bar{P}_{\mathbf{K}}}{1-\tau^{\|\mathbf{K}\|}}\ . \tag{2.8}$$

Thus the topological entropy of the superstable kneading sequence $\mathbf{K}$ can be determined by the smallest positive root of $\bar{P}_{\mathbf{K}}(\tau)$.

Introducing the truncation operator $\bar{\varphi}$ for a MSS sequence $\mathbf{Q}=Q_1Q_2\cdots Q_qC$ as the complement of shift operator φ [7]:

$$\bar{\varphi}^lQ=Q_1Q_2\cdots Q_l=\bar{\varphi}^{\,l}_{q+1},\quad l\in\mathbb{Z}^+ \tag{2.9}$$

(convention $\bar{\varphi}^0\mathbf{Q}=\varnothing$), for the invariant coordinate defined by (2.1) we have

$$\begin{aligned}\Theta(\bar{\varphi}^l\mathbf{Q})&=\Theta^l_{q+1}\ ,\\ \Theta(\bar{\varphi}^0\mathbf{Q})&=\Theta^0_{q+1}\equiv+1\ .\end{aligned} \tag{2.10}$$

The subscript $(q+1)$ in (2.9) and (2.10) denotes the period of $\mathbf{Q}$. Thus it can be easily proved that the invariant coordinate in $\mathbf{Q}*\mathbf{S}$ under the truncation operator $\bar{\varphi}$ is factorized

$$\Theta^{l+k(q+1)}_{(q+1)(s+1)}=\Theta^l_{q+1}\Theta^k_{s+1}\ ,$$

$$l=0,1,\ldots,q,\quad k=0,1,\ldots,s\ . \tag{2.11}$$

From (2.4), (2.7), and (2.11), we have

$$\bar{P}_{\mathbf{Q}*\mathbf{S}}(\tau)=\bar{P}_{\mathbf{Q}}(\tau)\bar{P}_{\mathbf{S}}(\tau^{\|\mathbf{Q}\|})\ . \tag{2.12}$$

Substituting $\mathbf{K}$ in (2.8) by $\mathbf{Q}*\mathbf{S}$, we obtain

$$P_{\mathbf{Q}*\mathbf{S}}(\tau)=\frac{\bar{P}_q(\tau)\bar{P}_s(\tau^{\|\mathbf{Q}\|})}{1-\tau^{\|\mathbf{Q}\|\|\mathbf{S}\|}}\ . \tag{2.13}$$

Considering the relation between the topological entropy and the smallest positive root of each factor (2.12), it is immediately found that

$$h(\mathbf{Q}*\mathbf{S})=\max\left\{h(\mathbf{Q}),\frac{h(\mathbf{S})}{\|\mathbf{Q}\|}\right\}\ . \tag{2.14}$$

Because the topological entropies of MSS sequences are monotonic

$$h(\mathbf{K})\le h(\mathbf{W})\quad \text{if } \mathbf{K}<\mathbf{W},\ \mathbf{K},\mathbf{W}\in\text{MSS}\ ,$$

the topological entropies of the maximal MSS sequence RL^∞ and the period-doubling bifurcation sequences R^{*n} are, respectively,

$$h(RL^\infty)=\ln 2,\quad h(R^{*n})=0,\quad n=1,2,\ldots\ .$$

(2.14) can be expressed explicitly as

$$h(\mathbf{Q}*\mathbf{S})=\begin{cases} h(\mathbf{Q}) & \text{if } \mathbf{Q}\neq R^{*n}\ ,\\ \dfrac{1}{2^n}h(\mathbf{S}) & \text{if } \mathbf{Q}=R^{*n}\ . \end{cases} \tag{2.15}$$

This formula generalizes the result of Collet, Crutchfield, and Eckmann [8],

$$h(R^{*m}*\mathbf{S})=\frac{1}{2^m}h(\mathbf{S})\ , \tag{2.16}$$

and is directly verified by our numerical experiment. By (2.15), it can be seen that

$$h(\mathbf{Q}^{*n})=h(\mathbf{Q}),\quad n=1,2,\ldots\ . \tag{2.17}$$

This important result reveals that the bifurcation processes (period doubling or period n-tupling) and the processes of transition to chaos at an accumulation point are processes preserving topological entropy.

B. ETEC of the finite word

As L^∞ is the minimal and RL^∞ the maximal for all admissible words (superstable or nonsuperstable kneading sequences), we use the notation $\mathcal{W}=(L^\infty,RL^\infty)$ to denote the interval of the set of admissible words. For MSS sequences, the primitive words cannot be decomposed by the $*$ composition operation. Let $\mathcal{P}_\pi$ be the set of all primitive words. From the formula (2.15) the concept of the equal topological entropy class for a finite word $\mathbf{Q}$, $\mathbf{Q}=R^{*n}$, $n=1,2,\ldots$, or $\mathbf{Q}\in\mathcal{P}_\pi$, can be abstracted.

When $\mathbf{Q}=R^{*n}$, $n=1,2,\ldots,\infty$, the series of these words describes Feigenbaum period-doubling bifurcation processes, and forms the equivalence class of zero topological entropy, i.e., $h(R^{*n})=0$. Denoting the periodic window R^{*n} by Δ'_n, this Feigenbaum equal topological entropy class can be written as

$$\Delta_F=\bigcup_{n=0}^{\infty} R^{*n}*[L^{\infty},R^{\infty}]=\bigcup_{n=1}^{\infty}\Delta'_n \ , \tag{2.18}$$

which is connected in the set of admissible words. In the parameter interval corresponding to the class the dynamic system does not exhibit complexity.

When $\mathbf{Q}\in\mathcal{P}_\pi$, because $h(\mathbf{Q}*\mathbf{W})=h(\mathbf{Q})\neq 0$ by (2.15), then $\mathbf{Q}*\mathbf{W}$, $\mathbf{W}\in\mathcal{W}$, forms an equal topological entropy class of nonzero topological entropy, and has the primitive word $\mathbf{Q}$ as the label of the class. Considering the continuity of map $f(x,\lambda)$ on the parameter λ and for a specific $\mathbf{Q}\in\mathcal{P}_\pi$, the values of the parameter of the ETEC $\mathbf{Q}*\mathbf{W}$ ($\mathbf{W}\in\mathcal{W}$) distribute in a connected interval on λ axis. In the entropy plot h-λ (Fig. 1), the ETEC $\mathbf{Q}*\mathbf{W}$ forms a step $H_{\mathbf{Q}}$, with a constant topological entropy $h(\mathbf{Q})$.

As the $*$ composition rule keeps MSS order, the words contained in the step $H_{\mathbf{Q}}$ can be expressed approximately as $(\mathbf{Q},\mathbf{Q}*RL^{\infty})$. The cases on two margins $\mathbf{Q}$ and $\mathbf{Q}*RL^{\infty}$ need considering carefully. Corresponding to the lower margin $\mathbf{Q}$, as a superstable word, there is a periodic window $(\mathbf{Q}^-,\mathbf{Q},\mathbf{Q}^+)$. Here, $\mathbf{Q}^-=(Q_1Q_2\cdots Q_qL^{t(\mathbf{Q})})^{\infty}$ and $\mathbf{Q}^+=(Q_1Q_2\cdots Q_qR^{t(\mathbf{Q})})^{\infty}$ are nonsuperstable sequences in the periodic window, called the lower sequence and the upper sequence of superstable word $\mathbf{Q}$, respectively [5,9]. It is proved readily that

$$h(\mathbf{Q}^-)=h(\mathbf{Q}^+)=h(\mathbf{Q}) \ . \tag{2.19}$$

Thus the minimal word in the step $H_{\mathbf{Q}}$ extends to $\mathbf{Q}^-$. The upper margin $\mathbf{Q}*RL^{\infty}$, however, is the limit point of $\mathbf{Q}*RL^n$, $n\to\infty$, with no extension. Therefore, all of the words contained in the step $H_{\mathbf{Q}}$ can be expressed exactly as

$$[\mathbf{Q}^-,\mathbf{Q}*RL^{\infty}]=\mathbf{Q}*[L^{\infty},RL^{\infty}] \ . \tag{2.20}$$

Denoting the parameter of a word $\mathbf{W}\in\mathcal{W}$ by $\lambda_{\mathbf{W}}$, we introduce the Lebesgue measure in space $\mathcal{W}$

$$|[\mathbf{W}_1,\mathbf{W}_2]|=\lambda_{\mathbf{W}_2}-\lambda_{\mathbf{W}_1},\quad \mathbf{W}_1<\mathbf{W}_2 \ . \tag{2.21}$$

Then the Lebesgue measure of the step $H_{\mathbf{Q}}$ is

$$\Lambda_{h(\mathbf{Q})}=|\mathbf{Q}*(L^{\infty},RL^{\infty})| \ . \tag{2.22}$$

The order of admissible words is isomorphic with the order of parameters (i.e., the order of real numbers). The one-to-one correspondence between λ and $\mathbf{W}$ maintains the continuity, ie..,

$$\lim_{\mathbf{W}_1\to\mathbf{W}_2}\lambda_{\mathbf{W}_1}=\lambda_{\mathbf{W}_2} \ . \tag{2.23}$$

The compactness of a closed interval on the λ axis means that $\mathcal{W}$ is a compact closed set with cardinal number $\aleph_1$. Owing to the order-preserving property of the DGP $*$ composition, it is easy to prove the compressibility of the operation $\mathbf{W}*$, $\mathbf{W}\in\mathcal{W}$:

$$|\mathbf{W}*[L^{\infty},RL^{\infty}]|<|[L^{\infty},RL^{\infty}]| \ . \tag{2.24}$$

Furthermore, we notice that $\mathcal{W}$ contains words of infinite length as parts of admissible words. To sum up, we have therefore obtained that the operation $\mathbf{Q}*$, $\mathbf{Q}\in\mathcal{P}_\pi$ is a compression operator which compresses the space $\mathcal{W}$ of all admissible words to an ETEC $H_{\mathbf{Q}}$, and the ETEC $H_{\mathbf{Q}}$ is a compact set with cardinal number $\aleph_1$, into which words of infinite length are embedded.

C. Infinite words

The word without a repeatedly characteristic byte must have infinite length, and is called an infinite word.

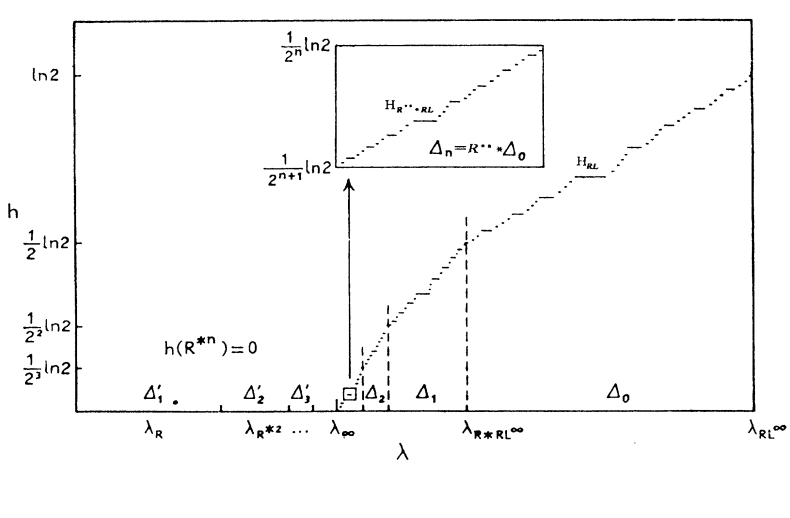

FIG. 1. The complete devil's staircase of topological entropy is a continuous multifractal curve of interval similarity in the plot of h vs λ, which is shown schematically. The higher the resolution in the plot, the more the steps may be seen.

There are many infinite words, which occupy a quite large measure in $\mathcal{W}$ [10]. For an ETEC step $H_{\mathbf{Q}}$ ($\mathbf{Q}\in\mathcal{P}_\pi$), when the period $\|\mathbf{Q}\|=q+1$ increases and extends to infinity, $\mathbf{Q}*L^\infty$ and $\mathbf{Q}*RL^\infty\to\mathbf{Q}$, $\Lambda_{h(\mathbf{Q})}\to 0$. Thus $\mathbf{Q}*L^\infty$ and $\mathbf{Q}*RL^\infty$ will coincide at $\mathbf{Q}$, and the step $H_{\mathbf{Q}}$ approaches a single point in h-λ space. In other words, an infinite word is exhibited as a single point in the entropy plot h-λ, not a step of an ETEC. Then how does the $\|\mathbf{Q}\|$ extend to infinity? This is a complicated problem. The following classes have now been studied.

(1) Eventually periodic sequences. Topological entropy of the type of kneading sequences $\rho\lambda^\infty$, which are generated by means of the generalized composition rule [11], can be calculated with the generalized characteristic polynomial [9]. There are many series of kneading sequences which converge to $\rho\lambda^\infty$. The series of $\rho\lambda^n|_C$, $n=1,2,\ldots$, is the simplest (the notation $\rho\lambda^n|_C$ means the last letter of $\rho\lambda^n$ is replaced by C). It is found that there is no uniform universal convergence rate δ in the series of $\rho\lambda^n|_C$, $n=1,2\ldots$ [12]. When the parameter of $f(x,\lambda)$ adopts values in the generating set [13], there also occurs another infinite kneading sequence, the type of $\mathbf{A}(\mathbf{B})^\infty$, which does not always satisfy the generalized composition rule. The topological entropy of the type of kneading sequence $\mathbf{A}(\mathbf{B})^\infty$ can be directly calculated with the Milnor-Thurston characteristic polynomial or the Stefan matrix. These two kinds of infinite kneading sequences are eventually periodic sequences, and can be approached by appropriately choosing the series of finite kneading sequences like $\rho\lambda^n|_C$, $\mathbf{AB}^n|_C$, $n=1,2,\ldots$, which are located in different ETEC steps. Since infinite words $\rho\lambda^\infty$ and $\mathbf{A}(\mathbf{B})^\infty$ are analogous to the word RL^∞, it is believed that they describe coarse-grained chaos [11]. Most likely they are located at the right-hand margin of an ETEC.

(2) Intermittency sequences. The intermittent transition to chaos before a tangent bifurcation. The well-known example is the one of the period-3 window $((RLR)^\infty,RLC,(RLL)^\infty)$. The sequences $R[(LRR)^kRR]^\infty$, $k=1,2,\ldots$, exhibit a chaotic regime with intermittent period-3 behavior. Clearly, the sequences $R[(LRR)^kRR]^\infty$ belong to the type of $\rho\lambda^\infty$ [11]. Most likely they are located at the left-hand margin of an ETEC.

(3) Fibonacci sequences. Tsuda [14] and Hao [9] found the Fibonacci bifurcations in the Belouzov-Zhabotinsky reaction system. Shibayama [15] set the rule for constructing recursively superstable periodic sequences whose periods increase as the Fibonacci numbers, i.e., $p_n=p_{n-1}+p_{n-2}$. For example, starting from RC,RLC, the Fibonacci sequences are $RLLRC$, $RLLRRRLC$, $RLLRRRLRRLLRC,\ldots$. Obviously, these superstable sequences belong to different ETEC steps, whose limit is an infinite word and is expressed as a single point.

There must be other kinds of infinite words which may or may not be formed according to a certain construction rule. It is important that each of the infinite words corresponds to a single point in h-λ space, which distributes in the gap between two consecutive ETEC steps. It may be concluded that these points of infinite words will densely combine ETEC steps into the continuous topological entropy curve $h(\lambda)$. We will return to the infinite words formed at random in the conclusion.

III. THE STRUCTURE OF ENTROPY DEVIL'S STAIRCASE

A. Entropy devil's staircase

The totality of ETEC steps for finite primitive words and single points for infinite words forms a devil's staircase in h-λ space (Fig. 1). According to the connectivity of the bifurcation diagram, all ETEC steps are divided into two classes of zero and nonzero entropies. The largest step in the entropy devil's staircase is the zero-entropy ETEC $\Delta_F=\cup_{n=1}^{\infty}\Delta'_n$, which extends from the left-hand to the accumulation point $R^{*\infty}$. The nonzero-entropy interval is $[R^{*\infty},RL^\infty]$. The minimal primitive word is the limit word $RLR^\infty=R*RL^\infty$, which is the merging point of the $1I$ band to the $2I$ band. Therefore we can redivide the nonzero-entropy interval into an infinite number of subintervals Δ_m by the $*$ composition rule:

$$[R^{*\infty},RL^\infty]=\bigcup_{m\in\mathbb{Z}^+}\Delta_m$$
$$=\bigcup_{m\in\mathbb{Z}^+}R^{*n}*[RLR^\infty,RL^\infty]\ . \quad (3.1)$$

The typical representative of Δ_m is the subinterval

$$\Delta_0=[RLR^\infty,RL^\infty]\ , \quad (3.2)$$

which contains an infinite number of ETEC steps:

$$H_{\mathbf{Q}_i}=\mathbf{Q}_i*[L^\infty,RL^\infty],\quad \mathbf{Q}_i\in\mathcal{P}_\pi,\quad i=1,2,\ldots\ . \quad (3.3)$$

The structure of Δ_0 is a generic prototype of the structure of the entropy devil's staircase.

B. The fractal structure of Δ_0

The numbers $N(n)$ of primitive words with period n in $\mathcal{W}$ are expressed by the following recursive formula [16]:

$$N(n)=\frac{1}{2n}\left[2^n-2-\sum_{i=1}^{p}2^i\right.$$
$$\left.-2\sum_{\{d:d|n\}}^{1<d<n}Q_0\prod_j d_jN(d_j)Q(d_j)\right]\ , \quad (3.4)$$

where $Q_0=2^k$ $(k\geq 0)$ and $Q(d)$ is defined recursively in the form

$$Q(d)=2+\sum_{i=1}^{q}2^i+2\sum_{\{c:c|d\}}^{1<c<d}Q_0\prod_k c_kN(c_k)Q(c_k)\ . \quad (3.5)$$

When $n=p$ is a prime number, (3.4) becomes quite a simple formula [17]

$$N(p)=(2^{p-1}-1)/p\ . \quad (3.6)$$

If we take $N(p)$ as the lower bound of the cardinal number of the primitive word subset with period $n=p$, we know immediately that the primitive word set $\mathcal{P}_\pi$ has the cardinal number $\aleph_1$ of continuum.

When steps in Δ_0 are observed with finite resolution, only the steps of the primitive words with period $\leq n$ (integer) are displayed; the steps of the primitive words with period $>n$ are treated as single points. Under this condition, $N_n=\sum_{k=1}^{n}N(k)$ is finite, and primitive words $\mathbf{Q}_i$, $i=1,2,\ldots,N_n$, i.e., N_n steps in Δ_0, can be ordered; meanwhile the points between two consecutive steps $H_{\mathbf{Q}_i}$ and $H_{\mathbf{Q}_{i+1}}$ correspond to finite primitive words with period $>n$ and infinite words. The higher the resolution, the more steps are displayed in Δ_0. Thus Δ_0 has a progressive embedding fractal structure. The fractal dimension of Δ_0 will be discussed in the following paper.

C. Similarity between subinterval of entropy devil's staircase

Subinterval $\Delta_0=[RLR^{\infty},RL^{\infty}]$ corresponds to the parametric length $[\lambda_{RLR^{\infty}},\lambda_{RL^{\infty}}]$ on the λ axis and the length $[\frac{1}{2}\ln 2,\ln 2]$ on the h axis because of $h(RL^{\infty})=\ln 2$ and $h(RLR^{\infty})=h(R*RL^{\infty})=\frac{1}{2}\ln 2$. Subinterval Δ_1 is a compression of subinterval Δ_0:

$$\Delta_1=R*[RLR^{\infty},RL^{\infty}]=R*\Delta_0\ . \tag{3.7}$$

By (2.15), the topological entropies at the lower and upper boundary of subinterval Δ_1, respectively, are

$$h(R*RLR^{\infty})=\tfrac{1}{4}\ln 2,\quad h(R*RL^{\infty})=\tfrac{1}{2}\ln 2\ . \tag{3.8}$$

We see that the upper boundary of Δ_1 is connected with the lower boundary of Δ_0.

The relation between subintervals is Δ_{m+1} and Δ_m, $m=1,2,\ldots$, is similar to the one between Δ_1 and Δ_0. From the above, it can be seen that the compression ratio of compression operator $R*$ along the h axis is $\frac{1}{2}$. By numerical calculation in the quadratic map $f(x)=1-\lambda x^2$, the compression ratio along the λ axis is the universal Feigenbaum constant

$$\lim_{m\to\infty}\frac{|R^{*m}*\mathbf{Q}_i*[L^{\infty},RL^{\infty}]|}{|R^{*m+1}*\mathbf{Q}_i*[L^{\infty},RL^{\infty}]|}=\delta(R)=4.6692\ldots\ . \tag{3.9}$$

That is, there is similarity between subintervals, $\Delta_0\sim\Delta_1\sim\Delta_2\sim\cdots$ in the entropy devil's staircase, and there exists the same fractal structure in subintervals Δ_m, $m=1,2,\ldots$ as that in Δ_0. The entropy devil's staircase is completely different from the Jensen-Bak-Bohr devil's staircase in the circle map [4]. The analysis of the ETEC and the entropy devil's staircase is established on the basis of the universality of MSS sequences, and therefore the structure of the entropy devil's staircase exists in all 1D unimodal maps. This is a global regularity in 1D unimodal maps.

IV. UNIVERSALITY AND NONUNIFORM COMPRESSION OF STAIR-CLIMBING SEQUENCE

In Sec. II A, we know that the main Feigenbaum bifurcation sequence, including the sequence from an accumulation point to chaos, $\mathbf{Q}^{*n}$, $n=1,2,\ldots$, is one that preserves topological entropy. From (2.15), the associated Feigenbaum bifurcation sequences $\mathbf{Q}^{*m}*\mathbf{S}$, $\mathbf{Q}*\mathbf{S}^{*m}$, $m=1,2,\ldots$, also preserve topological entropy. In our numerical experiment, we found that the convergence rates in the above sequences are $\delta(\mathbf{Q})$, $\delta(\mathbf{Q})$, and $\delta(\mathbf{S})$, respectively. This shows that various universal convergence rates are confined in different ETEC steps. Especially, the universal Feigenbaum convergence rate $\delta(R)=4.6692\ldots$ is confined in the zero-entropy step Δ_F. Essentially, convergence rates $\delta(R)$, $\delta(\mathbf{Q})$, and $\delta(\mathbf{S})$ exhibit the compressibilities of operators $R*$, $\mathbf{Q}*$, and $\mathbf{S}*$, respectively. It is then quite natural that the convergence rate $\delta(R)$ appears in the series of $\Delta_0,\Delta_1,\Delta_2,\ldots$ as the compression ratio. This is an inevitable result of the uniform compressibility of the operator $R*$, which forms a uniformly convergent *geometric* series. By comparison, the Fibonacci sequences only show a tendency to universality [15]. In this case, the Fibonacci sequences are a kind of stair-climbing compression sequence and form a nonuniformly convergent *hypergeometric* series. By numerical calculation in the quadratic map, taking RC and RLC as the first two stable periodic orbits to form the Fibonacci sequences and calculating parameter value λ_ns, the convergence rate is [15]

$$\delta=\lim_{n\to\infty}\frac{(\lambda_n-\lambda_\infty)(\lambda_{n+2}-\lambda_\infty)}{(\lambda_{n+1}-\lambda_\infty)^2}\approx 0.62\ldots\ . \tag{4.1}$$

In the series $\rho\lambda^n|_C$, $n=1,2,\ldots$, there is no universal convergence rate δ in the Feibenbaum sense [12]. However, it is still a stair-climbing compression sequence. It is possible that there are many ways to form new convergence rates. A new convergence rate will depend on double compressibilities, one comes from the compressibility in a step, i.e., the Feigenbaum compression, the other from the actual way in the stair-climbing sequences. The former is uniform and the latter is nonuniform. Therefore the universal convergence rate $\delta(\mathbf{Q})$ originates from the uniform compressibility of the operator $\mathbf{Q}*$. On the other hand, we can use nonuniformly convergent hypergeometric series to form new convergence rates in various stair-climbing sequences. We shall discuss the details of this interesting problem in another paper.

V. CONCLUSION

From the above, we can divide the set $\mathcal{W}$ of all admissible sequences for symbolic dynamics (L,R) of 1D unimodal maps into three subsets,

$$\mathcal{W}=\mathcal{P}\cup\mathcal{P}_e\cup\mathcal{A}\ .$$

(1) $\mathcal{P}$ is the set of regular periodic sequences, including superstable sequences, nonsuperstable sequences in periodic windows, and compound sequences by a finite number of grammatical rules (e.g., the DGP $*$ composition rule). They are located inside the steps in the entropy plot h-λ and have nonpositive Lyapunov exponents (≤ 0). There is no chaos in the set $\mathcal{P}$. (2) $\mathcal{P}_e$ is the set of sequences of infinite length constructed by a finite number of grammatical rules, such as the eventually periodic sequences, $\rho\lambda^{\infty}$ type sequences, $\mathbf{A}(\mathbf{B})^{\infty}$ type sequences, and the limits of intermittency sequences or the Fibonac-

ci sequences. Each of these members is only a single point located outside the steps in the entropy plot and has a positive Lyapunov exponent (>0). Hence they are coarse-grain chaos because of the finite number of grammatical rules. (3) $\mathcal{A}$ is the set of entirely irregular aperiodic sequences, called the Bernoulli-Chaitin-Ford infinite sequences. None of these symbolic sequences can be generated by a finite number of grammatical rules, but all of them can pass any random test. That is to say, when L and R in a Bernoulli-Chaitin-Ford sequence are transformed into 0 and 1, the sequence becomes a binary sequence and is a random real number or at least a normal number [18]. This kind of kneading sequence starts from the critical point C and extends aperiodically and infinitely. They are single points outside the steps in the entropy plot and have positive Lyanunov exponents (>0). These sequences have very considerable measure in $\mathcal{W}$, which has been argued by the Martin-Löf theorem [10] and by our numerical experiment. They are fine-grain chaos in symbolic dynamics.

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